Derived categories and stability structures

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Outline

1. **Categories**
   - Derived categories
   - t-structures

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   - Derived categories
   - t-structures

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Outline

The definition

Let **A** be an abelian category (e.g., **mod**-R, right R-modules, R an ass. ring with unity, and **Coh**(X)).

Define $C(\mathbf{A})$ to be the (abelian) **category of complexes** of objects in **A**. In particular:

- Objects:
  $$M^\bullet := \{\cdots \to M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \to \cdots\}$$
- Morphisms: sets of arrows $f^\bullet := \{f^i\}_{i\in\mathbb{Z}}$ making commutative the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{M^\bullet}^{i-2}} & M^{i-1} & \xrightarrow{d_{M^\bullet}^{i-1}} & M^i & \xrightarrow{d_{M^\bullet}^{i}} & M^{i+1} & \xrightarrow{d_{M^\bullet}^{i+1}} & \cdots \\
 & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\
\cdots & \xrightarrow{d_{L^\bullet}^{i-2}} & L^{i-1} & \xrightarrow{d_{L^\bullet}^{i-1}} & L^i & \xrightarrow{d_{L^\bullet}^{i}} & L^{i+1} & \xrightarrow{d_{L^\bullet}^{i+1}} & \cdots
\end{array}$$

The definition

For a complex $M^\bullet \in C(\mathbf{A})$, its i-th cohomology is

$$H^i(M^\bullet) := \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})} \in \mathbf{A}.$$

A morphism of complexes is a **quasi-isomorphisms** (qis) if it induces isomorphisms on cohomology.

Definition 1

The **derived category** $\mathrm{D}(\mathbf{A})$ of the abelian category $\mathbf{A}$ is such that:

- Objects: $\mathrm{Ob}(C(\mathbf{A})) = \mathrm{Ob}(\mathrm{D}(\mathbf{A}))$;
- Morphisms: (very) roughly speaking, obtained 'by inverting qis in $C(\mathbf{A})$'.

Remarks

Important!

The category $\mathrm{D}(\mathbf{A})$ is triangulated. In particular, it has a shift functor $[i]$, for any $i \in \mathbb{Z}$, and a set of *distinguished or exact* triangles.

If we just consider bounded complexes, we get the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$. Other possibilities are $\mathrm{D}^{-}(\mathbf{A})$ (bounded above complexes) and $\mathrm{D}^{+}(\mathbf{A})$ (bounded below complexes).

Exercise 2

Describe the bounded derived category of a closed point.

Geometry

If X is a smooth projective variety over a field k (always assume $k = \overline{k}$!), set $\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$.

Exercise 3

Let C be a smooth complex curve. Show that any $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(C)$ is isomorphic to the direct sum of shifts of sheaves.

Proposition 4

If X is a smooth projective variety over k, then $\bigoplus_i \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{F}[i])$ is finite dimensional, for any $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$.

In this case, we say that $\mathrm{D}^{\mathrm{b}}(X)$ is **of finite type** over k.

Geometry

Define the **Grothendieck group** $K(X)$ of $\mathrm{D}^{\mathrm{b}}(X)$ as the free abelian group generated by the isomorphism classes of objects of $\mathrm{D}^{\mathrm{b}}(X)$ modulo the relation $[\mathcal{E}] = [\mathcal{F}] + [\mathcal{G}]$ for a distinguished triangle $\mathcal{F} \to \mathcal{E} \to \mathcal{G}$.

Exercise 5

Show $K(X) = K(\mathbf{Coh}(X))$ (more generally, for any abelian category **A**...)

Using this, define the **Euler-Poincaré pairing**

$$\chi : K(X) \times K(X) \to \mathbb{Z}$$

by $\chi([\mathcal{E}], [\mathcal{F}]) := \sum_i (-1)^i \dim \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{F}[i])$.

Derived functors

Given a functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories, it is not straightforward to 'extend' it to $\mathrm{D}^{\mathrm{b}}(\mathbf{A}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{B})$.

This is not automatic already for left or right exact functors.

Nevertheless, in the geometric setting, all the 'basic functors' can be *derived*, i.e. defined on the level of the bounded derived categories. For example, for X, Y smooth finite-dimensional noetherian schemes:

- Tensor product: $- \overset{L}{\otimes} - : \mathrm{D}^{\mathrm{b}}(X) \times \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$;
- For a proper morphism $f : X \to Y$, $Rf_* : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$;
- For f as above, $Lf^* : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$.

Derived functors

For X, Y smooth projective varieties, special exact functors $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ are those of **Fourier–Mukai type**. That is, those which are isomorphic to the functor

$$\Phi_{\mathcal{E}}(-) := Rp_* \left(\mathcal{E} \overset{L}{\otimes} q^*(-) \right),$$

for $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and p, q the natural projections.

Remark 6

Many classes of functors have been proved to be of Fourier-Mukai type at different levels of generalities. Among the authors who contributed to this, we mention: Orlov (+Bondal-Van den Bergh), Kawamata, Canonaco-S. and Ballard.

Serre functor

Definition 7

For **A** an abelian category, a **Serre functor** of $\mathrm{D^b}(\mathbf{A})$ is an exact equivalence $S : \mathrm{D^b}(\mathbf{A}) \to \mathrm{D^b}(\mathbf{A})$ such that, for any $\mathcal{E}, \mathcal{F} \in \mathrm{D^b}(\mathbf{A})$, there is an isomorphism

$$\eta_{\mathcal{E},\mathcal{F}} : \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(\mathcal{F}, S(\mathcal{E}))^{\vee}$$

of k-vector spaces which is functorial in $\mathcal{E}$ and $\mathcal{F}$.

Some basic properties of Serre functors are the following:

- They commute with equivalences (i.e., for $F : \mathrm{D^b}(\mathbf{A}) \to \mathrm{D^b}(\mathbf{B})$ an equivalence, $S_{\mathbf{B}} \circ F \cong F \circ S_{\mathbf{A}}$);
- For $\mathrm{D^b}(\mathbf{A})$ of finite type, a Serre functor, if it exists, is unique up to isomorphism.

Serre functor

In the geometric setting, we can be more precise:

Proposition 8

If X is a smooth projective variety defined over k, then the autoequivalence $S_X : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ such that

$$S_X(-) := (-) \otimes \omega_X[\dim(X)],$$

where ω_X is the dualizing line bundle, is a Serre functor.

Exercise 9

Use the Serre functor to show that, if X has trivial canonical bundle, then χ is symmetric if $\dim(X)$ is even and is skewsymmetric otherwise.

Outline

The definition

Question: Given the triangulated category $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$, can we produce abelian subcategories $\mathbf{B} \subseteq \mathrm{D}^{\mathrm{b}}(\mathbf{A})$, possibly such that $\mathbf{A} \neq \mathbf{B}$?

Definition 10

A ***t*-structure** on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is a pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ of full subcategories such that, if we put $\mathbf{D}^{\leq n} := \mathbf{D}^{\leq 0}[-n]$ and $\mathbf{D}^{\geq n} := \mathbf{D}^{\geq 0}[-n]$, we have

- $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{A})}(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 1}) = 0$;
- $\mathbf{D}^{\leq 0} \subseteq \mathbf{D}^{\leq 1}$ and $\mathbf{D}^{\geq 1} \subseteq \mathbf{D}^{\geq 0}$;
- For any $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ there exist $\mathcal{F} \in \mathbf{D}^{\leq 0}$, $\mathcal{G} \in \mathbf{D}^{\geq 1}$ and an exact triangle
$$\mathcal{F} \to \mathcal{E} \to \mathcal{G}.$$

The definition

Definition 11

A t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is **bounded** if

$$\mathrm{D}^{\mathrm{b}}(\mathbf{A}) = \cup_{i,j\in\mathbb{Z}}(\mathbf{D}^{\leq 0}[i] \cap \mathbf{D}^{\geq 0}[j]).$$

Definition 12

The **heart** of a t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is the full subcategory $\mathbf{B} := \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$.

Proposition 13

The heart $\mathbf{B}$ is an abelian category.

The standard t-structure

For $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ we can define the two full subcategories

$$\mathbf{D}^{\leq 0} := \{\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A}) : H^i(\mathcal{E}) = 0 \text{ for } i > 0\}$$
$$\mathbf{D}^{\geq 0} := \{\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A}) : H^i(\mathcal{E}) = 0 \text{ for } i < 0\}.$$

The pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ defines a bounded t-structure whose heart is again **A**.

This is usually called the **standard *t*-structure** on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$.

Tiltings (after Happel-Reiten-Smalo)

Definition 14

A **torsion pair** in an abelian category **A** is a pair of full subcategories $(\mathbf{T}, \mathbf{F})$ of **A** which satisfy $\mathrm{Hom}_{\mathbf{A}}(\mathcal{T}, \mathcal{F}) = 0$, for $\mathcal{T} \in \mathbf{T}$ and $\mathcal{F} \in \mathbf{F}$, and such that, for every $\mathcal{E} \in \mathbf{A}$, there are $\mathcal{T} \in \mathbf{T}$ and $\mathcal{F} \in \mathbf{F}$ and a short exact sequence

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0.$$

Proposition 15

If $(\mathbf{T}, \mathbf{F})$ is a torsion pair in $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$, then the full subcategory

$$\mathbf{B} := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A}) : \begin{array}{l} \bullet\ H^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathbf{F} \text{ and } H^0(\mathcal{E}) \in \mathbf{T} \end{array} \right\}$$

is the heart of a bounded t-structure.

Outline

The definition

Warning: For simplicity, we restrict ourselves to the case of stability conditions on derived categories!

A **stability condition** on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : K(\mathrm{D}^{\mathrm{b}}(\mathbf{A})) \to \mathbb{C}$ is a linear map (the **central charge**)
- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

The definition

(B1) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E})\exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

(B2) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(B3) $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(B4) Any $0 \neq \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ admits a **Harder–Narasimhan filtration** given by a collection of distinguished triangles

$$\mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{A}_i$$

with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

Further definitions

- The non-zero objects in the abelian category $\mathcal{P}(\phi)$ are the **semistable** objects of phase ϕ . The objects $\mathcal{A}_i$ in (B4) are the **semistable factors** of $\mathcal{E}$.
- The minimal objects of $\mathcal{P}(\phi)$ (i.e. those with no proper subobjects) are called **stable** of phase ϕ.
- The category $\mathcal{P}((0,1])$, generated by the semistable objects of phase in $(0,1]$, is called the **heart** of σ.

Basic properties

One could alternative start with an abelian category **A** and a **slope function** $Z : K(\mathbf{A}) \to \mathbb{C}$ such that, for $0 \neq \mathcal{E} \in \mathbf{A}$,

$$Z([\mathcal{E}]) \in \{z \in \mathbb{C} \setminus \{0\} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}.$$

Define

$$\phi(\mathcal{E}) := \frac{1}{\pi} \arg(Z(\mathcal{E})) \in (0, 1].$$

An object $\mathcal{E} \in \mathbf{A}$ is **semistable** if

$$\phi(\mathcal{F}) \leq \phi(\mathcal{E})$$

for any proper subobject $\mathcal{F} \subseteq \mathcal{E}$.

A slope function has the **Harder–Narasimhan property** if it has HN-filtrations with semistable factors.

Basic properties

Proposition 16

To exhibit a stability condition on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$, it is enough to give

- a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ with heart $\mathbf{B}$;
- a group homomorphism $Z : K(\mathbf{B}) \to \mathbb{C}$ such that $Z(\mathcal{E}) \in \mathbb{H}$, for all $0 \neq \mathcal{E} \in \mathbf{B}$, and with the Harder–Narasimhan property.
  (Here $\mathbb{H} := \{z \in \mathbb{C} \setminus \{0\} : z = |z| \exp(i\pi\phi),\ 0 < \phi \leq 1\}$.)

All stability conditions are assumed to be **locally finite**. Hence every object in $\mathcal{P}(\phi)$ has a finite **Jordan–Hölder filtration**.

$\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ is the set of locally finite stability conditions.

Basic properties

$\mathrm{Stab}(\mathrm{D^b}(\mathbf{A}))$ carries a natural topology with the following important property:

Theorem 17 (Bridgeland)

For each connected component $\Sigma \subseteq \mathrm{Stab}(\mathrm{D^b}(\mathbf{A}))$, there is a linear subspace $V(\Sigma) \subseteq \mathrm{Hom}\,(K(\mathrm{D^b}(\mathbf{A})), \mathbb{C})$ with a well defined topology and a local homeomorphism $\mathcal{Z} : \Sigma \to V(\Sigma)$ which maps a stability condition $(Z, \mathcal{P})$ to its central charge Z.

As explained later in the examples, for $\mathbf{A} = \mathbf{Coh}(X)$, (up to some modifications...) $\mathrm{Stab}(\mathrm{D^b}(X))$ is a finite dimensional complex manifold.

Group actions

There are two groups acting naturally on $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ (and whose actions commute):

- The left action of the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ of exact autoequivalences of $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$. Indeed, $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbf{A}))$ sends $(Z, \mathcal{P})$ to $(Z', \mathcal{P}')$, where

$$Z'([\mathcal{E}]) = Z([\Phi^{-1}(\mathcal{E})]) \qquad \mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi)).$$

- The right action of the universal cover $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ of $\mathrm{Gl}_2^+(\mathbb{R})$. $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ is the set of pairs (T, f) where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 \setminus 0)/\mathbb{R} > 0$ are the same. So $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

Outline

Preliminaries

For X a smooth projective variety (defined over $\mathbb{C}$), define the **numerical Grothendieck group** to be the quotient

$$\mathcal{N}(X) := K(X)/K(X)^{\perp},$$

where $\perp$ is with respect to the pairing χ.

A stability condition is **numerical** if Z factors through $v(-) := \mathrm{ch}\,(-) \cdot \sqrt{\mathrm{td}(x)} : K(X) \to \mathcal{N}(X)$. $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ is the finite dimensional complex manifold parametrizing numerical stability conditions and $\dim_{\mathbb{C}} \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X)) = \dim_{\mathbb{C}}(\mathcal{N}(X) \otimes \mathbb{C})$.

Example 18

If X is a smooth curve than $\mathcal{N}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and so $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ has dimension 2.

Examples of stability conditions (Bridgeland)

Let C be a smooth curve of genus $g > 0$ defined over $\mathbb{C}$. The abelian category **Coh**(C) is the heart of a bounded t-structure.

As $\mathcal{N}(C) = H^0(C, \mathbb{Z}) \oplus H^2(C, \mathbb{Z})$, define $Z : \mathcal{N}(C) \to \mathbb{C}$ as

$$\mathcal{E} \mapsto -\deg(\mathcal{E}) + i\,\mathrm{rk}\,(\mathcal{E}).$$

Exercise 19

Show that Z as above is a slope function.

The HN-property follows easily from the existence of HN-filtrations for the slope stability (recall that $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}\,(\mathcal{E})}$).

The space of stability conditions

Theorem 20 (Bridgeland, Macrì)

If C is a curve of genus $g > 0$ defined over $\mathbb{C}$, then the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ on $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ is free and transitive. In particular, $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X)) \cong \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$.

Note: The case of $\mathbb{P}^1$ was treated independently by Okada and Macrì.

Sketch of the proof

- **Gorodentsev–Kuleshov–Rudakov:** If $\mathcal{E} \in \mathbf{Coh}(C)$ sits in a triangle
$$\mathcal{F} \to \mathcal{E} \to \mathcal{G},$$
with $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(C)$ and $\mathrm{Hom}^{\leq 0}(\mathcal{F}, \mathcal{G}) = 0$, then $\mathcal{E}, \mathcal{G} \in \mathbf{Coh}(C)$ as well.
- From this one deduces that the skyscraper sheaves $\mathcal{O}_x$ are all stable in any stability condition. Indeed, one proves that $\mathcal{O}_x$ is semistable and all its stable factors are isomorphic. By the above results they are in $\mathbf{Coh}(C)$ and so isomorphic to $\mathcal{O}_x$.
- By the same argument it follows that all line bundles are stable in all stability conditions.

Sketch of the proof

- Take $\sigma = (Z, \mathcal{P})$ and a line bundle L. Let ϕ and ψ be the phases of the stable objects L and $\mathcal{O}_x$.

- The existence of the maps $L \to \mathcal{O}_x$ and $\mathcal{O}_x \to L[1]$ gives the inequalities $\psi - 1 \leq \phi \leq \psi$. This implies that Z (seen as a map $\mathcal{N}(C) \otimes \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$) is an orientation preserving isomorphism.

- Hence by acting by $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$, we can assume that $Z = -\deg(\mathcal{E}) + i\,\mathrm{rk}\,(\mathcal{E})$ and that all skyscraper sheaves are stable of phase 1. This implies that $\mathcal{P}((0,1])$, the heart of the stability condition, is $\mathbf{Coh}(C)$.

Outline

Geometry

Definition 21

A **K3 surface** is a smooth Kähler (complex) surface X such that:

- X is simply connected.
- The canonical bundle ω_X is trivial.

Some examples are

- Quartics in $\mathbb{P}^3$ and double covers of $\mathbb{P}^2$ ramified along a sextic.
- Kummer surfaces (i.e. crepant resolutions of the quotient of an abelian surface by the involution $a \mapsto -a$).

Note: We restrict ourselves to projective ones!

Geometry

For X a K3, $\mathcal{N}(X) \cong \mathbb{Z}^{\oplus\rho}$, with $3 \leq \rho \leq 22$. All values are realized!

$\mathcal{N}(X)$ is actually the algebraic part of the total cohomology.

$H^*(X, \mathbb{Z})$ is endowed with a natural symmetric bilinear form, called **Mukai pairing**:

$$\langle\alpha, \beta\rangle := \alpha_2 \cup \beta_2 - \alpha_0 \cup \beta_4 - \alpha_4 \cup \beta_0,$$

for $\alpha = (\alpha_0, \alpha_2, \alpha_4)$ and $\beta := (\beta_0, \beta_2, \beta_4)$ in $H^0 \oplus H^2 \oplus H^4$.

Stability - Bad news

The main difference with the curve case is:

Proposition 22

If X is a smooth complex projective variety of dimension $d \geq 2$, then there are no numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ with heart $\mathbf{Coh}(X)$.

Reason: After reducing to the case $d = 2$, one observes that it is already impossible to have a slope function on $\mathbf{Coh}(X)$.

Stability - Examples

Let X be a K3 surface and let $\beta, \omega \in \mathrm{Pic}(X) \otimes \mathbb{Q}$. Assume moreover ω to be ample.

Define $Z_{\beta,\omega} : K(X) \to \mathbb{C}$ as

$$Z(\mathcal{E}) := \langle \exp(\beta + i\omega), v(\mathcal{E}) \rangle.$$

Let $\mathbf{T}, \mathbf{F} \subseteq \mathbf{Coh}(X)$ be full additive subcategories:

- The non-trivial objects in **T** are the sheaves such that their torsion-free part have μ_ω-semistable Harder–Narasimhan factors of slope $\mu_\omega > \beta \cdot \omega$.
- A non-trivial sheaf $\mathcal{E}$ is an object in **F** if $\mathcal{E}$ is torsion-free and every μ_ω-semistable Harder–Narasimhan factor of $\mathcal{E}$ has slope $\mu_\omega \leq \beta \cdot \omega$.

One shows that $(\mathbf{T}, \mathbf{F})$ defines a torsion pair.

Stability - Examples

Define the heart of the induced t-structure as the abelian category

$$\mathbf{A}_{\beta,\omega} := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \bullet\ H^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathbf{F}, \\ \bullet\ H^0(\mathcal{E}) \in \mathbf{T} \end{array} \right\}.$$

Lemma 23

Assume $\beta, \omega \in \mathrm{Pic}\,(X) \otimes \mathbb{Q}$ and ω ample such that $\omega \cdot \omega > 2$. The map $Z_{\beta,\omega}$ is a stability function on $\mathbf{A}_{\beta,\omega}$ with the HN property. Moreover, it defines a numerical locally finite stability condition $\sigma_{\beta,\omega}$.

Note: one could impose a weaker condition on $Z_{\beta,\omega}$.

The main result

Define:

- $\mathcal{P}(X) \subseteq \mathcal{N}(X) \otimes \mathbb{C}$ consisting of those vectors whose real and imaginary parts span positive definite two-planes in $\mathcal{N}(X) \otimes \mathbb{R}$;
- $\mathcal{P}^+(X) \subset \mathcal{P}(X)$ denote the connected component containing vectors of the form $\exp(\beta + i\omega)$, where $\omega \in \mathrm{Pic}\,(X) \otimes \mathbb{Q}$ is ample;
- $\Delta(X) = \{\delta \in \mathcal{N}(X) : \langle \delta, \delta \rangle = -2\}$;
- $\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subseteq \mathcal{N}(X) \otimes \mathbb{C}$.
- Any autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ induces an Hodge isometry on cohomology. Denote by $\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(X))$ the subgroup acting trivially.

The main result

Theorem 24 (Bridgeland)

There is a connected component $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ mapped by $\mathcal{Z}$ onto $\mathcal{P}_0^+(X)$. Moreover, the induced map $\mathcal{Z} : \mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathcal{P}_0^+(X)$ is a covering map, and the subgroup of $\mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(X))$ which preserves the connected component $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ acts freely on $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ and is the group of deck transformations of $\mathcal{Z}$.

Conjecture 25 (Bridgeland)

The action of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ on $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ preserves the connected component $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$. Moreover $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X))$ is simply-connected.

Remarks

Huybrechts-Macrì-S.: The conjecture has been verified for

- Generic non-algebraic K3 surfaces (i.e. such that $\mathrm{Pic}\,(X) = 0$);
- Generic projective twisted K3 surfaces (the twist is given by an element of the Brauer group of the surface).

Bridgeland: As a consequence of the conjecture we get the following short exact sequence

$$1 \to \pi_1(\mathcal{P}_0^+(X)) \to \mathrm{Aut}\,(\mathrm{D^b}(X)) \to \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \to 1,$$

where $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ is the group of orientation preserving Hodge isometries of the total cohomology of X.

Remarks

The morphism $\Pi : \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\widetilde{H}(X, \mathbb{Z}))$ sends an autoequivalence to the induced Hodge isometry.

The fact that Π should factor through a surjective morphism onto $\mathrm{O}_{+}(\widetilde{H}(X, \mathbb{Z}))$ was previously conjectured by Szendoi based on some results by Orlov, Mukai,...

Huybrechts-Macrì-S.: Szendroi's conjecture holds true.

Warning: To prove this, we need anyhow a (tiny) part of Bridgeland's theory of stability conditions!

Outline

The definition (after Kontsevich-Soibelman)

Denote by **C** an ind-constructible weakly unital triangulated A_∞-category over a field k.

A data **stability structure** is given by the data:

- An ind-constructible homomorphism $\mathrm{cl} : K(\mathbf{C}) \to \Gamma$, where $\Gamma \cong \mathbb{Z}^n$ is a free abelian group of finite rank endowed with a bilinear form $\langle -, - \rangle : \Gamma \times \Gamma \to \mathbb{Z}$ such that for any two objects $\mathcal{E}, \mathcal{F} \in \mathrm{Ob}(\mathbf{C})$,

$$\langle \mathrm{cl}(\mathcal{E}), \mathrm{cl}(\mathcal{F}) \rangle = \chi(\mathcal{E}, \mathcal{F});$$

- An additive map $Z : \Gamma \to \mathbb{C}$, called the **central charge**;
- A collection $\mathbf{C}^{ss}$ of (isomorphism classes of) non-zero objects in **C** called semistable, such that $Z(\mathcal{E}) \neq 0$ for any $\mathcal{E} \in \mathbf{C}^{ss}$;
- A choice of a phase for $Z(\mathcal{E})$, where $\mathcal{E} \in \mathbf{C}^{ss}$.

The definition (after Kontsevich-Soibelman)

The data must satisfy the following axioms:

(KS1) For all $\mathcal{E} \in \mathbf{C}^{ss}$ and for all $n \in \mathbb{Z}$, $\mathcal{E}[n] \in \mathbf{C}^{ss}$ and $\phi(Z(\mathcal{E}[n])) = \phi(Z(\mathcal{E})) + n$;

(KS2) For all $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{C}^{ss}$ with $\phi(\mathcal{E}_1) > \phi(\mathcal{E}_2)$ we have $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$;

(KS3) For any $\mathcal{E} \in \mathrm{Ob}(\mathbf{C})$, there exist $n \geq 0$ and a chain of morphisms $0 = \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_n = \mathcal{E}$ (HN filtration) such that $\mathcal{F}_i := \mathrm{Cone}(\mathcal{E}_{i-1} \to \mathcal{E}_i)$, for $i = 1, \ldots, n$ are semistable and $\phi(\mathcal{F}_1) > \phi(\mathcal{F}_2) > \cdots > \phi(\mathcal{F}_n)$;

The definition (after Kontsevich-Soibelman)

(KS4) For each $\gamma \in \Gamma \setminus \{0\}$, the set of isomorphism classes of a $\mathbf{C}_\gamma^{ss} \subset \mathrm{Ob}(\mathbf{C})_\gamma$ consisting of semistable objects $\mathcal{E}$ defined over $\overline{k}$ and such that $\mathrm{cl}(\mathcal{E}) = \gamma$ and $\phi(\mathcal{E})$ is fixed, is a constructible set;

(KS5) **(Support Property)** For a norm $\| - \|$ on $\Gamma \otimes \mathbb{R}$, there exists $C > 0$ such that for all $\mathcal{E} \in \mathbf{C}^{ss}$ one has $\| \mathrm{cl}(\mathcal{E}) \| \leq C|Z(\mathcal{E})|$.

Remarks

- The forgetting map $\mathrm{Stab}(\mathbf{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C})$ sending a stability structure to Z is a local homeomorphism.
- Hence, $\mathrm{Stab}(\mathbf{C})$ is a complex manifold, not necessarily connected.
- Due to the support property, all stability structures are locally finite.