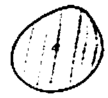
François Loeser — **Motivic Milnor fibre**

## §1. Milnor fibre

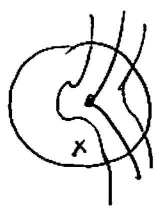
$f: X \to \mathbb{C}$ non constant, $X$ smooth, $d$-dim'l, connected, cplx. var.

$x \in X$, $f(x) = 0$.

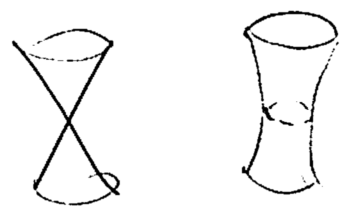
- $f$ smooth at $x$ : $f$ loc. const.



- $f$ sing. at $x$ : $df(x) = 0$



$\rightsquigarrow$ vanishing cycle, e.g.



$x^2+y^2=0$ $\quad$ $x^2+y^2=c\neq 0$

**Thm (Milnor)**
If $0 < \eta \ll \varepsilon \ll 1$ then $\tilde{f}: D(x,\varepsilon) \cap f^{-1}(D(0,\eta)\setminus\{0\}) \to D(0,\eta)^* \;(= D\setminus\{0\})$ is a fibration wrt. cplx. topol.

**Def:** Milnor fibre $F_x = F_{\varepsilon,\eta} := \tilde{f}^{-1}(\eta)$ + automorphism $M_x$ on $F_x$.
$M_x \curvearrowright H^i(F_x, \mathbb{Q})$. $\quad$ [proved by Milnor only for isolated sing'ty].

**Expl:** $(x^2+y^3=0) \cap S(0,\varepsilon)$ is a trefoil knot.

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**A la Deligne:**

$$\begin{array}{ccccc} \tilde{X}^* & \to & X & \leftarrow & X_0 \\ \downarrow & \lrcorner & f\downarrow & & \downarrow \\ \tilde{D}^* & \xrightarrow{j} & D & \xleftarrow{i} & \{0\} \end{array}$$

$\tilde{D}^* \to D^*$ univ. cover
$X_0 = f^{-1}(0)$.
$\tilde{X}^* = X \times_D \tilde{D}^*$.

$\mathcal{F}$ constr. sheaf $/X$ $\quad$ (or $\in D^b_{cons}(X)$)

One considers $i^* R j_* j^* \mathcal{F} = R\psi \mathcal{F}$ ; has monodromy trf. $M$ [induced by Deck trf. on $\tilde{X}^*$]

**Exercise:** $(R^q\psi \underline{\mathbb{C}})_x \simeq H^q(F_x)$, $M$-equiv. iso.

FL2

Natural triangle $\quad i^*\mathcal{F} \longrightarrow R\psi\mathcal{F} \longrightarrow R\phi\mathcal{F}$ $\leftarrow$ vanishing cycle functor.

<u>Monodromy theorem</u>: The monodromy action on $H^q(\mathcal{F}_x)$ is quasi-unipotent: Eigenvalues are roots of unity (Jordan blocks of size $\leq q+1$).

[$h: Y \to X$ log resolution of $(X, X_0)$; $Y$ smooth, $h$ proper, $h^{-1}(X_0)$ NCD, $h$ an isom outside $h^{-1}(X_0)$.
On $Y$ consider $g := f \circ h$. Is locally monomial, so monodromy action on $R\psi_g \underline{\mathbb{C}}_Y$ is finite. $R^q\psi_f \underline{\mathbb{C}}_X = Rh_* R\psi_g \underline{\mathbb{C}}_Y$. Implies statement by exercise in derived nonsense.]

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## §2. Motivic Milnor fibre

If $n \in \mathbb{N}$, $\quad \mathcal{L}_n(X)(\mathbb{C}) := X(\mathbb{C}[t]/t^{n+1})$

$\mathcal{L}(X)(\mathbb{C}) := X(\mathbb{C}[[t]])$

<u>Expl</u>: $X = V(F)$, $\quad F = F(x_0, \ldots, x_n)$.

$x_i(t) = \sum_{j \geq 0} a_{ij} t^j$, $\quad a_{ij} \in \mathbb{C}$

Now expand $F(x_0(t), \ldots, x_n(t)) = \sum_{\ell \geq 0} F_\ell(a_{ij}) t^\ell$.

$\mathcal{L}(X)$ is defined by $F_\ell(-) = 0 \ \forall \ell$. (infinitely many eqns in infin. many var's).

$X$ smooth $\Rightarrow$ $\mathcal{L}_n(X) \to X$ is a fibration with fibre $\mathbb{A}^{nd}$.

$K_0(Var) :=$ quotient of free abelian group of isom. classes of cplx. var's. modulo cut and paste: $[X] = [X'] + [X \setminus X']$, $X' \subset X$ closed.

<u>Variants</u>: $K_0(Var_S)$: version over $S$.

Equvt. version: $G$ alg. group. $G$ acts trivially on $S$, $G \curvearrowright \begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}$ (TL3)

$K_0(Var_S^G)$ additional relation:
$$\begin{tikzcd} A \arrow[dr] & & A' \arrow[dl] \\ & X \arrow[d] & \\ & S & \end{tikzcd}$$
$A$ and $A'$ affine bdls over $X$ of the same rank

lift the same $G$-action on $X$

then $[A \to S] = [A' \to S]$ in $K_0(Var_S^G)$.

[cf. Toen]

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$X \xrightarrow{f} \mathbb{C}$

Fact: There are (at least) three different connections of $\mathcal{L}_n(X)$ with the monodromy.

Consider: $\mathcal{X}_n := \{\varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n + \dots\}$, $\mathcal{X}_{n,x} := \{\varphi \in \mathcal{X}_n \mid \varphi(0) = x\}$.

Thm: (Denef-L.) If $n \geq 1$ $\quad Eu(\mathcal{X}_{n,x}) = tr(M_x^n, H^*(F_x))$.

[Rem: Cf. with Floer cohomology? ($\rightsquigarrow$ Seidel).]

Pf: Compute both sides on a resolution $h: Y \to X$.

Challenge: Find a geometric (or physics) proof.

$\mu_n$: $n$-th root of unity, $\hat{\mu} := \varprojlim \mu_n$.

$K_0(Var_S^{\hat{\mu}}) := \varinjlim_n K_0(Var_S^{\mu_n})$

Now $\mathcal{X}_n$ is endowed with a $\mu_n$-action.

$\mathbb{C}^*$ acts on $\mathcal{L}(X)$, $\mathcal{L}_n(X)$: $\varphi(t) \mapsto \varphi(\lambda t)$.

Def: Motivic zeta-fct: $\boxed{Z_f(T) := \sum_{n \geq 1} [\mathcal{X}_n] T^n \in K_0(Var_{X_0}^{\hat{\mu}})[[T]]}$

For $n > 1$ $\quad \mathcal{X}_n \to X_0 = f^{-1}(0)$.

FL4

Thm: (Denef-L.) $Z_f(T)$ is a rat'l fct. of $T$. (→ talk II).

Monodromy Conjecture: Let $r$ be a rat'l number. If $\mathbb{L}^r$ is a pole of $Z_f(T)$ [$\mathbb{L} = \mathbb{A}^1_S$ in $K_0(Var_S^{\hat{\mu}})$ with trivial action] then $\exp(2\pi i r)$ is an eigenvalue of monodromy acting on the cohomology of $H^q(F_x)$, some $x \in X_0$, some $q$.