DERIVED CATEGORIES AND QUIVERS

[Examples for Paolo's lecture]

Plan:

1. Ex. 0, $\mathbb{P}^N$
2. Quivers / Quivers w/ potential
3. Ex. 1, $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$
4. Ex. 2, $\mathcal{O}_{\mathbb{P}^2}(-3)$

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1. Projective spaces [Beilinson, Bondal, Rickard, ... etc.]

Want to describe $D^b(\mathbb{P}^N) := D^b(\mathrm{coh}\, \mathbb{P}^N)$.

[~~Idea: Find a "good" resolution of $\mathcal{O}_\Delta$~~]

Lemma:
$\exists$ l. free resol.

$$0 \to \mathcal{O}(-N) \boxtimes \Omega^N(N) \to \mathcal{O}(-N+1) \boxtimes \Omega^{N-1}(N-1) \to \_\_ \to \mathcal{O}(-1) \boxtimes \Omega(1) \xrightarrow{\Delta^\vee} \mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^N}$$

$$\to \mathcal{O}_\Delta \to 0 \qquad [*]$$

$p_1^*(-) \otimes p_2^*(-)$

sheaf of diff'ls on $\mathbb{P}^N$

str. sheaf of $\mathbb{P}^N \xrightarrow{\Delta} \mathbb{P}^N \times \mathbb{P}^N$

$$\begin{tikzcd} & \mathbb{P}^N \times \mathbb{P}^N \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^N & & \mathbb{P}^N \end{tikzcd}$$

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Pf (sketch): Write $\mathbb{P}^N = \mathbb{P}(V)$, $[\ell], [\ell'] \in \mathbb{P}(V)$.

v. space of dim $N+1$

$\rightsquigarrow$ Euler ~~sequence~~ $\cancel{0 \to \Omega(1) \to V^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0}$

$\rightsquigarrow$ ~~restrict to $[\ell] \in \mathbb{P}(V)$~~ $\cancel{0 \to \ell^\perp \to V^\vee \to \ell^\vee \to 0}$

$\rightsquigarrow$ ~~get section~~ $\cancel{s \in H^0(\mathbb{P}^N \times \mathbb{P}^N, \underbrace{\mathcal{O}(1) \boxtimes T(-1)}_{[\mathcal{O}(-1) \boxtimes \Omega(1)]^\vee})}$

$$\overset{\vee}{s}([\ell],[\ell']) \in \left(\mathcal{O}(-1)_{[\ell]} \otimes \Omega(1)_{[\ell']}\right)^\vee$$

$$\hookrightarrow = \quad (x \otimes \varphi) \mapsto \varphi(x)$$

$x \in \ell$

$\varphi|_{\ell'} = 0$

$\mathcal{O}(-1)_{[\ell]} \equiv \ell$

$\Omega(1)_{[\ell']} = (\ell')^\perp$

[via Euler seq.]

$\rightsquigarrow$ get $[*]$ as Koszul complex assoc. to $\overset{\vee}{s}$, since

$$\operatorname{coker}\left(\overset{\vee}{s} : \mathcal{O}(-1) \boxtimes \Omega(1) \longrightarrow \mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^N}\right) \cong \mathcal{O}_\Delta$$

$$\left[e_{i_0} \wedge \_ \wedge e_{i_k} \longmapsto \sum (-1)^\ell \overset{\vee}{s}(e_{i_\ell}) e_{i_0} \wedge \_ \wedge \hat{e}_{i_\ell} \wedge \_ \wedge e_{i_k} \right]$$

▲

<u>Set</u> $E := \mathcal{O}_{\mathbb{P}^N} \oplus \_ \oplus \mathcal{O}_{\mathbb{P}^N}(-N)$

f.gen. right A-modules

$$\underline{\Phi} := R\mathrm{Hom}(E, \_) : D^b(\mathbb{P}^N) \longrightarrow D^b(A) := D^b(\text{mod-}A)$$

where $A := \mathrm{End}(E)$ f. dim'l assoc. algebra

Thm:

$\underline{\Phi}$ is an equivalence of triang. categories.

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Pf (sketch):

- E satisfies $R\mathrm{Hom}(E,E) \simeq \mathrm{Hom}(E,E) = A$
- <u>Set</u> $\Psi(\_) := \_ \otimes_A E \; : D^b(A) \to D^b(\mathbb{P}^N)$

We have $(\underline{\Phi}\Psi)(M) = R\mathrm{Hom}(E, M \otimes_A E) \cong M \otimes_A R\mathrm{Hom}(E,E) \simeq M \otimes_A A \simeq M$

$\forall M \in D^b(A)$.

left adjoint

- $\Psi \dashv \Phi \Longrightarrow \exists\ (\Psi\Phi)(F) \xrightarrow{\text{nat}} F$

$\rightsquigarrow$ get triangle in $D^b(\mathbb{P}^N)$ $\quad (\Psi\Phi)(F) \to F \to C$

$\rightsquigarrow$ apply Φ and using prev. result $\quad \Phi(F) \xrightarrow{\sim} \Phi(F) \to \Phi(C)$

$\rightsquigarrow \Phi(C) \cong 0$, i.e. $R\mathcal{H}om(E, C) = 0$.

$\underset{\text{Lemma}}{\rightsquigarrow} C \cong 0 \rightsquigarrow (\Psi\Phi)(F) \cong F \rightsquigarrow \Phi$ equiv.

$\blacksquare$

The equiv. Φ is very useful for several purposes (e.g. study v.bdles on $\mathbb{P}^N$).
Here: we'll construct stability cond's on some local CY.
First: Find a presentation for $A \rightsquigarrow$ quiver w/ relations.

2. Quivers

k – field

Def | (finite) quiver $Q := (Q_0, Q_1, s, t)$ $\quad s, t : Q_1 \to Q_0$, $\quad Q_0, Q_1$ finite sets

(4-uple; Q_0: vertices; Q_1: arrows; s: source map; t: target map)

Exs

- $Q_0 = \{1\}$
  $Q_1 = \{a\}$
  $s(a) = t(a) = 1$

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop right, "a"] \end{tikzcd}$$

- $Q_0 = \{1, 2\}$
  $Q_1 = \{a, b\}$
  6 $\quad s(a) = s(b) = 1$
  $t(a) = t(b) = 2$

$$\begin{tikzcd} 1\ \circ \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \circ\ 2 \end{tikzcd}$$

$\blacktriangle$

Representation : $\forall x \in Q_0 \rightsquigarrow V_x$ $\mathbb{k}$-v.space

$\forall a \in Q_1 \rightsquigarrow a : V_x \to V_y$ $\mathbb{k}$-lin. map

$s(a)=x$
$t(a)=y$

$\rightsquigarrow$ abelian category $\operatorname{Rep} Q$ [morph's : obvious ones]

Path-algebra : • $R := \mathbb{k}^{Q_0}$ + pointwise mult.

• $A := \mathbb{k}^{Q_1}$ R-bimod : $e \in R$, $f \in A \rightsquigarrow$ $(ef)(a) = e(sa)\cdot f(a)$, $(fe)(a) = f(a)\cdot e(ta)$ $\forall a \in Q_1$

• $\mathbb{k}(Q) := \bigoplus_{d=0}^{\infty} A^d$, $A^d = \underbrace{A \otimes_R \cdots \otimes_R A}_{d\text{-times}}$ [alg. gen. by paths + idemp. at vertices & mult. = comp. of paths]

• $Q \rightsquigarrow Q^*, R^*, A^*$ (A^*: R-bimod; Q^*: reverse all arrows)

$\rightsquigarrow \operatorname{Rep} Q \cong \operatorname{mod-}\mathbb{k}Q$.

Ex $\mathbb{P}^1 \rightsquigarrow \operatorname{End}(\mathcal{O} \oplus \mathcal{O}(1)) \cong \mathbb{k}Q$, w/ $Q : \bullet \rightrightarrows \bullet$

Quivers w/ relations : (Q, J), where $J \subseteq \mathbb{k}Q$ (quiver)

$\rightsquigarrow \mathbb{k}(Q,J) := \mathbb{k}Q / \langle J \rangle$ ← ideal gen. by

Ex $\mathbb{P}^N$, $N > 1$ $\rightsquigarrow \operatorname{End}(E) \cong \mathbb{k}(Q,J)$, w/

$Q_0 = \{0, \dots, N\}$ $\quad$ Arrows$(i \to i+1) = \{\varphi_i^j,\ j = 0, \dots, N\}$

$\overset{0}{\bullet} \Rrightarrow \overset{1}{\bullet} \cdots \overset{N-1}{\bullet} \Rrightarrow \overset{N}{\bullet}$ $\quad$ $J = \{\varphi_i^j \varphi_{i+1}^k - \varphi_i^k \varphi_{i+1}^j\}$

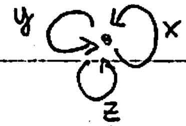
when interested in 3CYs, relations given by potential

Quivers w/ potentials:

- potential $W \in \mathbb{K}Q_{cyc}$ — spanned by paths $a_1 \cdots a_d$ w/ $s(a_1) = t(a_d)$ — better to say $a_1^\vee \cdots a_d^\vee$
- cyclic derivative $\forall \xi \in A^* \rightsquigarrow \partial_\xi : \mathbb{K}Q_{cyc} \longrightarrow \mathbb{K}Q$

$$\partial_\xi (a_1 \cdots a_d) := \sum_{k=1} \xi(a_k)\, a_{k+1} \cdots a_d \cdot a_1 \cdots a_{k-1}$$

$$(Q, W) \rightsquigarrow \mathbb{K}(Q,W) = \mathbb{K}(Q) / \langle \partial_\xi W \rangle$$

Ex. Q:  , $W = xyz - xzy$

$\rightsquigarrow A^* \equiv A$ and $\partial_x W = yz - zy$, $\partial_y W = zx - xz$, $\partial_z W = xy - yx$

$\rightsquigarrow \mathbb{K}(Q,W) \simeq \mathbb{K}[x,y,z]$

▲

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3. $X = |\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)|$ [from here on, $\mathbb{K} \equiv \mathbb{C}$]

$$\begin{array}{c} X \xrightarrow{\pi} \mathbb{P}^1 \\ \uparrow i \;\; \text{0-sect.} \\ \mathbb{P}^1 \end{array}$$

Set $E := \mathcal{O}_Y \oplus \mathcal{O}_Y(1)$, $\mathcal{O}_Y(1) := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$

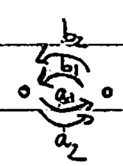
$A := \mathrm{End}(E)$

Cor. 1 [VdBergh]

$\Phi := R\mathrm{Hom}_Y(E, \_) : D^b(X) \longrightarrow D^b(A)$ equivalence

~~Pf. Coroll. of Thm. 1~~ ?

■

Rmk $A \simeq \mathbb{C}(Q,W)$, where Q:  , $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$

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▲

<u>Stability structures</u> : $D^b_0(X) \subseteq D^b(X)$

cohomology sheaves
i.e. topol. supported on 0-section

$\rightsquigarrow \quad \Phi : D^b_0(X) \xrightarrow{\sim} D^b_{nilp}(A)$

cohomology modules
nilpotent, i.e. $\exists\, d \geq 0$ w/ $M \cdot A^d = 0$
$M \in \operatorname{mod} A$ s.th.

$\rightsquigarrow$ $\operatorname{mod}_{nilp}$-$A$ is <u>heart</u> of a bd. t-structure on $D^b_0(X)$
generated by <u>exts.</u> by $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)[1]$

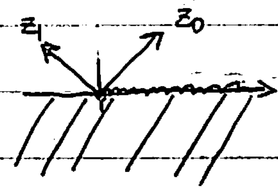
$\rightsquigarrow$ of <u>finite length</u> (<u>art.</u> + <u>noeth.</u>)

<u>Set</u> $\quad Z : K(D^b_0(X)) \cong \mathbb{Z}^{\oplus 2} \longrightarrow \mathbb{C}$

$[\mathcal{O}_{\mathbb{P}^1}] \longmapsto z_0$

$[\mathcal{O}_{\mathbb{P}^1}(-1)[1]] \longmapsto z_1$

$z_0, z_1 \in \mathbb{H} \cup \mathbb{R}_{<0}$



$\rightsquigarrow \sigma = (Z, \operatorname{mod}_{nilp}$-$A)$ <u>stability structure</u> on $D^b_0(X)$

$[\mathrm{Stab}^{\circ}(D^b_0(X)) \to \mathbb{C} \setminus \mathbb{Z}$
covering$]$

<u>Rmks</u> • [Toda] can classify all stability str. on $D^b_0(X)$ in this way

• can study all semistable objs :

[NN] if $\arg(z_0) < \arg(z_1)$, then the σ-stable objs in $\operatorname{mod}_{nilp}$-$A$ are

- $\mathcal{O}_{\mathbb{P}^1}(n)$, $n \geq 0$
- $\mathcal{O}_{\mathbb{P}^1}(n)[1]$, $n < 0$
- $\mathbb{C}(x)$, skyscraper sheaves at $x \in \mathbb{P}^1$.

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4. $X = |\mathcal{O}_{\mathbb{P}^2}(-3)|$

$$\begin{array}{c} X \xrightarrow{\pi} \mathbb{P}^2 \\ \uparrow i \\ \mathbb{P}^2 \end{array}$$

Set $E := \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(2)$, $\mathcal{O}_Y(n) := \pi^* \mathcal{O}_{\mathbb{P}^2}(n)$

$A := \mathrm{End}_Y(E)$

Cor 2 [Bridgeland]

$\Phi := R\mathrm{Hom}_Y(E, -) : D^b(X) \longrightarrow D^b(A)$ equiv.

~~Pf: Coroll. of Thm.~~

$\square$

Rmk $A \cong \mathbb{C}(Q, W)$, where Q:
$$\begin{tikzcd} & \bullet \arrow[dl, "c_0 c_1 c_2"'] & \\ \bullet \arrow[rr, Rightarrow, "a_0 a_1 a_2"'] & & \bullet \arrow[ul, "b_0 b_1 b_2"'] \end{tikzcd}$$
, $W = \sum \varepsilon^{ijk} a_i b_j c_k$

$\left[= (a_0 b_1 - a_1 b_0) c_2 + (a_1 b_2 - a_2 b_1) c_0 + (a_2 b_0 - a_0 b_2) c_1 \right]$

Stability structures : $D^b_0(X) \xrightarrow{\sim} D^b_{nilp}(A)$

$\rightsquigarrow$ $\mathrm{mod}_{nilp}\text{-}A$ heart gen. by ext's by $\mathcal{O}_{\mathbb{P}^2}(-1)[2]$, $\Omega_{\mathbb{P}^2}(1)[1]$, $\mathcal{O}_{\mathbb{P}^2}$

$\rightsquigarrow$ of finite length.

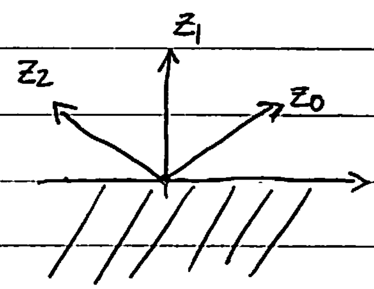
Set $Z : K(D^b_0(X)) \longrightarrow \mathbb{C}$

$[\mathcal{O}_{\mathbb{P}^2}(-1)[2]] \longmapsto z_0$

$[\Omega_{\mathbb{P}^2}(1)[1]] \longmapsto z_1$

14 $[\mathcal{O}_{\mathbb{P}^2}] \longmapsto z_2$

$z_0, z_1, z_2 \in$



$\rightsquigarrow$ $\sigma = (Z, \mathrm{mod}_{nilp}\text{-}A)$ stability str. on $D^b_0(X)$.

Rmks • [Bridgeland, Bayer-M.] can classify all stab. str's on $D^b_0(X)$ but need to consider also stab. str. defined as for K3 surfaces

• can study semistable objs (e.g., if $\arg(z_0) > \arg(z_1) > \arg(z_2)$, all σ-semistable objs are push-forwards from $\mathbb{P}^2$).