$\longrightarrow$ need constructibility

YSY

Talk II.

Approaches to BPS invts (= "DT-invts"):

1) Stability data on graded Lie algebras $\rightsquigarrow \Omega(\gamma),\ \gamma \in \Gamma$.
2) 3 CY categories $\longrightarrow$ motivic DT-invts
3) Cohomological Hall algebras (BPS-states algebras) – motivic DT-invts.
   (Q,W) quiver + potential

Today: (2) $\rightsquigarrow$ (1).

$\mathcal{C}$ 3-CY category $/k$, char $k=0$.

On $K_0(\mathcal{C})$: $\chi(E,F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E,F)$ skew-symmetric.

$\Gamma \simeq \mathbb{Z}^n$ free abelian, $\langle , \rangle : \Gamma \otimes \Gamma \longrightarrow \mathbb{Z}$ skew-symmetric bilinear

Assume: $cl: K_0(\mathcal{C}) \longrightarrow \Gamma$, compatible with χ and $\langle , \rangle$.

<u>Rem:</u> $\Gamma \longrightarrow \Gamma \oplus \Gamma^\vee \Rightarrow$ We may assume that $\langle , \rangle$ is symplectic.

$(\Gamma, \langle , \rangle) \rightsquigarrow$ graded Lie alg. $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \ ,\ \mathfrak{g}_\gamma \simeq \mathbb{Q} \cdot e_\gamma$

$\Gamma \otimes \mathbb{G}_m \simeq \mathbb{G}_m^n$

Poisson alg. of fcts: $\begin{cases} [e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1+\gamma_2} \\ e_{\gamma_1} \cdot e_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1+\gamma_2} \end{cases}$

Quantization gives "quantum torus":

parameter $q = [\mathbb{A}^1] = \mathbb{L}$, or rather $q^{1/2} = \mathbb{L}^{1/2}$

ground ring $C := \mathrm{Mot}_S^{\hat{\mu}}$, $S = \mathrm{Ob}(\mathcal{C})$: ring of motivic stack functions

Need condition on $\mathcal{C}$: ind-constructible. [some more, technical conditions on C]

$R_{\mathcal{C}}$ = motivic quantum torus (assoc. C-algebra)

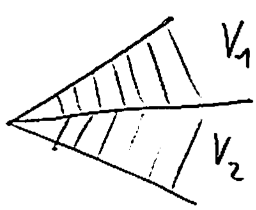
$$\hat{e}_{\gamma_1} \cdot \hat{e}_{\gamma_2} = \mathbb{L}^{\frac{1}{2}\langle \gamma_1, \gamma_2 \rangle} \hat{e}_{\gamma_1+\gamma_2}$$

[quasi-classical limit $\mathbb{L}^{1/2} = -1/2$ + replace coefficients]

Motivic DT-invts will be a collection (A_V^{mot}), $A_V^{mot} \in R_{\mathcal{C}}^{\times}$,

V is a strict sector in $\mathbb{R}^2$: 

satisfying factorization property:



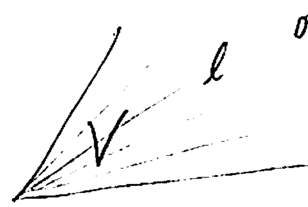
$$\Rightarrow A_V^{mot} = A_{V_1}^{mot} \cdot A_{V_2}^{mot}. \qquad (FP)$$

Construction of A_V^{mot} involves the following choices:

1) Stability condition on $\mathcal{C}$

2) Orientation data: choice of sqrt. of $\mathrm{sdet}\, \mathrm{Ext}^{\bullet}(E,E)$.
$$\downarrow$$
$$\mathrm{ob}(\mathcal{C})$$

$$FP \Rightarrow A_V^{mot} = \overrightarrow{\prod_{l \subset V \text{ ray}}} A_l^{mot}$$



Expl: $\mathcal{C} = \langle E \rangle$ 3-CY $\Rightarrow$ E spherical: $\mathrm{Ext}^{\bullet}(E,E) \simeq H^{\bullet}(S^3, k)$

$K_0(\mathcal{C}) = \mathbb{Z} \cdot [E]$

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$$A_V^{mot} = A^{mot} = \sum_{n\geq 0} \frac{\mathbb{L}^{n^2/2}}{(\mathbb{L}^n-1)\cdots(\mathbb{L}^n-\mathbb{L}^{n-1})} \hat{e}^n_{[E]} = \mathrm{Li}_2^{quant}(\hat{e}_{[E]}).$$

<u>transformation</u>: $T_V^{mot}(x) = A_V^{mot} \cdot x \cdot (A_V^{mot})^{-1}$, $x \in R_{\mathcal{C}}$.

Conj: T_V^{mot} does not have poles at $\mathbb{L}^n = 1$ $\forall n$ <u>"Absence of poles"</u>

In particular, could specialize to $\mathbb{L}^{1/2} = -1$ and obtain formal symplectomorphism T_V of $\Gamma \otimes \mathbb{G}_m$. Fulfills also $T_V = T_{V_1} \cdot T_{V_2}$ and $T_V = \prod_{\ell \subset V} T_\ell$

Let $T_\gamma : e_\mu \longmapsto (1-e_\gamma)^{\langle\gamma,\mu\rangle} e_\mu = \exp[\mathrm{Li}_2(e_\gamma)\cdot](e_\mu)$, $\mathrm{Li}_2(t) = \sum_{m\geq 1} \frac{t^m}{m^2}$, $\gamma \in \Gamma$.

(is birat'l symplectomorphism).

Then $T_\ell = \prod_{Z(\gamma)\in \ell} T_\gamma^{\Omega(\gamma)}$, $\Omega(\gamma) \in \mathbb{Q}$. (for generic stability condition)

<u>Def</u>: $\Omega(\gamma)$ are called <u>BPS-invts</u> of $\mathcal{C}$, or <u>numerical DT-invariants</u>

<u>Conj</u>: $\Omega(\gamma) \in \mathbb{Z}$. <u>Rem</u>: Refined BPS: $H_{\mathcal{X}}(-\gamma)$ $\longleftrightarrow$ motivic DT-invts? ($\to$ physics).

! $\Omega(\gamma) e_\gamma \in \mathfrak{g}_\gamma$, $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$, $[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$

$\{\Omega(\gamma) e_\gamma\}$ defines stability condition on $\mathfrak{g}$, i.e.

$\exists$ CY-cat. + data $\rightsquigarrow$ stability data on $\mathfrak{g}$.

<u>Def</u>: (stability data on $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}_\gamma$ gr. LA):

a) $Z : \Gamma \to \mathbb{C}$ central charge

b) collection $a(\gamma) \in \mathfrak{g}_\gamma$, $\gamma \neq 0$. $a = (a_\gamma)$

axiom: (support property) $\exists$ quadr. form Q on $\Gamma\otimes\mathbb{R}$ s.th.

$Q|_{\ker Z} < 0$ and $Q(\gamma) \geq 0$ if $a(\gamma) \neq 0$.

$\mathrm{Stab}(\mathfrak{g})$ = space of stability data on $\mathfrak{g}$ carries a Hausdorff topology:

$$\begin{array}{ccc} \mathrm{Stab}(\mathfrak{g}) & & \sigma = (Z, a) \\ \pi \downarrow & & \downarrow \\ \mathrm{Hom}(\Gamma, \mathbb{C}) & & Z \end{array}$$

Prop: π is a local homeom.

$T_V = \prod_{\ell \subset V} \exp\left(\sum_{Z(\gamma)\in\ell} a(\gamma)\right) \in$ pronilpotent group fulfills $T_V = T_{V_1} T_{V_2} = \overrightarrow{\prod_{\ell\subset V}} T_\ell$.

Topology on $\mathrm{Stab}(\mathfrak{g})$ $\rightsquigarrow$ T_V doesn't change for $\sigma\in\mathrm{Stab}(\mathfrak{g})$ as long as $Z(\gamma)$ doesn't cross ∂V for $\gamma\in\mathrm{supp}$

Wall crossing formula is now a triviality:

Expl: $\mathbb{Q}[[x,y]]$, $T^{(k)}_{a,b}: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x(1-(-1)^{kab}x^a y^b)^{-kb} \\ y(1-(-1)^{kab}x^a y^b)^{ka} \end{pmatrix}$, $a,b\geq 0$, $a+b\geq 1$

WCF: $k=1$ $\quad T^{(1)}_{1,0}T^{(1)}_{0,1} = T^{(1)}_{0,1}T^{(1)}_{1,1}T^{(1)}_{1,0}$.

$k=2$ $\quad T^{(2)}_{1,0}T^{(2)}_{0,1} = T^{(2)}_{0,1}T^{(2)}_{1,2}T^{(2)}_{2,3}\cdots\left(T^{(2)}_{1,1}\right)^{-2}\cdots T^{(2)}_{2,1}T^{(2)}_{1,0}$.

Have also quantum version