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I. Motivation

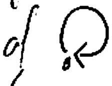
Generating functions:

a) $\sum_{n\geq 0} q^n = \frac{1}{1-q} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} q^n\right) =: \varphi_1(q)$

b) $\sum_{\lambda \text{ 2d part.}} q^{|\lambda|} = \sum_{n\geq 0} p_2(n) q^n = \prod_{n\geq 1} \frac{1}{1-q^n} = \exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{q^n}{1-q^n}\right) =: \varphi_2(q)$

c) $\sum_{\pi \text{ 3d part}} q^{|\pi|} = \sum_{n\geq 0} p_3(n) q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^n} = \exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{q^n}{(1-q^n)^2}\right)$ McMahon fct.

Interpretation: (a) $\longleftrightarrow$ v.bs. over a pt. $\longleftrightarrow$ $\bullet$

(b) $\longleftrightarrow$ torsion sheaves on $\mathbb{C}$ supp. at $x=0$ $\longleftrightarrow$ cont. repr. of $\mathbb{C}[[x]]$ = cont. repr. of 

[Indeed, x acts by JNF $\left\{ \begin{pmatrix} \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} & \\ & \end{pmatrix} \end{pmatrix} \right\}$, $\sum n_i = n$]
$n_1 \quad \cdots \quad n_k$

(c) $\longleftrightarrow$ torsion sheaves on $\mathbb{C}^3$, supp. at origin $\longleftrightarrow$ $\mathbb{C}[[x,y,z]]$-cont. repr.
$\longleftrightarrow$ cont. repr. of  Q_3
$xy=yx,\ xz=zx,\ yz=zy$

[or rather $W \in \mathbb{C}Q_3/[\mathbb{C}Q_3, \mathbb{C}Q_3]$, $W = xyz - xzy$ $\longleftrightarrow$ $\mathbb{C}Q_3 = \mathbb{C}\langle x,y,z\rangle / I$.
Then $I = \partial W$, : $\frac{\partial W}{\partial x} = yz - zy$, $\frac{\partial W}{\partial y} = zx - xz$, $\frac{\partial W}{\partial z} = xy - yx$ (cyclic derivatives)]

Unifying pt. of view: $Z = \sum_{n\geq 0} q^n \left(\int_{\mathcal{M}_n} \xi_n\right)$ ← char. class of bdl. on $\mathcal{M}_n$
↑ moduli space

Donaldson – Thomas (6d theory): X 3-CY mfld $\left(\begin{array}{l}\text{cplx, compact, 3d,} \\ \Lambda^{top}_X T^*_X = \mathcal{O}_X \leftrightarrow \Omega^{3,0}\text{-hol.}\end{array}\right)$

$E \downarrow X$ vb.

Chern-Simons $CS_{\mathbb{C}}(\underset{\substack{\uparrow \\ (0,1)\text{-conn. on } E}}{A = A_0 + a}) = \int_X \left(\frac{1}{2} \mathrm{Tr}(\bar\partial_{A_0} a, a) + \frac{1}{3} \mathrm{Tr}(a \wedge a \wedge a)\right) \wedge \Omega^{3,0}$

X 3-CY $\Rightarrow$ virt. dim. of $\mathrm{Crit}\, CS_{\mathbb{C}}$ is zero.

$\mathrm{Crit}(CS_{\mathbb{C}}) = \{\text{holom. connect. on } E\} \Rightarrow$ counting of crit pts $\overset{}{=}$ counting of holom. vb's of same top. type

Bad moduli space $\Rightarrow$ impose stability conditions ((count semi-) stable bdls).

Use Donaldson / Uhlenbeck / Yau: Fix Kähler metric ω, solve HYM eqns $\begin{cases} F_A^{0,2} = 0 \\ \omega \wedge F_A \wedge \omega = \text{const} \end{cases}$

Counting $\to$ walls in $\{[\omega]\}$.

Philosophy from HMS: for counting pblm look for underlying category.

<u>Here:</u> Underlying dg-category

$(\Omega^{0,*}(X), \bar\partial)$ – dg modules, which are projective as graded modules (with C^∞-topol.)

dg-algebra $\Downarrow$ holom. vector bundles

$\downarrow$

triangulated envelope of a dg-algebra $\rightsquigarrow$ $\mathrm{Perf}(X)$ category of perfect sheaves on X (X projective)

[or $D^b(X)$ – bd. derived categ. of coh. sheaves]

<u>Rem:</u> Bondal - van den Bergh theorem: $D^b(X) = \langle P \rangle$ (generator) $\Rightarrow D^b(X) \simeq D^b(A\text{-mod})$, $A = \mathrm{End}(P)^{op}$.

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$\mathcal{C}$ - dg-category $/k$:

$m_1 : \mathrm{Hom}^\bullet(E,E) \to \mathrm{Hom}^\bullet(E,E)[1]$, $m_1^2 = 0$ $\quad$ ($\deg m_1 = 1$, $E \in \mathrm{Ob}(\mathcal{C})$),

$m_2 : \mathrm{Hom}^\bullet(E,E) \otimes \mathrm{Hom}^\bullet(E,E) \to \mathrm{Hom}^\bullet(E,E)$, $m_2(f,g) = fg$

Rel. $(m_1 + m_2)^2 = 0$.

3-CY condition means: $\exists\, \mathrm{Tr} = (\,,\,) : \mathrm{Hom}^i(E,E) \otimes \mathrm{Hom}^{3-i}(E,E) \to k$

Can write: $W_E(\alpha) = \frac{1}{2}(m_1(\alpha),\alpha) + \frac{1}{3}(m_2(\alpha,\alpha),\alpha)$ potential, $\alpha \in \mathrm{Hom}^1(E,E)$.

Further generalization:

A_∞-category $\mathcal{C}/k$: $m_1, m_2, \dots$

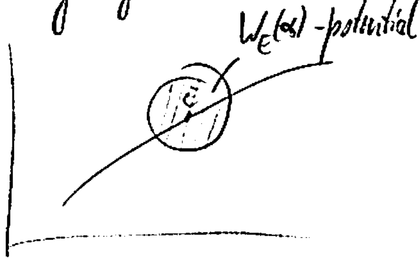
$m_n : \bigotimes \mathrm{Hom}(E_i, E_{i-1}) \to \mathrm{Hom}(E_1, E_n)$, $\deg m_n = 2-n$. $(E_1, \dots, E_n \in \mathrm{Ob}(\mathcal{C}))$.

Rel: $(\sum m_i)^2 = 0$.

3-CY struct. $(\cdot,\cdot) : \mathrm{Hom}^i(E,F) \otimes \mathrm{Hom}^{3-i}(F,E) \to \mathbb{C}$

$\Rightarrow W_E(\alpha) = \sum_{n \geq 2} \frac{(m_{n-1}(\alpha, \dots, \alpha), \alpha)}{n}$ (formal sum), $\alpha \in \mathrm{Ext}^1(E,E)$

" tangent space to formal def's of E.



$\mathcal{C}$ - 3CY A_∞-category, $\gamma \in K_0(\mathcal{C})$

Theory of stability conditions on triangulated categories (Bridgeland)

$\Rightarrow \mathcal{C}^{ss}$ - (semi-) stable objects

$\mathcal{C}^{ss}_\gamma = \{$ semistables E with $[E] = \gamma \}$

Partition function $Z^{mot}(\mathcal{C}) = \sum_\gamma \#[\mathcal{C}^{ss}_\gamma]\, q^\gamma$ ← "motive of the space of semistables" $\in K_0(\mathrm{Var}_k)$