François Loeser — Motivic Milnor fibre II

$\mathcal{X}_n := \{\varphi \in \mathcal{L}_n(X),\ f(\varphi) = t^n + \text{higher order}\}$

$\mathbb{L} := [\mathbb{A}^1_S] \in K_0(Var_S^{\hat{\mu}})$.

$\mathcal{M}_S^{\hat{\mu}} := K_0(Var_S^{\hat{\mu}})[\mathbb{L}^{-1}]$

$d = \dim X$.

$Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_n]\mathbb{L}^{-nd}T^n \in \mathcal{M}_S^{\hat{\mu}}[[T]]$ [This is the correct def'n!].

Take $h: Y \longrightarrow X$ a log resolution of (X, X_0) as in talk I.

Write $h^{-1}(f^{-1}(0)) = \sum_{i\in J} N_i E_i$, E_i smooth, $|h^{-1}(X_0)| = \bigcup_{i\in J} E_i$ ($X_0 = f^{-1}(0)$)

If $I \subset J$ set $E_I = \bigcap_{i\in I} E_i$, $\mathring{E}_I = E_I \setminus \bigcup_{j\notin I} E_j$.

$h^* K_X = K_Y + \sum_{i\in I}(n_i - 1)E_i$, $n_i \geq 1$

ν_{E_i} normal bdl., $\nu_{\mathring{E}_I}$ normal bdl.

associated $\mathbb{G}_m^{|I|}$-bdl over $\mathring{E}_I$: $V_I \subset \nu_{\mathring{E}_I}$ (throw away 0-section)

Consider $f\circ h: Y \longrightarrow \mathbb{A}^1$.

Fix I. Locally on $\mathring{E}_I$, if E_i given by $z_i = 0$: $f\circ h = \prod_{i\in I} z_i^{N_i}\cdot u$ (unit).

"Divide by $\prod_{i\in I} z_i^{N_i}$" gets well-defined (!) $f_I: V_I \longrightarrow \mathbb{G}_m$ s.th.

$$f_I(\lambda x) = \lambda^{\sum_{i\in I} N_i}\cdot f_I(x).$$

Now consider $f_I^{-1}(1) =: V_I^1 \subset V_I$ (is endowed with a μ_N-autom., $N = \sum_{i\in I} N_i$)

Thm (Denef-Loeser): $\boxed{Z_f(T) = \sum_{\emptyset\neq I\subset J} [V_I^1] \prod_{i\in I} \frac{1}{T^{-N_i}\mathbb{L}^{n_i} - 1}}$ holds in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$

Proof based on the following

<u>Proposition</u>: (Denef-L.) $h: Y \to X$ proper birat'l, X, Y smooth var's.

Fix $e \in \mathbb{N}$, consider

$$\Delta_e := \{\varphi \in \mathcal{L}(Y) \,/\, \operatorname{ord} h^*(K_X)(\varphi) = e\}.$$

For $n \geq 2e$,

(1) the image $\Delta_{e,n}$ of Δ_e in $\mathcal{L}_n(Y)$ is a union of fibres of $h_n: \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$.

(2) the restriction of h_n to $\Delta_{e,n}$ is a piecewise Zariski fibration onto its image with fibre $\mathbb{A}^e$.

[In particular, if $W \subset h_n(\Delta_{n,e})$: $[W] = [h_n^{-1}(W)]\,\mathbb{L}^{-e}$].

<u>Rem</u>: $Z \to W$ piecewise Zariski fibration with fibre $A \Rightarrow [Z] = [A][W]$.

$$\lim_{T \to \infty} \frac{1}{T^{-N}\mathbb{L}^{n} - 1} = -1 .$$

So $\lim_{T\to\infty} Z_f(T) \in \mathcal{M}_{X_0}^{\hat{\mu}}$ makes sense.

<u>Def</u>: $\mathcal{S}_f := \lim_{T\to\infty} Z_f(T)$ motivic Milnor fibre.

Given h, $\mathcal{S}_f = -\sum (-1)^{|I|}\,[V_I^1]$

<u>Q</u>: Why is this a meaningful definition? ⤵

<u>Betti realization</u>

Consider $\zeta = ((\exp \frac{2\pi i}{n})_n) \in \hat{\mu}$. Have $\mu_{nm} \to \mu_n$, $z \mapsto z^m$.

Given $x \in X_0$: $\mathcal{M}_{X_0}^{\hat{\mu}} \xrightarrow{i_x^*} \mathcal{M}^{\hat{\mu}} = K_0(\mathrm{Var}^{\hat{\mu}})[\mathbb{L}^{-1}]$ (take the fibre over x)

$$\mathcal{M}_{X_0}^{\hat{\mu}} \xrightarrow{i_x^*} \mathcal{M}^{\hat{\mu}} \xrightarrow{Eu} K_0(\mathrm{Vect}^{\mathrm{autom.}})$$

X var. with $\hat{\mu}$ action $\longmapsto \sum_i (-1)^i [H_c^i(X) \circlearrowleft T]$ ← action of ξ

Lemma: $Eu(i_x^* \mathcal{S}_f) = $ class of $(H^* \mathcal{F}_{f,x}) \circlearrowleft M$

Similar game with Hodge structures: $H : K_0(\mathrm{Var}^{\hat{\mu}}) \to K_0(HS^{\mathrm{autom.}})$

Again $H(i_x^* \mathcal{S}_f)$ is the class of the mixed HS on $\mathcal{F}_x$ with the monodromy automorphism.

Recall: A canonical MHS was constructed by Steenbrink (isolated sig) and by Navarro Aznar (generic).

$\mathcal{S}_f \leftrightarrow R\psi$: Let us consider $\mathcal{S}_{f,x} := i_x^* \mathcal{S}_f$.

$$\mathcal{S}_{f,x}^{\phi} := (-1)^{d-1} (\mathcal{S}_{f,x} - 1).$$

Than-Sebastiani: (allows uniform treatment of Brieskorn-Pham sing's etc)

Take X_1 and X_2 smooth, $f_i : X_i \to \mathbb{C}$, $i = 1,2$.

$$f_1 \oplus f_2 : X_1 \times X_2 \to \mathbb{C}, \ (x_1, x_2) \mapsto f_1(x_1) + f_2(x_2).$$

Thm (T.-S.): $\mathcal{F}_{f_1 \oplus f_2, (x_1, x_2)}$ is the topological join of $\mathcal{F}_{f_1, x_1}$ and $\mathcal{F}_{f_2, x_2}$

Convolution: $\mathcal{M}^{\hat{\mu}} \times \mathcal{M}^{\hat{\mu}} \to \mathcal{M}^{\hat{\mu}}$, $(a,b) \mapsto a * b$ commutative and associative.

Notation: G finite group scheme acting on X and Y, then

$$X \times^G Y := X \times Y / (gx, y) \sim (x, gy)$$ endowed with diagonal G-action.

Fix $n \geq 1$: $F_1^n = \{(x_1, x_2) \in \mathbb{G}_m^2 \mid x_1^n + x_2^n = 1\}$ with $\mu_n \times \mu_n$-action.

$F_0^n = \{ \quad " \quad \mid \quad " = 0\}$

Now if X and Y have μ_n-action, define

$$[X] * [Y] = -[(X \times Y) \times^{\mu_n \times \mu_n} F_1^n] + [(X \times Y) \times^{\mu_n \times \mu_n} F_0^n].$$

This induces

$$* : \mathcal{M}^{\mu_n} \times \mathcal{M}^{\mu_n} \longrightarrow \mathcal{M}^{\mu_n},$$

and taking $n \to \infty$:

$$* : \mathcal{M}^{\hat{\mu}} \times \mathcal{M}^{\hat{\mu}} \longrightarrow \mathcal{M}^{\hat{\mu}}.$$

Thm (Motivic Thom-Sebastiani) $f_1 : X_1 \to \mathbb{C}$, $f_2 : X_2 \to \mathbb{C}$

$$\Rightarrow \varphi^{\phi}_{f_1 \oplus f_2, (x_1, x_2)} = \varphi^{\phi}_{f_1, x_1} * \varphi^{\phi}_{f_2, x_2}.$$

Heuristically: Thom-Sebastiani is a version of

$$\int \exp(\lambda f_1 \oplus f_2) = \int (\exp \lambda f_1) \cdot \int (\exp \lambda f_2) \quad , \quad \lambda \text{ param.}$$

Rem: l-adic version $\nearrow$ talk by Deligne, cf. paper by Massey using constructible sheaves