Updates on the Ubiquity Conjecture

Max Pitz
With N. Bowler, C. Elbracht, J. Erde, P. Gollin, K. Heuer and M. Teegen

University of Hamburg, Germany

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The ubiquity question
Potential infinity vs. actual infinity

The Ubiquity Question:

- Fix your favourite connected graph $G$.
- Suppose have a host graph $\Gamma$ which contains arbitrarily many disjoint copies of $G$.
- Can you find infinitely many disjoint copies of $G$ in $\Gamma$?

If yes for all possible host graphs $\Gamma$, we say $G$ is ubiquitous.
Small detour: What do we mean by ‘copies of $G$ in $\Gamma$’?

Embeddings as subgraph, topological minor and minor

When saying ‘a copy of $G$ in $\Gamma$’, written $G \prec \Gamma$, we could mean:

- $G$ embeds as subgraph ($G \subseteq \Gamma$)
Small detour: What do we mean by ‘copies of $G$ in $\Gamma$’?

Embeddings as subgraph, topological minor and minor

When saying ‘a copy of $G$ in $\Gamma$’, written $G \triangleleft \Gamma$, we could mean:

- $G$ embeds as subgraph ($G \subseteq \Gamma$)
- $G$ embeds as topological minor ($G \leq \Gamma$)
Small detour: What do we mean by ‘copies of $G$ in $\Gamma$’?

Embeddings as subgraph, topological minor and minor

When saying ‘a copy of $G$ in $\Gamma$’, written $G \triangleleft \Gamma$, we could mean:

- $G$ embeds as *subgraph* ($G \subseteq \Gamma$)
- $G$ embeds as *topological minor* ($G \leq \Gamma$)
- $G$ embeds *as a minor* ($G \preceq \Gamma$)
Finite graphs are ubiquitous w.r.t. all relations
We simply pick copies greedily.

\[ H_1 \subseteq \Gamma \]

1. Pick first copy \( H_1 \subseteq \Gamma \) of \( G \).
Finite graphs are ubiquitous w.r.t. all relations
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1. Pick first copy $H_1 \subset \Gamma$ of $G$.
2. Know that $\Gamma$ contains $|H_1| + 1$ disjoint copies of $G$. Pick second copy $H_2 \subset \Gamma$ of $G$ disjoint from $H_1$. 
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3. Know that $\Gamma$ contains $|H_1| + |H_2| + 1$ disjoint copies of $G$.
4. Continue...
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- Any ray from a given layer might intersect all rays from all other layers.
- Halin’s idea:
  - If rays don’t intersect → pick greedily.
  - If rays do intersect → re-route onto the next layer.
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Our infinitely many rays use finite subpaths from the layers, but otherwise have little in common with our original rays!
Bad news for subgraph and topological minor relation

Counterexamples due to Andreae, Lake and Woodall.

Figure: A graph which is not $\subseteq$-ubiquitous.

Figure: A graph which is not $\leq$-ubiquitous.
Overview of known ubiquity results

\(\subseteq\)-Ubiquity:  
- ✓ Finite graphs
- ✓ Ray / Double ray (Halin, ’65/’70)
- ✗ Infinite comb

\(\preceq\)-Ubiquity:  
- ✓ Finite graphs
- ✓ Trees with \(\Delta \leq 3\) (Halin, ’75)
- ✓ Locally finite trees (Andreae, ’79)
- ✗ Infinite comb with triangles

\(\preceq\)-Ubiquity:  
- ✓ Finite graphs
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\(\succeq\)-Ubiquity: ✓ Finite graphs ✓ Countable trees (Halin, ’75)
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The Ubiquity Conjecture (Andreae, ’01): All locally finite connected graphs are minor-ubiquitous.
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☞ All trees, all cardinalities (BEEGHPT ’18+)

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- \(\Rightarrow\) All graphs of bounded treewidth (BEEGHPT '18+)

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The Ubiquity Conjecture (Andreae, ’01): All locally finite connected graphs are minor-ubiquitous.
Plan: Show ubiquity ideas in a simple class of examples

Let’s take an infinite graph $G$ which is glued together from a sequence of finite connected graphs $(G_n)_{n \in \mathbb{N}}$ along 1-separators.
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Let’s take an infinite graph $G$ which is glued together from a sequence of finite connected graphs $(G_n)_{n \in \mathbb{N}}$ along 1-separators.

We may also fix a representative ray $R \subset G$ for later use. Note that $R$ passes through each 1-separator precisely once.
Concentrated families
A simple yet crucial new idea:

- Your task is to hide copies of $G$ in $\Gamma$ such that
  - $n \cdot G \triangleleft \Gamma$ for all $n \in \mathbb{N}$, and such that
  - not easy for me to find infinitely many copies of $G$ in $\Gamma$. 
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- For every finite vertex set $X \subseteq V(\Gamma)$, at most $|X|$ graphs from each layer can meet $X$.
- Still $n \cdot G \triangleleft \Gamma - X$ for all $n \in \mathbb{N}$.
- If $\exists_{\infty}$ components $C$ of $\Gamma - X$ with $G \triangleleft C$ then gameover.
- Ow/, $\exists$ component $C$ of $\Gamma - X$ with $n \cdot G \triangleleft C$ for all $n \in \mathbb{N}$ (pigeon hole).
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I now apply the following strategy:

- If possible, pick finite $X_1 \subset \Gamma$ s.t. in $\Gamma - X_1$ there exist components $C_1 \neq D_1$ with
  - $n \cdot G \triangleleft C_1$ for all $n \in \mathbb{N}$,
  - $D_1$ contains a copy $H_1$ of $G$. 
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- If possible, pick finite $X_2 \subset C_1$ s.t. in $C_1 - X_1$...
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- If this process doesn’t stop, then $\{H_n : n \in \mathbb{N}\} \rightarrow \text{gameover.}$
Concentrated families

A simple yet crucial new idea:

**Lesson:** Place $G$-copies in $\Gamma$ s.t. $\forall X \subseteq V(\Gamma)$ finite, $\exists !$ component $C_X$ of $\Gamma \setminus X$ such that ‘almost all’ copies of $G$ are contained in $C_X$. 
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**Observation:** The family $(C_X)_X$ satisfies

\[ X \subseteq X' \rightarrow C_X \supseteq C_{X'}. \]

Such a choice of components $(C_X)_X$ is called a *direction in* $\Gamma$. 
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**Observation:** The family $(C_X)_X$ satisfies

$$X \subseteq X' \rightarrow C_X \supseteq C_{X'}.$$  

Such a choice of components $(C_X)_X$ is called a *direction in $\Gamma$*. (Diestel and Kühn have shown that *directions* and *ends* are the same thing.)
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- A ray $S \subset \Gamma$ agrees with $(C_X)_X$ if $S$ has a tail in every $C_X$. 
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A ray $S \subset \Gamma$ *agrees with* $(C_X)_X$ if $S$ has a tail in every $C_X$.

Fix a ray $R$ in our graph $G$.

For every $G$-copy $H$ in $\Gamma$, the lifted ray $H(R)$ either agrees with $(C_X)_X$ or not.

Pigeon hole: May assume that $H(R)$ either agrees with $(C_X)_X$ always or never, uniformly for all $G$-copies $H$ in $\Gamma$. 
In the never-agree case, can again pick copies greedily

1. Pick first copy $H_1 \subset \Gamma$ of $G$. 
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2. Find $X_1$ where $H_1(R)$ disagrees with $C_{X_1}$. 
In the never-agree case, can again pick copies greedily

1. Pick first copy $H_1 \subset \Gamma$ of $G$.
2. Find $X_1$ where $H_1(R)$ disagrees with $C_{X_1}$.
3. Pick second copy $H_2 \subset C_{X_1}$ of $G$. 
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2. Find $X_1$ where $H_1(R)$ disagrees with $C_{X_1}$.
3. Pick second copy $H_2 \subset C_{X_1}$ of $G$.
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with $C_{X_2}$. 
In the never-agree case, can again pick copies greedily

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2. Find $X_1$ where $H_1(R)$ disagrees with $C_{X_1}$.
3. Pick second copy $H_2 \subset C_{X_1}$ of $G$.
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with $C_{X_2}$.
5. Pick third copy $H_3 \subset C_{X_2}$ of $G$. 
In the never-agree case, can again pick copies greedily

1. Pick first copy $H_1 \subset \Gamma$ of $G$.
2. Find $X_1$ where $H_1(R)$ disagrees with $C_{X_1}$.
3. Pick second copy $H_2 \subset C_{X_1}$ of $G$.
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with $C_{X_2}$.
5. Pick third copy $H_3 \subset C_{X_2}$ of $G$.
6. Continue....
In the always-agree case, use well-quasi-ordering theory

Using the Robertson-Seymour result on wqo of finite graphs

- Colour the left cut-vertex of each $G_n$ with 1 and the right cut-vertex with 2.

- Labelled wqo of finite graphs (Robertson-Seymour): $\exists N \in \mathbb{N}$ s.t. every $G_n$ for $n > N$ embeds into infinitely many $G_i$. 

\[ \cdots \]
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- Labelled wqo of finite graphs (Robertson-Seymour): $\exists N \in \mathbb{N}$ s.t. every $G_n$ for $n > N$ embeds into infinitely many $G_i$.

- May assume $N = 1$, i.e. can find every blob but the first again and again.
In the always-agree case, use well-quasi-ordering theory

The construction – a picture proof for a one-ended example
In the always-agree case, use well-quasi-ordering theory

The construction – a picture proof for a one-ended example

![Diagram](image-url)
In the always-agree case, use well-quasi-ordering theory

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(copy of $G_2$)
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Lesson 1: As with Halin’s rays, our $G$-copies use finite blobs from the layers, but otherwise have little in common with original copies!
In the always-agree case, use well-quasi-ordering theory

The construction – a picture proof for a one-ended example

Lesson 1: As with Halin’s rays, our $G$-copies use finite blobs from the layers, but otherwise have little in common with original copies!

Lesson 2: If you place your $G$-copies all over the host graph $\Gamma$, then easy for me to win. And if you place them so that they are concentrated, you will inadvertently create lots of new $G$-copies due to wqo which I may exploit.
For the details see....

Bowler, Elbracht, Erde, Gollin, Heuer, Pitz, Teegen:

- Ubiquity in graphs III: Ubiquity of a class of locally finite graphs, preprint available soon.
- Ubiquity in graphs IV: Ubiquity of graphs of bounded tree-width, at some point.