Talk 4: (Q,W) quiver w/ potential ⇝ fin. dim'l repr. ⇝ $\boxed{\text{cohomological Hall algebra}}$ = Cf. Harvey-Moore: BPS-algebra (90's)

$\updownarrow$

3 CY category with cluster gen's

today.

YS11

I - vertices, $i \xrightarrow{a_{ij}} j$

dim'l vector ⇝ $\gamma=(\gamma^i)_{i\in I}\in\mathbb{Z}^I_{\geq 0}=:\Gamma_+$. , , $k=\mathbb{C}$

$$M_\gamma = \prod_{i,j\in I}\mathbb{C}^{a_{ij}\gamma^i\gamma^j} \circlearrowleft \underline{G_\gamma}=\prod_i GL(\gamma^i,\mathbb{C}) \to \overset{EG_\gamma}{\downarrow}{BG_\gamma}=\prod_i BGL(\gamma^i,\mathbb{C})$$

$$=\prod_i Gr(\gamma^i,\mathbb{C}^\infty).$$

$$M^{univ}_\gamma = EG_\gamma\times M_\gamma/G_\gamma.$$ (W=0 for the time being).

$$\mathcal{H}=\bigoplus_{\gamma\in\Gamma_+}\mathcal{H}_\gamma=\bigoplus_{\gamma\in\Gamma_+}H^\bullet_{G_\gamma}(M_\gamma)=\bigoplus_i H^i(M^{univ}_\gamma)$$

$$m=\bigoplus m_{\gamma_1,\gamma_2}$$

$$H^\bullet_{G_{\gamma_1}}(M_{\gamma_1})\otimes H^\bullet_{G_{\gamma_2}}(M_{\gamma_2}) \longrightarrow H^{\bullet+?}_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1+\gamma_2})$$

$$\parallel$$

$$H_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2}) \xrightarrow{\simeq} H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1+\gamma_2})$$

$$\xrightarrow{push} H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) \xrightarrow{push} H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma)$$

[$M_{\gamma_1,\gamma_2}\subset M_\gamma$ subrep., $\gamma=\gamma_1+\gamma_2$

$\circlearrowleft$

$G_{\gamma_1,\gamma_2}=\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ parabolic

$\prod_i\mathbb{C}^{\gamma_1^i}\subset\prod_i\mathbb{C}^{\gamma_1^i+\gamma_2^i}$ is subrepr.]

Thm: m makes $\mathcal{H}$ into an associative algebra.

Expls $Q_d=$  d loops, $W=0$.

$$\mathcal{H}=\bigoplus_{n\geq 0}H^\bullet(BGL(n))[(d-1)n^2]$$ ↖ shift of grading

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$$(f_1\cdot f_2)(x_1,\dots,x_{n+m}) = \sum_{\substack{i_1<\dots<i_n\\ j_1<\dots<j_m\\ \{i_1,\dots,i_n,j_1,\dots,j_m\}=\{1,\dots,n+m\}}} f_1(x_{i_1},\dots,x_{i_n})\cdot f_2(x_{j_1},\dots,x_{j_m}) \prod_{k=1}^{n}\prod_{l=1}^{m}(x_{j_l}-x_{i_k})^{d-1}$$

[special case: Feigin–Odesski shuffle algebra]

Bigrading $\deg(P_k(x_1,\dots,x_n)) := (n, 2k+(1-d)n^2)$

- $d=0 \Rightarrow \mathcal{A}$ is a free fermion algebra $\simeq \Lambda^\bullet(\mathbb{C}[x])$
- $\updownarrow$ $d=1 \Rightarrow \mathcal{A}$ is a free boson algebra $\simeq \mathrm{Sym}^\bullet(\mathbb{C}[x])$.

<u>Q:</u> Define a structure of Fock space so that $\updownarrow$ becomes a boson-fermion correspondence.

Symmetric case: $\mathcal{A} = \bigoplus_{\gamma,k} \mathcal{A}_{\gamma,k}$, $\mathcal{A}_{\gamma,k} = H^{k-\chi_Q(\gamma,\gamma)}(BG_\gamma)$, $\chi_Q(\gamma_1,\gamma_2) = \sum_i \gamma_1^i\gamma_2^i - \sum_{i,j} a_{ij}\gamma_1^i\gamma_2^j$ — Euler form.

<u>Conj:</u> In symmetric case, $\mathcal{A}$ is a free supercomm. algebra generated by a graded vector space $V^{prim}\otimes\mathbb{C}[x]$, where the bidegree of x is $(0,2)$, and for any γ the space $V^{prim}_{\gamma,k}$ is $\neq 0$ only for finitely many k. (for any Q)

<u>For Q_d:</u>
$$P_d(z,q^{1/2}) = \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \dim \mathcal{A}_{n,m}\, z^n q^{m/2} = \sum_{n\geq 0} \frac{q^{(1-d)\frac{n^2}{2}}}{(1-q)\dots(1-q^n)} z^n \in \mathbb{Z}(q^{1/2})[[z]].$$

<u>Cor:</u> $\exists\, \delta(n,m) = \delta^{(d)}(n,m) \in \mathbb{Z}_{\geq 0}$, $n\geq 1$, $m\in(d-1)n+2\mathbb{Z}$ s.th.
$$P_d = \prod_{n\geq 1}\prod_{m\in\mathbb{Z}} \left((-1)^m q^{\frac{m}{2}} z^n; q\right)_\infty^{(-1)^{n-1}\delta(n,m)},$$ where $(x;q) = (1-x)(1-xq)(1-xq^2)\dots$

<u>Rem:</u> $P_d(x,q^{-1})$ is a motivic DT-invt for 3-CY category associated with quiver Q and potential W.

- In non-symm. case, $\mathcal{A}$ is graded by Heisenberg group assoc. with $\chi_Q(\gamma_1,\gamma_2)$. $m_{\gamma_1\gamma_2}: \mathcal{A}_{\gamma_1}\otimes\mathcal{A}_{\gamma_2}\to\mathcal{A}_{\gamma_1+\gamma_2}\otimes\mathbb{T}^{\otimes\chi_Q(\gamma_1,\gamma_2)}$.

<u>Now $W\neq 0$</u>: Replace $H^\bullet(X)$ by $H^\bullet(X,f)$:

Technically, use "rapid decay cohomology": $f: X\to\mathbb{C}$ -hol., then

$$H^\bullet(X,f) := \lim_{t\to-\infty} H^\bullet(X, f^{-1}(t)).$$

(stabilizes for $|t|\gg 0$).

$$\mathcal{A} = \bigoplus_\gamma \mathcal{A}_\gamma = \bigoplus_\gamma H^\bullet\Big(\underset{\overset{\shortparallel}{EG_\gamma\times M_\gamma/G_\gamma}}{M_\gamma^{univ}},\ \underset{\overset{\shortparallel}{\mathrm{Tr}(W)|_{M_\gamma}}}{W_\gamma^{univ}}\Big)$$

<u>Rem:</u> May also use $H^\bullet_{crit}(X,f) = \bigoplus_{z\in\mathbb{C}} H^\bullet(f^{-1}(z), \varphi_{-f|z}\mathbb{Z}_X[-1])$ ← sheaf of vanish. cycles.

[but may give something different, depending on conditions on f, cf. Sabbah]

<u>$W=0$ again</u>. Relation to stability?

$\mathcal{M}^{HN}_{\gamma,\gamma'}\subset\mathcal{M}_\gamma$

↖ reprs with fixed HN-type $(\gamma_1',\dots,\gamma_n')$

<u>Thm:</u> $\mathcal{M}^{HN}_{\gamma,\gamma'} = \mathcal{M}_{\gamma,\gamma'}\setminus\bigcup_{\gamma'_\bullet<\gamma'}\mathcal{M}_{\gamma,\gamma'_\bullet}$ [$\Rightarrow$ spectral sequence converges to $\mathcal{A}_\gamma$]

↑ rep's E of dim γ which admit $0=E_0<E_1<\dots<E_n=E$, $d(E_i/E_{i-1})$ given by deg γ'

Prop: $H^\bullet_{G_\gamma}(M_\gamma^{ss})$ concentr. in even degrees ⇒ spectral sequence collapses in the 1st step, which is $\bigotimes \bigoplus_{\substack{n\geq 0 \\ \gamma_1,\dots,\gamma_n\in\mathbb{Z}^I_{\geq 0}\setminus\{0\} \\ \mathrm{Arg}(\gamma_1)>\mathrm{Arg}(\gamma_2)>\dots}} \bigotimes_{i=1}^{n} H^i_{G_{\gamma_i}}(M^{ss}_{\gamma_i})$

Cor: Leads to motivic WCF

Expl: $\bullet \longrightarrow \bullet$ gives subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ and $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_1$ depending on stability s.th.

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A} = \mathcal{A}_2 \otimes \mathcal{A}_3 \otimes \mathcal{A}_1 .$$

Going over to dimensions gives identity: $E(x_1)E(x_2) = E(x_2)E(x_3)E(x_1)$, $E(x) = \mathrm{Li}_{2,q}(x)$.

More generally, $A := \sum_{\gamma\in\mathbb{Z}^I_{\geq 0}} \left(\sum_{k\geq 0} (-1)^k H^k_{G_\gamma}(M_\gamma)\right) \cdot \hat{e}_\gamma$ ← gen. of quantum torus asso. with χ_Q.

Prop: spectral sequ. ⇒ $A = \overrightarrow{\prod_j} A^{ss}_{\gamma_j}$, where (γ_j) is a collection of primitive elements of Γ_+.

$$A^{ss}_{\gamma_j} = \sum_{n\geq 0} H^\bullet_{G_\gamma}(M^{ss}_{n\gamma_j})\,\hat{e}^{\,n}_{\gamma_j} \quad \longleftrightarrow \quad \left(A_V = \overrightarrow{\prod} A_{\gamma_j}\right).$$