§1: Stable pair theory (Pandharipande - Thomas)

X ·· smooth proj C.Y.3 $/\mathbb{C}$. $(K_X = \mathcal{O}_X)$

- DT-type inv, counting # of obj in $D^b \mathrm{Coh} X$
- Mathematical definition of Gopakumar-Vafa inv

~~...~~ counting obj in $\mathrm{Coh}_{\le 1} X$

$\mathrm{Coh}_{\le 1} X := \{ E \in \mathrm{Coh} X : \dim \mathrm{Supp} E \le 1 \}$

Def: (F, s) is a stable pair $\Leftrightarrow$

- $F \in \mathrm{Coh}_{\le 1} X$, pure i.e. $\nexists Q \hookrightarrow F$ $\dim \mathrm{Supp} Q = 0$
- $s: \mathcal{O}_X \to F$ $\dim \mathrm{Supp} \mathrm{Cok}(s) = 0$

Ex: $C \hookrightarrow X$ smooth curve $D \subset C$ divisor

$\Rightarrow s: \mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \hookrightarrow \mathcal{O}_C(D)$ $(\mathcal{O}_C(D), s)$ is a stable pair.

$\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$

$P_n(X, \beta) =$ moduli space of stable pairs (F, s)

$[F] = \beta$, $\chi(F) = n$

$\uparrow$ projective scheme

with 0-dim virtual cycle.

$$P_{n,\beta} := \int_{[P_n(X,\beta)]^{vir}} 1 \quad \in \mathbb{Z}$$

$$PT(X) := \sum P_{n,\beta}\, q^n t^\beta \qquad PT_\beta(X) := \sum_n P_{n,\beta}\, q^n$$

Conj (PT)

(1) $PT_\beta(X)$ is the Laurent expansion of a rat func of q, invariant under $q \leftrightarrow 1/q$ (rationality conj)

(2) $PT(X) = \exp\left(\sum N^{GW}_{g,\beta}\, \lambda^{2g-2} t^\beta\right)$ via $q = -e^{i\lambda}$ (GW/PT correspondence)

$N^{GW}_{g,\beta}$: # of stable maps $\Sigma \xrightarrow{f} X$, $g(\Sigma) = g$, $f_*[\Sigma] = \beta$

Assuming (2), $\exists \{n_{g,\beta}\}_{g \geq 0, \beta > 0} \in \mathbb{Z}$ s.t.

$$PT(X) = \prod_\beta \prod_{j \geq 1} \left(1 - (-q)^j t^\beta\right)^{j n_{0,\beta}} \prod_{\substack{g \geq 1 \\ 0 \leq k \leq 2g-2}} \left(1 - (-q)^{g-1-k} t^\beta\right)^{(-1)^{k+g}\binom{2g-2}{k} n_{g,\beta}} \quad \cdots (*)$$

(1.5) $PT(X)$ has the form $(*)$ (Strong rationality conj)

$\{n_{g,\beta}\}$ .. math. def of GV inv

Thm:

$\exists\, N_{n,\beta} \in \mathbb{Q}$ $\quad L_{n,\beta} \in \mathbb{Q}$ ~~s.t.~~ satisfying

- $N_{n,\beta} = N_{-n,\beta}$, $\quad L_{n,\beta} = L_{-n,\beta}$
- $\exists\, d \in \mathbb{Z}$ s.t. $N_{n,\beta} = N_{n+d,\beta}$
- $L_{n,\beta} = 0$ for $|n| \gg 0$ $\quad$ s.t.

$$PT(X) = \prod_{n \geq 0, \beta > 0} \exp\left((-1)^{n-1} n N_{n,\beta}\, q^n t^\beta\right) \left(\sum_{n,\beta} L_{n,\beta}\, q^n t^\beta\right)$$

Cor: (1) Conj (1) is true

(2) Conj (1.5) is true if and only if

$$N_{n,\beta} = \sum_{r \mid (n,\beta)} \frac{1}{r^2} N_{1,\beta/r}.$$

Roughly

$N_{n,\beta}$ = # of ω-semistable $E \in \mathrm{Coh}_{\leq 1} X$, $ch_2 E = \beta$, $ch_3 E = n$

$L_{n,\beta}$ = # of "perverse sheaves" ~~on~~ $E \in D^b \mathrm{coh} X$, $ch E = (1, 0, -\beta, -n)$, semistable w.r.t self dual stab cond.

. generalized PT-inv
. doesn't depend on ω.

Strategy: Study WCF on

$$D = \langle \mathrm{Coh}_{\leq 1} X, \mathcal{O}_X \rangle_{tr} \hookrightarrow D^b \mathrm{coh} X$$

smallest triangulated subcat which contains $\mathcal{O}_X$, $\mathrm{Coh}_{\leq 1} X$.

($\exists$ another pf by Bridgeland, without using stability cond)

§2: Stability conditions on D

Lem: $\exists\ \mathcal{A} \subset D$ heart of bounded t-str s.t.

$$\mathcal{A} = \langle \mathrm{Coh}_{\leq 1} X[-1],\ \mathcal{O}_X \rangle_{ex} \subset D$$

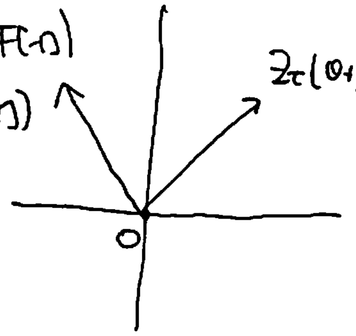
$(\mathcal{O}_X \xrightarrow{s} F) \in$ ext-closed sub cat which contains $\mathrm{Coh}_{\leq 1} X[-1]$, $\mathcal{O}_X$.

Fix ample div ω on X.

For $t \in \mathbb{C}$, $\mathrm{Im}\, t > 0$, set

$$Z_t : K(D) \to \mathbb{C}, \quad E \mapsto \mathrm{ch}_3 E - i\omega\, \mathrm{ch}_2 E + t \cdot \mathrm{ch}_0 E$$

Lem: $(Z_t, \mathcal{A}) \in \mathrm{Stab}\, D$.

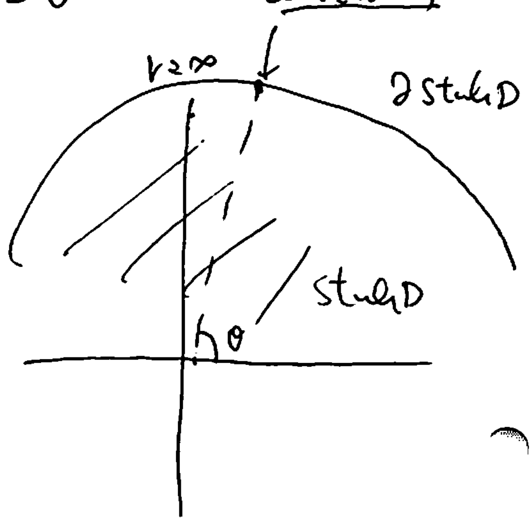


Simplification:

$t = r e^{i\pi\theta}$ $\quad 0 < \theta < 1 \quad 0 \neq E \in \mathcal{A}$

$$\frac{1}{\pi} \lim_{r \to \infty} \arg Z_t(E) = \begin{cases} \theta & \text{if } \mathrm{rk}\, E > 0 \\ \frac{1}{\pi} \arg Z_0(E) & \text{if } \mathrm{rk}\, E = 0 \end{cases}$$

$=: \phi_\theta(E)$



Def: $0 \neq E \in \mathcal{A}$ is θ-(semi)stable

$\iff \forall\ 0 \to F \to E \to G \to 0$,

$$\phi_\theta(F) \underset{(\leq)}{<} \phi_\theta(G)$$

$v \in H^*(X, \mathbb{Q})$

$\mathcal{M}_\theta(v) :=$ moduli stack of $E \in \mathcal{A}$, θ-semistable with
$\mathrm{ch}\, E = v$.

If $v = (r, 0, -, -) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6$

$r = 0$ or 1

$\Rightarrow \mathcal{M}_\theta(v)$ .. alg stack of finite type.

<u>Thm</u> :

(1) If $0 < 1-\theta \ll 1$,

$$\mathcal{M}_\theta(1, 0, -\beta, -n) = \left[P_n(X,\beta) / \mathbb{G}_m \right] \qquad \mathbb{G}_m \curvearrowright P_n(X,\beta) \text{ trivial}$$

(2) For $\theta = \frac{1}{2}$, we have

- $\mathcal{M}_{\frac{1}{2}}(r, 0, -\beta, -n) \xrightarrow{\cong} \mathcal{M}_{1/2}(r, 0, -\beta, n)$, $r = 0, 1$

  $E \longmapsto R\mathcal{H}om(E, \mathcal{O}_X)$

- $\mathcal{M}_{\frac{1}{2}}(0, 0, -\beta, -n) \xrightarrow[\otimes \mathcal{L}]{\cong} \mathcal{M}_{\frac{1}{2}}(0, 0, -\beta, -n-d)$

  $\mathcal{L} \in \mathrm{Pic}\, X$ $\quad c_1(\mathcal{L}) = \omega$, $d = \omega \cdot \beta$

- $\mathcal{M}_{1/2}(1, 0, -\beta, -n) = \emptyset$ for $|n| \gg 0$.

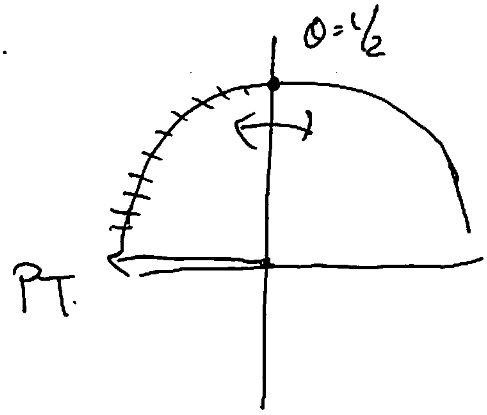
Set $N_{n,\beta} =$ # of obj in $\mathcal{M}_{\frac{1}{2}}(0, 0, -\beta, -n) \cong$ (ω-Gieseker semistable $F \in \mathrm{Coh}_1(X)$, $[F] = \beta$, $\chi(F) = n$)

$L_{n,\beta} =$ # : $\mathcal{M}_{\frac{1}{2}}(1, 0, -\beta, -n)$

~~Thm~~ $N_{n,\beta}$ $L_{n,\beta}$ satisfy desired property.

WCF from $\theta = 1/2$ to $\theta = 1$

$\Rightarrow$ desired formula.



§3 : Example.

$f : X \to S$ elliptic fib i.e. gen fib is elliptic curve.

$$PT(X/S) := \sum_{\substack{n,\beta \\ f_*\beta = 0}} P_{n,\beta}\, q^n t^\beta$$

Thm: (1) $f_*\beta = 0 \Rightarrow N_{n,\beta} = \sum_{r \mid (n,\beta)} \frac{1}{r^2} N_{(1,\beta/r)}$. In particular strong rationality holds for $PT(X)$

(2) If f -- flat, S -- smooth

$\Rightarrow n_{0,\beta} = -\chi(X)$ $\quad n_{1,\beta} = -\chi(S)$ $\quad n_{g,\beta} = 0$ $\quad g \geq 2$

i.e.

$$PT(X/S) = \prod_{\substack{j \geq 1 \\ f_*\beta = 0}} \left(1 - (-q)^j t^\beta\right)^{-\chi(X)} \prod_{k \geq 1} \left(1 - t^k\right)^{-\chi(S)}$$

$$DT_\theta := \sum_{n,\beta} \#\left(M_{\theta}(1, 0, -\beta, -n)\right) q^n t^\beta$$

- $\lim_{\theta \to 1} DT_\theta = PT$

- $DT_{\frac{1}{2}} = L$

$$\prod_{\substack{n,\beta \\ -n+\beta\cdot\omega i \\ \in \mathbb{R}_{>0}\,\cdots}} \exp\left((-1)^{n-1} n N_{n,\beta}\, q^n t^\beta\right)$$