Derived categories and stability structures

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Outline

1. **Categories**
   - Derived categories
   - t-structures

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   - Derived categories
   - t-structures

2. **Stability structures**
   - Bridgeland's definition
   - Example 1: curves
   - Example 2: K3's
   - KS definition

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- Objects:
$$M^\bullet := \{\cdots \to M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \to \cdots\}$$
- Morphisms: sets of arrows $f^\bullet := \{f^i\}_{i\in\mathbb{Z}}$ making commutative the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{M^\bullet}^{i-2}} & M^{i-1} & \xrightarrow{d_{M^\bullet}^{i-1}} & M^i & \xrightarrow{d_{M^\bullet}^{i}} & M^{i+1} & \xrightarrow{d_{M^\bullet}^{i+1}} & \cdots \\
 & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\
\cdots & \xrightarrow{d_{L^\bullet}^{i-2}} & L^{i-1} & \xrightarrow{d_{L^\bullet}^{i-1}} & L^i & \xrightarrow{d_{L^\bullet}^{i}} & L^{i+1} & \xrightarrow{d_{L^\bullet}^{i+1}} & \cdots
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Exercise 2

Describe the bounded derived category of a closed point.

Geometry

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If X is a smooth projective variety over k, then $\bigoplus_i \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{F}[i])$ is finite dimensional, for any $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$.

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In this case, we say that $\mathrm{D}^{\mathrm{b}}(X)$ is **of finite type** over k.

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Define the **Grothendieck group** $K(X)$ of $\mathrm{D}^{\mathrm{b}}(X)$ as the free abelian group generated by the isomorphism classes of objects of $\mathrm{D}^{\mathrm{b}}(X)$ modulo the relation $[\mathcal{E}] = [\mathcal{F}] + [\mathcal{G}]$ for a distinguished triangle $\mathcal{F} \to \mathcal{E} \to \mathcal{G}$.

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Using this, define the **Euler-Poincaré pairing**

$$\chi : K(X) \times K(X) \to \mathbb{Z}$$

by $\chi([\mathcal{E}], [\mathcal{F}]) := \sum_i (-1)^i \dim \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{F}[i])$.

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- For f as above, $Lf^* : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$.

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For X, Y smooth projective varieties, special exact functors $\mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ are those of **Fourier–Mukai type**. That is, those which are isomorphic to the functor

$$\Phi_{\mathcal{E}}(-) := Rp_* \left(\mathcal{E} \overset{L}{\otimes} q^*(-) \right),$$

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Remark 6

Many classes of functors have been proved to be of Fourier-Mukai type at different levels of generalities. Among the authors who contributed to this, we mention: Orlov (+Bondal-Van den Bergh), Kawamata, Canonaco-S. and Ballard.

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For **A** an abelian category, a **Serre functor** of $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is an exact equivalence $S : \mathrm{D}^{\mathrm{b}}(\mathbf{A}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ such that, for any $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A})$, there is an isomorphism

$$\eta_{\mathcal{E},\mathcal{F}} : \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{A})}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{A})}(\mathcal{F}, S(\mathcal{E}))^{\vee}$$

of k-vector spaces which is functorial in $\mathcal{E}$ and $\mathcal{F}$.

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- They commute with equivalences (i.e., for $F : \mathrm{D}^{\mathrm{b}}(\mathbf{A}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{B})$ an equivalence, $S_{\mathbf{B}} \circ F \cong F \circ S_{\mathbf{A}}$);

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- For $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ of finite type, a Serre functor, if it exists, is unique up to isomorphism.

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Proposition 8

If X is a smooth projective variety defined over k, then the autoequivalence $S_X : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ such that

$$S_X(-) := (-) \otimes \omega_X[\dim(X)],$$

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Exercise 9

Use the Serre functor to show that, if X has trivial canonical bundle, then χ is symmetric if $\dim(X)$ is even and is skewsymmetric otherwise.

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Question: Given the triangulated category $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$, can we produce abelian subcategories $\mathbf{B} \subseteq \mathrm{D}^{\mathrm{b}}(\mathbf{A})$, possibly such that $\mathbf{A} \neq \mathbf{B}$?

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Definition 10

A ***t*-structure** on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is a pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ of full subcategories such that, if we put $\mathbf{D}^{\leq n} := \mathbf{D}^{\leq 0}[-n]$ and $\mathbf{D}^{\geq n} := \mathbf{D}^{\geq 0}[-n]$, we have

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- $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{A})}(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 1}) = 0$;
- $\mathbf{D}^{\leq 0} \subseteq \mathbf{D}^{\leq 1}$ and $\mathbf{D}^{\geq 1} \subseteq \mathbf{D}^{\geq 0}$;
- For any $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ there exist $\mathcal{F} \in \mathbf{D}^{\leq 0}$, $\mathcal{G} \in \mathbf{D}^{\geq 1}$ and an exact triangle
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The **heart** of a t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is the full subcategory $\mathbf{B} := \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$.

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Proposition 13

The heart $\mathbf{B}$ is an abelian category.

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For $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ we can define the two full subcategories

$$\mathbf{D}^{\leq 0} := \{\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A}) : H^i(\mathcal{E}) = 0 \text{ for } i > 0\}$$
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The pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ defines a bounded t-structure whose heart is again **A**.

This is usually called the **standard *t*-structure** on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$.

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Definition 14

A **torsion pair** in an abelian category $\mathbf{A}$ is a pair of full subcategories $(\mathbf{T}, \mathbf{F})$ of $\mathbf{A}$ which satisfy $\mathrm{Hom}_{\mathbf{A}}(\mathcal{T}, \mathcal{F}) = 0$, for $\mathcal{T} \in \mathbf{T}$ and $\mathcal{F} \in \mathbf{F}$, and such that, for every $\mathcal{E} \in \mathbf{A}$, there are $\mathcal{T} \in \mathbf{T}$ and $\mathcal{F} \in \mathbf{F}$ and a short exact sequence

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Proposition 15

If $(\mathbf{T}, \mathbf{F})$ is a torsion pair in $\mathrm{D^b}(\mathbf{A})$, then the full subcategory

$$\mathbf{B} := \left\{ \mathcal{E} \in \mathrm{D^b}(\mathbf{A}) : \begin{array}{l} \bullet\ H^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathbf{F} \text{ and } H^0(\mathcal{E}) \in \mathbf{T} \end{array} \right\}$$

is the heart of a bounded t-structure.

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- $Z : K(\mathrm{D}^{\mathrm{b}}(\mathbf{A})) \to \mathbb{C}$ is a linear map (the **central charge**)

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- $Z : K(\mathrm{D}^{\mathrm{b}}(\mathbf{A})) \to \mathbb{C}$ is a linear map (the **central charge**)
- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

The definition

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(B1) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

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Further definitions

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- The minimal objects of $\mathcal{P}(\phi)$ (i.e. those with no proper subobjects) are called **stable** of phase ϕ.

- The category $\mathcal{P}((0, 1])$, generated by the semistable objects of phase in $(0, 1]$, is called the **heart** of σ.

Basic properties

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One could alternative start with an abelian category **A** and a **slope function** $Z : K(\mathbf{A}) \to \mathbb{C}$ such that, for $0 \neq \mathcal{E} \in \mathbf{A}$,

$$Z([\mathcal{E}]) \in \{z \in \mathbb{C} \setminus \{0\} : z = |z| \exp(i\pi\phi),\ 0 < \phi \leq 1\}.$$

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A slope function has the **Harder–Narasimhan property** if it has HN-filtrations with semistable factors.

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$\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ is the set of locally finite stability conditions.

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Theorem 17 (Bridgeland)

For each connected component $\Sigma \subseteq \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$, there is a linear subspace $V(\Sigma) \subseteq \mathrm{Hom}\,(K(\mathrm{D}^{\mathrm{b}}(\mathbf{A})), \mathbb{C})$ with a well defined topology and a local homeomorphism $\mathcal{Z} : \Sigma \to V(\Sigma)$ which maps a stability condition $(Z, \mathcal{P})$ to its central charge Z.

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As explained later in the examples, for $\mathbf{A} = \mathbf{Coh}(X)$, (up to some modifications...) $\mathrm{Stab}(\mathrm{D^b}(X))$ is a finite dimensional complex manifold.

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- The left action of the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ of exact autoequivalences of $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$. Indeed, $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ sends $(Z, \mathcal{P})$ to $(Z', \mathcal{P}')$, where

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$$Z'([\mathcal{E}]) = Z([\Phi^{-1}(\mathcal{E})]) \qquad \mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi)).$$

- The right action of the universal cover $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ of $\mathrm{Gl}_2^+(\mathbb{R})$. $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ is the set of pairs (T, f) where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 \setminus 0)/\mathbb{R} > 0$ are the same. So $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

Outline

Preliminaries

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For X a smooth projective variety (defined over $\mathbb{C}$), define the **numerical Grothendieck group** to be the quotient

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A stability condition is **numerical** if Z factors through $v(-) := \mathrm{ch}(-) \cdot \sqrt{\mathrm{td}(x)} : K(X) \to \mathcal{N}(X)$. $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ is the finite dimensional complex manifold parametrizing numerical stability conditions and $\dim_{\mathbb{C}} \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) = \dim_{\mathbb{C}}(\mathcal{N}(X) \otimes \mathbb{C})$.

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Example 18

If X is a smooth curve than $\mathcal{N}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and so $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ has dimension 2.

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The HN-property follows easily from the existence of HN-filtrations for the slope stability (recall that $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}\,(\mathcal{E})}$).

The space of stability conditions

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Theorem 20 (Bridgeland, Macrì)

If C is a curve of genus $g > 0$ defined over $\mathbb{C}$, then the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ on $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ is free and transitive. In particular, $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \cong \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$.

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Note: The case of $\mathbb{P}^1$ was treated independently by Okada and Macrì.

Sketch of the proof

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- **Gorodentsev–Kuleshov–Rudakov:** If $\mathcal{E} \in \mathbf{Coh}(C)$ sits in a triangle
$$\mathcal{F} \to \mathcal{E} \to \mathcal{G},$$
with $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(C)$ and $\mathrm{Hom}^{\leq 0}(\mathcal{F}, \mathcal{G}) = 0$, then $\mathcal{E}, \mathcal{G} \in \mathbf{Coh}(C)$ as well.

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- From this one deduces that the skyscraper sheaves $\mathcal{O}_x$ are all stable in any stability condition. Indeed, one proves that $\mathcal{O}_x$ is semistable and all its stable factors are isomorphic. By the above results they are in $\mathbf{Coh}(C)$ and so isomorphic to $\mathcal{O}_x$.

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- By the same argument it follows that all line bundles are stable in all stability conditions.

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- Hence by acting by $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, we can assume that $Z = -\deg(\mathcal{E}) + i\,\mathrm{rk}\,(\mathcal{E})$ and that all skyscraper sheaves are stable of phase 1. This implies that $\mathcal{P}((0, 1])$, the heart of the stability condition, is **Coh**(C).

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Note: We restrict ourselves to projective ones!

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$H^*(X, \mathbb{Z})$ is endowed with a natural symmetric bilinear form, called **Mukai pairing**:

$$\langle \alpha, \beta \rangle := \alpha_2 \cup \beta_2 - \alpha_0 \cup \beta_4 - \alpha_4 \cup \beta_0,$$

for $\alpha = (\alpha_0, \alpha_2, \alpha_4)$ and $\beta := (\beta_0, \beta_2, \beta_4)$ in $H^0 \oplus H^2 \oplus H^4$.

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Proposition 22

If X is a smooth complex projective variety of dimension $d \geq 2$, then there are no numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ with heart $\mathbf{Coh}(X)$.

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Proposition 22

If X is a smooth complex projective variety of dimension $d \geq 2$, then there are no numerical stability conditions on $\mathrm{D^b}(X)$ with heart $\mathbf{Coh}(X)$.

Reason: After reducing to the case $d = 2$, one observes that it is already impossible to have a slope function on $\mathbf{Coh}(X)$.

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- The non-trivial objects in **T** are the sheaves such that their torsion-free part have μ_ω-semistable Harder–Narasimhan factors of slope $\mu_\omega > \beta \cdot \omega$.
- A non-trivial sheaf $\mathcal{E}$ is an object in **F** if $\mathcal{E}$ is torsion-free and every μ_ω-semistable Harder–Narasimhan factor of $\mathcal{E}$ has slope $\mu_\omega \leq \beta \cdot \omega$.

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Let $\mathbf{T}, \mathbf{F} \subseteq \mathbf{Coh}(X)$ be full additive subcategories:

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- A non-trivial sheaf $\mathcal{E}$ is an object in $\mathbf{F}$ if $\mathcal{E}$ is torsion-free and every μ_ω-semistable Harder–Narasimhan factor of $\mathcal{E}$ has slope $\mu_\omega \leq \beta \cdot \omega$.

One shows that $(\mathbf{T}, \mathbf{F})$ defines a torsion pair.

Stability - Examples

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Define the heart of the induced t-structure as the abelian category

$$\mathbf{A}_{\beta,\omega} := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \bullet \; H^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet \; H^{-1}(\mathcal{E}) \in \mathbf{F}, \\ \bullet \; H^0(\mathcal{E}) \in \mathbf{T} \end{array} \right\}.$$

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Lemma 23

Assume $\beta, \omega \in \mathrm{Pic}\,(X) \otimes \mathbb{Q}$ and ω ample such that $\omega \cdot \omega > 2$. The map $Z_{\beta,\omega}$ is a stability function on $\mathbf{A}_{\beta,\omega}$ with the HN property. Moreover, it defines a numerical locally finite stability condition $\sigma_{\beta,\omega}$.

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Note: one could impose a weaker condition on $Z_{\beta,\omega}$.

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- Any autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ induces an Hodge isometry on cohomology. Denote by $\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(X))$ the subgroup acting trivially.

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Theorem 24 (Bridgeland)

There is a connected component $\mathrm{Stab}^{\dagger}(\mathrm{D^b}(X))$ of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ mapped by $\mathcal{Z}$ onto $\mathcal{P}_0^+(X)$. Moreover, the induced map $\mathcal{Z} : \mathrm{Stab}^{\dagger}(\mathrm{D^b}(X)) \to \mathcal{P}_0^+(X)$ is a covering map, and the subgroup of $\mathrm{Aut}^0(\mathrm{D^b}(X))$ which preserves the connected component $\mathrm{Stab}^{\dagger}(\mathrm{D^b}(X))$ acts freely on $\mathrm{Stab}^{\dagger}(\mathrm{D^b}(X))$ and is the group of deck transformations of $\mathcal{Z}$.

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Conjecture 25 (Bridgeland)

The action of $\mathrm{Aut}(\mathrm{D^b}(X))$ on $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ preserves the connected component $\mathrm{Stab}^\dagger(\mathrm{D^b}(X))$. Moreover $\mathrm{Stab}^\dagger(\mathrm{D^b}(X))$ is simply-connected.

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Bridgeland: As a consequence of the conjecture we get the following short exact sequence

$$1 \to \pi_1(\mathcal{P}_0^+(X)) \to \mathrm{Aut}\,(\mathrm{D^b}(X)) \to \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \to 1,$$

where $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ is the group of orientation preserving Hodge isometries of the total cohomology of X.

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Warning: To prove this, we need anyhow a (tiny) part of Bridgeland's theory of stability conditions!

Outline

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(KS1) For all $\mathcal{E} \in \mathbf{C}^{ss}$ and for all $n \in \mathbb{Z}$, $\mathcal{E}[n] \in \mathbf{C}^{ss}$ and $\phi(Z(\mathcal{E}[n])) = \phi(Z(\mathcal{E})) + n$;

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(KS3) For any $\mathcal{E} \in \mathrm{Ob}(\mathbf{C})$, there exist $n \geq 0$ and a chain of morphisms $0 = \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_n = \mathcal{E}$ (HN filtration) such that $\mathcal{F}_i := \mathrm{Cone}(\mathcal{E}_{i-1} \to \mathcal{E}_i)$, for $i = 1, \ldots, n$ are semistable and $\phi(\mathcal{F}_1) > \phi(\mathcal{F}_2) > \cdots > \phi(\mathcal{F}_n)$;

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(KS5) **(Support Property)** For a norm $\| - \|$ on $\Gamma \otimes \mathbb{R}$, there exists $C > 0$ such that for all $\mathcal{E} \in \mathbf{C}^{ss}$ one has $\| \mathrm{cl}(\mathcal{E}) \| \leq C|Z(\mathcal{E})|$.

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- Hence, $\mathrm{Stab}(\mathbf{C})$ is a complex manifold, not necessarily connected.
- Due to the support property, all stability structures are locally finite.