Talk III

YS8

Recall: 3 CY-category + data $\longrightarrow$ **motivic DT invts**

$(A_V^{mot}) \in R_{\mathcal{C}}^{\times}$ = motivic quantum torus $(\hat{e}_{\gamma_1} \cdot \hat{e}_{\gamma_2} = \mathbb{L}^{1/2\langle\gamma_1,\gamma_2\rangle} \hat{e}_{\gamma_1+\gamma_2})$

$V =$  $\subset \mathbb{Q}^2$

(FP) $A^{mot}_{V_1 \sqcup V_2} = A^{mot}_{V_1} A^{mot}_{V_2}$ 

$\updownarrow \; \mathbb{L}^{1/2} \to -1$

$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} e_\gamma, \; [e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$

$T_V = \lim_{\mathbb{L}^{1/2} \to -1} \mathrm{Ad}_{(A_V^{mot})} = \prod_{Z(\gamma) \in V} T_\gamma^{\Omega(\gamma)}$.

Today: 3 CY-category + data $\longrightarrow$ motivic Hall algebra $\overset{\varphi = \int}{\dashrightarrow}$ motivic DT invts

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Preparations

Ind-constructible A_∞-cat./k: $Ob(\mathcal{C}) = \coprod_{i \in I} X_i$ constr./k

$S \rightsquigarrow S(\bar{k}) \circlearrowleft \mathrm{Aut}(\bar{k}/k)$

Now add "Ind-constructible" to all notions:

E.g. $\mathcal{H}om^n$, $n \in \mathbb{Z}$, should be an Ind-constructible vector bundle

$\mathcal{H}om^n$
$\downarrow$
$Ob(\mathcal{C}) \times Ob(\mathcal{C})$

and $m_n : \bigotimes p^*_{i,i+1} \mathcal{H}om^{l_i} \longrightarrow p^*_{1,n+1} \mathcal{H}om^{l_1 + \ldots + l_n + 2 - n}$ for $l_1, \ldots, l_n \in \mathbb{Z}$

[higher products] hom. of Ind-constructible vb's.

Constructible stability structures: $\mathcal{C}_\gamma^{ss}$: constructible set for $\gamma \in \Gamma$, $\mathrm{Arg}(Z(\gamma))$ fixed.

(Y59)

- $\mathrm{Mot}(\mathrm{Ob}\,\mathcal{C}) = \bigoplus_i \mathrm{Mot}(X_i)$ elements: $[Y \to X]$
- equiv. version : $\mathrm{Mot}^G(X) \rightsquigarrow [(Y,H) \to (X,G)]$
  constr. $\downarrow$ $\swarrow$ affine alg gp.
- motivic stack functions: $\mathrm{Mot}_{st}(X, G)$

($\to$ [KS, §4] and [Joyce])

- $[(X,G)] = [X \times GL(n)/G]/[GL(n)] \in K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}, [GL(n)]^{-1}]$ $(G \subset GL(n))$.

$$\int_{[(X,G)]} [(X',G') \to (X,G)] = [X',G']$$

$\mathcal{C} = \coprod_{i \in I} X_i$, $GL(n_i) \circlearrowright X_i$

For $\mathcal{C}$ as above, $H(\mathcal{C}) := \bigoplus_i \mathrm{Mot}_{st}(X_i, GL(n_i))$ <u>motivic Hall algebra</u>

Product: $[Y_1 \xrightarrow{\pi_1} \mathrm{Ob}(\mathcal{C})] \cdot [Y_2 \xrightarrow{\pi_2} \mathrm{Ob}(\mathcal{C})] = \sum_n [W_n \to \mathrm{Ob}(\mathcal{C})] \cdot \mathbb{L}^{-n}$

$$W_n = \left\{ (y_1, y_2, \alpha) \,\middle|\, \begin{array}{l} y_i \in Y_i,\ i = 1,2;\ \alpha \in \mathrm{Ext}^1(\pi_2(y_2), \pi_1(y_1)) \text{ s.th.} \\ \underbrace{\sum_{i \le 0} (-1)^i \dim \mathrm{Ext}^i(\pi_2(y_2), \pi_1(y_1))}_{=: (\pi_2(y_2), \pi_1(y_1))_{\le 0}} = n \end{array} \right\}$$

<u>Thm</u> $\underset{\bigoplus_{\gamma \in \Gamma} H(\mathcal{C})_\gamma}{\overset{}{(H(\mathcal{C}), \cdot)}}$ is an associative algebra.

$Z : \Gamma \to \mathbb{C}$ central charge, $V \subset \mathbb{R}^2 \rightsquigarrow \underline{\mathcal{C}_V} \subset \mathcal{C}$
$\uparrow cl$
$K_0(\mathcal{C})$

generated by extensions of s.s. E s.th. $Z(E) \in V \cup 0$.

$A_V^{Hall} := 1 + \ldots = \sum_{i \in I} \mathbb{1}_{(\mathrm{Ob}(\mathcal{C}) \cap X_i, GL(n_i))}$

(cf. Toën: derived Hall algebra ?)

YS10

Thm: $A^{Hall}_{V_1 \sqcup V_2} = A^{Hall}_{V_1} \cdot A^{Hall}_{V_2}$

$\mathcal{C}$ is 3-CY., m_n higher compos., $(\cdot,\cdot)$: $Hom^\bullet(E,F) \otimes Hom(F,E) \to k[-3]$.

Assume also $(m_n(a_1,\dots,a_n),a_{n+1})$ is $\mathbb{Z}/n+1$ - inv.

(super-) potential $W_E(\alpha) := \sum_{n\geq 2} \frac{(m_n(\alpha,\dots,\alpha),\alpha)}{n}$, $\alpha \in Ext^1(E,E)$,

$MF(W_E^{min})$ motivic Milnor fibre of minimal potential: $W_E = \underset{\geq cubic}{W_E^{min}} \oplus \underset{quadratic}{Q_E} \oplus N$

[have to extend definition of MF to formal setup here]. [on ∞-dim'l space, generally]

$MF(W^{min}) \underset{orient. data}{\longrightarrow} w = w(E)$ <u>motivic weight</u>. ($\nearrow$ below)

Thm: The map $\phi = \int : H(\mathcal{C}) \to R_{\mathcal{C}}$, $\nu \mapsto (\nu, w) \cdot \hat{e}_\gamma$
is a hom of algebras. — pairing between motivic functions

Def: $A^{mot}_V := \phi(A^{Hall}_V)$.

<u>Discussion of $w(E)$:</u>

naive guess $w(E) = \mathbb{L}^{1/2 \cdot (E,E)_{\leq 1}} (1 - MF_0(W_E^{min}))$. with $(E,E)_{\leq 1} = \sum_{i \leq 1} (-1)^i \dim Ext^i(E,E)$

<u>Problem:</u> consistent choice of Q_E $\rightsquigarrow$ requires orientation data.

Then define $w(E) := \mathbb{L}^{1/2 (E,E)_{\leq 1}} (1 - MF_0(W^{min})) \underbrace{(1 - MF_0(Q_E))\, \mathbb{L}^{-\frac{1}{2} rk\, Q_E}}_{=: I(Q)}$

<u>Rem:</u> Motivic Thom-Sebastiani is needed to show ϕ is a hom.,
for well-definedness of $w(E)$ it suffices to have $I(Q_1 \oplus Q_2) = I(Q_1)\, I(Q_2)$
for quadratic forms Q_i.