Max Pitz:

Applications of order trees in infinite graphs

27. August 2020, ALGOS2020

Slides available under `https://www.math.uni-hamburg.de/home/pitz/`

§0: T-graphs: Definition and examples

§1: T-graphs, colouring number and forbidden minors

§2: T-graphs and wqo of infinite graphs

§3: Halin's end degree conjecture

§0: T-graphs: Definition and examples

Let's agree on the following notation regarding order trees:

- **Order tree**: A partially ordered set $(T, \leq)$ with unique minimal element (called the *root*) and all subsets of the form $\lceil t \rceil = \lceil t \rceil_T := \{t' \in T : t' \leq t\}$ are well-ordered. Write $\lfloor t \rfloor := \{t' \in T : t \leq t'\}$.
- **Branch**: A maximal chain in T (well-ordered).
- **Height**: The *height* of T is the supremum of the order types of its branches. The *height* of a point $t \in T$ is the order type of $\lceil \mathring{t} \rceil := \lceil t \rceil \setminus \{t\}$.
- **Level**: The set T^i of all points at height i is the ith *level* of T, and write $T^{<i} := \bigcup \{T^j : j < i\}$.
- **Successors and limits**: If $t < t'$, we write $[t, t'] = \{x : t \leq x \leq t'\}$ etc. If $t < t'$ but there is no point between t and t', we call t' a *successor* of t and t the *predecessor* of t'; if t is not a successor of any point it is called a *limit*.

Rooted graph-theoretic trees (connected, acyclic graphs) correspond to the order trees of height at most ω. Are there useful graphs on order trees? Well, the comparability graph; but the following concept is much more versatile:

Definition (Brochet & Diestel)**.** For an order tree $(T, \leq)$, a graph $G = (V, E)$ is a *T-graph* if $V = T$, the ends of any edge $e = tt'$ are comparable in T, and the neighbours of any $t \in T$ are cofinal in $\lceil \mathring{t} \rceil := \{t' \in T : t' < t\}$.

Example. (1) Rado ('78): Generalised path $\leftrightarrow$ T-graph for T an ordinal.

- Erdős & Rado ('78): Any countable complete graph K_ω where the edges have been coloured with $r \in \mathbb{N}$ many colours can be partitioned into r monochromatic paths / rays.
- D. Soukup ('16): Any complete graph K_κ where the edges have been coloured with $r \in \mathbb{N}$ colours can be partitioned into r monochromatic generalised paths.
- Bürger & Pitz ('18): Any complete bipartite graph $K_{\kappa,\kappa}$ where the edges have been coloured with $r \in \mathbb{N}$ colours can be partitioned into $2r - 1$ monochromatic gen. paths.

§0: T-graphs: Definition and examples

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Example. (1) Rado ('78): Generalised path $\leftrightarrow$ T-graph for T an ordinal.

(2) Fun fact: every ω_1-graph has a K_{ω_1} subdivision.

(3) Thomas ('88): Used T-graphs for certain binary trees of height $\omega + 1$ to construct examples that uncountable graphs are not well-quasi-ordered (more about that later).

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If a graph G is (isomorphic to) a T-graph for some order tree $(T, \leq)$, we say that $(T, \leq)$ is a *normal tree order* for G. When T has height at most ω, we say T is a *normal spanning tree* for G.

Open Problem. Which connected graphs admit a normal tree order?

- Not all graphs do: consider an uncountable clique where every edge has been subdivided once.
- Jung ('69): Every countable graph contains a normal spanning tree with any arbitrarily chosen vertex as the root.
- Brochet & Diestel ('95): Every connected graph G "almost" has a normal tree order: There is a contraction G' with normal tree order $(T, \leq)$ and branch sets $(V_t)_{t \in T}$ in G such that $|V_t| \leq \mathrm{cf}\,(\mathrm{height}(t))$ for all $t \in T$.

Can we say more about which graphs have a normal spanning tree?

Definition (Erdős & Hajnal)**.** The *colouring number* $\mathrm{col}(G)$ is the least cardinal μ such that $V(G)$ has a well-order $\preceq$ such that every vertex has $< \mu$ neighbours preceding it in $\preceq$.

- Observation: If G has a normal spanning tree, then $\mathrm{col}(G) \leq \aleph_0$.
- Converse: No (again: an uncountable clique where every edge has been subdivided once)
- BUT: Having an NST is a minor-closed property!

Conjecture (Halin, '98)**.** A connected graph G has a normal spanning tree if and only if every minor of G has countable colouring number.

Theorem (Pitz, '20$^+$)**.** *Halin's conjecture is true.*

Consequence: As there is a forbidden subgraph characterisation for having colouring number $\leq \mu$ (Bowler, Carmesin, Komjath, Reiher, '15), this yields a forbidden minor characterisation for the property of having a normal spanning tree!

§2: T-graphs and wqo of infinite graphs

- **Minor** H is a minor of G if there are disjoint connected vertex sets $\{V_h\colon h \in H\}$ in G such that G has a $V_h - V_{h'}$ edge whenever hh' is an edge in H. Write $G \preccurlyeq H$ if G is a minor of H.
- **Wqo**: A binary relation $\triangleleft$ on a set X is a *well-quasi-order* if it is reflexive and transitive, and for every sequence $x_1, x_2, \ldots \in X$ there is some $i < j$ such that $x_i \triangleleft x_j$.

Theorem (Roberton & Seymour, '80s)**.** *Finite graphs are well-quasi ordered under the minor relation $\preccurlyeq$.*

Open Problem. Are countable graphs well-quasi ordered by $\preccurlyeq$?

Theorem (Thomas '88)**.** *Graphs of size $2^{\aleph_0}$ are not well-quasi ordered by $\preccurlyeq$: There is a sequence $G_1, G_2, \ldots$ of binary trees with tops such that $G_i \not\preccurlyeq G_j$ whenever $i < j$.*

Theorem (Komjath '95)**.** *For every uncountable cardinal κ there is a family $\{G_i\colon i < 2^\kappa\}$ of κ-sized graphs such that $G_i \not\preccurlyeq G_j$ whenever $i \neq j$.*

Downside: Komjath's graphs are hard to define. Better:

Theorem (Pitz '20$^+$)**.** *For every uncountable regular κ there is a family $\{G_i\colon i < \kappa\}$ of T_κ with κ many tops such that $G_i \not\preccurlyeq G_j$ whenever $i \neq j$.*

Remark: Implies Komjath (take disjoint unions over subsets of indices $\subseteq \kappa$).

Theorem (Thomas '88)**.** *There are binary trees with $2^{\aleph_0}$ many tops $G_1, G_2, \dots$ such that $G_i \not\preccurlyeq G_j$ whenever $i < j$.*

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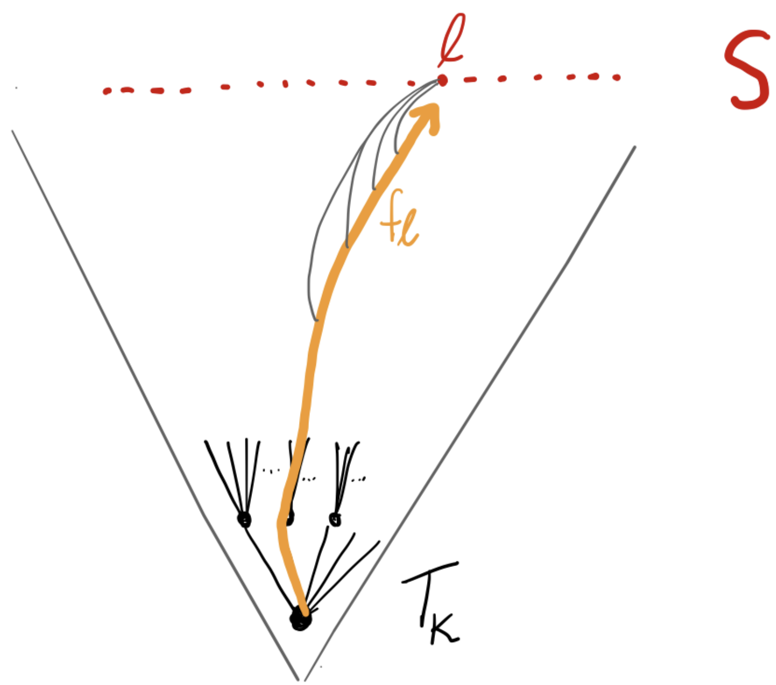
Idea for the construction: Let $\Lambda \subseteq \kappa$ denote the set of limit ordinals of countable cofinality. For every $\ell \in \Lambda$ pick an increasing cofinal sequence $f_\ell \colon \mathbb{N} \to \ell$, which we may interpret as a rooted ray in $T_\kappa = \kappa^{<\omega}$.
For $S \subseteq \Lambda$ let $T(S)$ be the tree order where we add for every $\ell \in S$ a top above every ray f_ℓ in T_κ, and $G(S)$ any $T(S)$-graph.

Proof: Show that if $S, R \subseteq \Lambda$ are disjoint stationary subsets of κ, then $G(S) \not\preccurlyeq G(R)$. □



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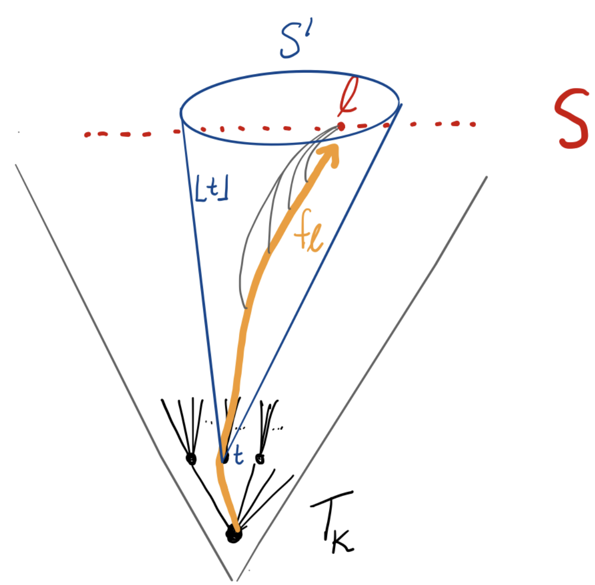
What is so interesting about $F(S) = \{f_\ell : \ell \in S\}$ for $S = \Lambda$ or $S \subseteq \Lambda$ stationary?

- Topological interpretation: The rays in T_κ naturally form a topological space $\kappa^{\mathbb{N}}$, the *Baire space of weight κ.* Stone ('63 & '72) has shown that $F(S)$ is not Borel in $\kappa^{\mathbb{N}}$, but each separable subspace of $F(S)$ is countable.
- Surprising connection to normal spanning trees: $G = G(S)$ doesn't have a normal spanning tree.

What makes the proofs work?

- The rays bunch up in a strange way:
- For $n \in \mathbb{N}$ arbitrary, by the pressing down-lemma, stationary many tops $S' \subseteq S$ agree on their first n coordinates.



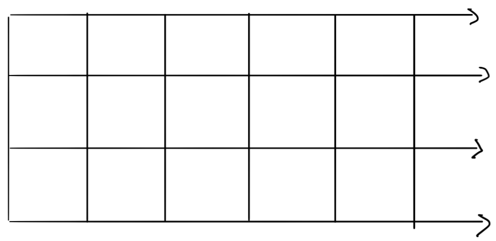
§3: Halin's end degree conjecture

Definition.

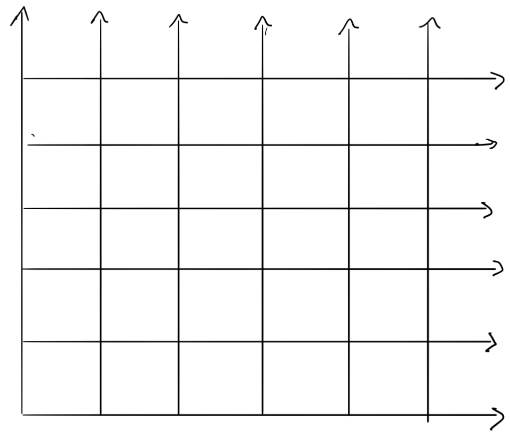
- An *end* ϵ of a graph G is an equivalence class of rays in G, where two rays $R_1 \sim R_2$ are equivalent if there are infinitely many disjoint $R_1 - R_2$ paths in G.
- The degree of an end ϵ is the maximal size of a collection of disjoint rays in ϵ (well-defined by a theorem of Halin).

Example.

- The $\{1, \dots, n\} \Box \mathbb{N}$ grid: $\deg(\epsilon) = n$.
- The $\mathbb{N} \Box \mathbb{N}$ grid: $\deg(\epsilon) = \aleph_0$.
- The star of rays $S_\kappa \Box \mathbb{N}$ with $\deg(\epsilon) = \kappa$.



The $\{1, \dots, n\} \Box \mathbb{N}$ grid



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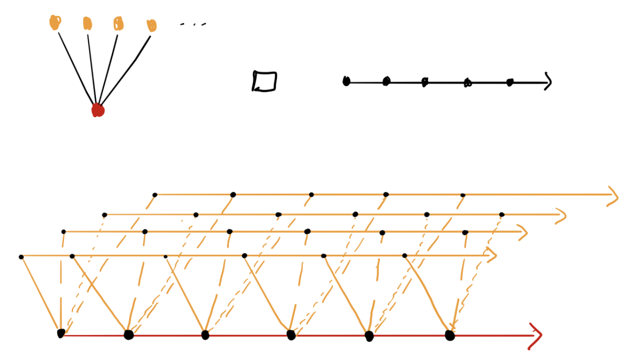

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Example.
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How typical are these examples?

Definition. Let $\mathcal{R}$ be a set of pairwise disjoint rays in an arbitrary end ϵ of G, and let $\mathcal{P}$ be a set of pairwise independent finite G such that each $P \in \mathcal{P}$ connects vertices from distinct rays in $\mathcal{R}$ and has no internal vertex in common with any ray from $\mathcal{R}$. The *ray graph* $G(\mathcal{R}, \mathcal{P})$ is the graph with vertex set $\mathcal{R}$ where two rays are adjacent if there are infinitely many disjoint $R_1 - R_2$ paths in $\mathcal{P}$.

Conjecture (Halin)**.** For any end ϵ there are $\mathcal{R} \subseteq \epsilon$ and $\mathcal{P}$ as above with $|\mathcal{R}| = \deg(\epsilon)$ such that $G(\mathcal{R}, \mathcal{P})$ is connected.

Remark.
- For $\deg(\epsilon) = \aleph_0$, this holds by Halin's grid theorem.
- For $\deg(\epsilon) = \kappa$ regular, one would find in $G(\mathcal{R}, \mathcal{P})$ a vertex of degree κ. To this vertex and its neighbours there would correspond a "central" ray R and κ neighbouring rays $(R_i \colon i < \kappa)$, all disjoint from each other, such that each R_i with R and the connecting paths from $\mathcal{P}$ forms a subdivision of the one-way infinite ladder – i.e. a subdivided $S_\kappa \square \mathbb{N}$ with some edges missing.

Theorem (Geschke, Kurkofka, Melcher, Pitz 20^+)**.** *Halin's conjecture fails for end degrees* $\deg(\epsilon) = \aleph_1$*, holds for all end degrees* $\aleph_2, \aleph_3, \ldots, \aleph_\omega$*, fails again for* $\deg(\epsilon) = \aleph_{\omega+1}$*, and is undecidable for the next* $\aleph_{\omega+n}$ *for* $n \in \mathbb{N}$*,* $n \geq 2$*.*

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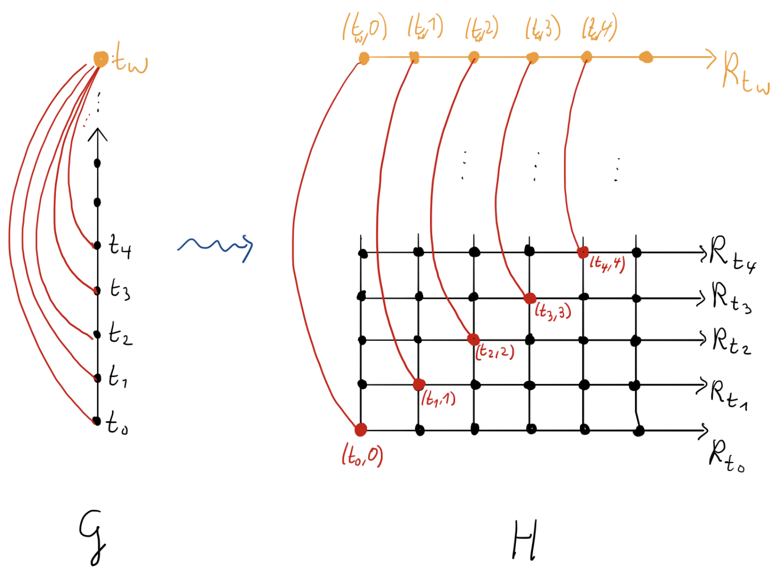
"Think" of Halin's conjecture: The 'only' way to build an end of degree κ is $T \square \mathbb{N}$ for some tree T with $|T| = \kappa$.

For our counterexamples at $\aleph_1$ and $\aleph_{\omega+1}$: A new idea to construct ends with prescribed degree based on T-graphs.

Definition. Let G be a T-graph where T be an order tree of height at most ω_1 where for every limit t, $N(t) \cap \lceil t \rceil$ has order type ω. The *ray-inflation* $G \sharp \mathbb{N}$ of G is the graph with vertex set $T \times \mathbb{N}$, and the following edges:

(1) For every $t \in T$ and $n \in \mathbb{N}$ we add the edge $(t, n)(t, n+1)$ (such that $R_t := \{t\} \times \mathbb{N}$ induces a ray).
(2) If $t \in T$ is a successor with predecessor t', we add all edges $(t, n)(t', n)$ for all $n \in \mathbb{N}$.
(3) If $t \in T$ is a limit with down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots$ in G we add the edges $(t, n)(t_n, n)$ for all $n \in \mathbb{N}$.

Example. The ray inflation of an $(\omega+1)$-graph:



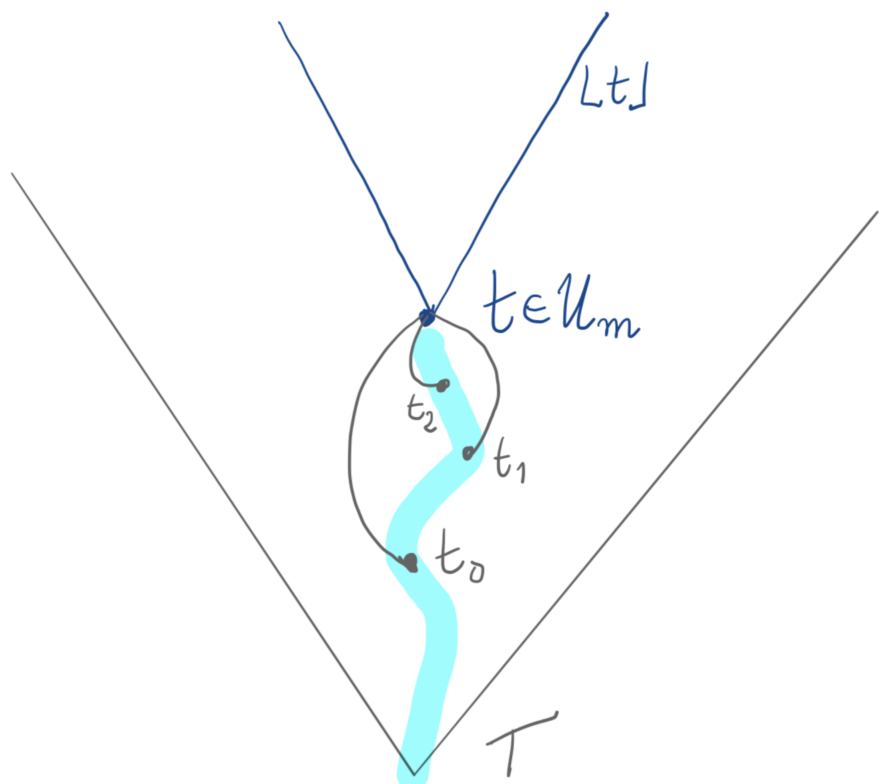
Lemma. *The ray inflation* $G \sharp \mathbb{N}$ *has one end, which has degree* $|T|$.

Theorem (GKMP 20^+)**.** *Let* T *be an Aronszajn tree and* G *a* T*-graph with property* $(\star)$. *Then* $G \sharp \mathbb{N}$ *contains no subdivided* $\aleph_1$*-star of rays; i.e. Halin's conjecture fails at* $\aleph_1$.

Theorem (GKMP 20^+)**.** *From an* $\aleph_\omega^+$*-scale on* $\Pi_{n<\omega} \aleph_n$ *one can obtain a tree* T *with* $|T^{<\omega}| = \aleph_\omega$ *plus* $\aleph_\omega^+$ *many tops, such that* $T \sharp \mathbb{N}$ *contains no subdivided* $\aleph_\omega^+$*-star of rays; i.e. Halin's conjecture fails at* $\aleph_{\omega+1}$.

Theorem (GKMP 20^+). *Let T be an Aronszajn tree and G a T-graph with property $(\star)$. Then $G \sharp \mathbb{N}$ contains no subdivided $\aleph_1$-star of rays; i.e. Halin's conjecture fails at $\aleph_1$.*

- Aronzsajn tree: $|T| = \aleph_1$, but all levels and branches countable.
- Property $(\star)$ relies on an idea of Diestel, Leader and Todorcevic: Pick a (special) Aronzsajn tree T with antichain partition $(U_n)_{n\in\mathbb{N}}$. Given a limit $t \in T$, pick down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots <_T t$ with $t_i \in U_{n_i}$ recursively such that each n_{i+1} is smallest possible.
- The resulting T-graph G has the following property $(\star)$: For each t there is a finite set $S_t \subseteq \lceil \mathring{t} \rceil$ such that *every* $s >_T t$ satisfies $N(s) \cap \lceil \mathring{t} \rceil \subseteq S_t$.



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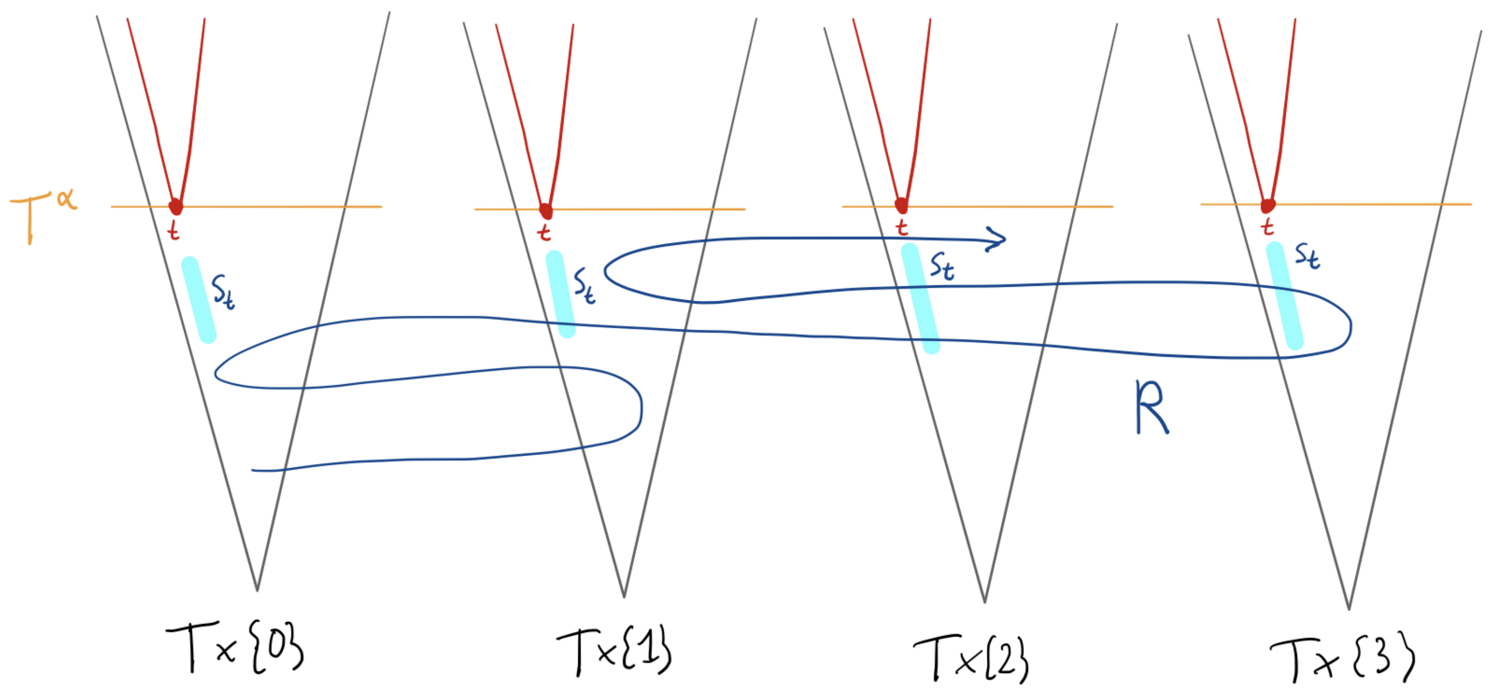
- Suppose there is a star of rays S in $G \sharp \mathbb{N}$ "central" ray R and $\aleph_1$ neighbouring rays $(R_i \colon i < \aleph_1)$. Since R is countable, there is $\alpha < \omega_1$ such that $R \subseteq T^{<\alpha} \times \mathbb{N}$, and wlog all $R_i \subseteq (T \setminus T^{<\alpha}) \times \mathbb{N}$. Components of the last graph are of the form $\lfloor t \rfloor \times \mathbb{N}$ for $t \in T^\alpha$. But now a component of $S - R$ that avoids $T^{\leq\alpha} \times \mathbb{N}$ yields a contradiction.



§3: Halin's end degree conjecture

Let $HC(\kappa)$ be the statement that Halin's conjecture holds for all ends of degree κ.

Theorem (Geschke, Kurkofka, Melcher, Pitz 20^{+})**.** *The following assertions about $HC(\kappa)$ are true:*

(1) *$HC(\aleph_1)$ fails, $HC(\aleph_n)$ holds for all $2 \leq n \leq \omega$, and $HC(\aleph_{\omega+1})$ fails again.*
(2) *More generally, $HC(\kappa)$ fails for all κ with $\mathrm{cf}(\kappa) \in \{\mu^+ : \mathrm{cf}(\mu) = \omega\}$.*
(3) *Under GCH, $HC(\kappa)$ holds for all cardinals not excluded by (2).*
(4) *However, $HC(\aleph_{\omega+\alpha+2})$ is also consistent false for every $\alpha < \omega_1$. Furthermore, $HC(\kappa)$ consistently fails for all κ with $\mathrm{cf}(\kappa)$ greater than the least fixed point of the $\aleph$ function.*

Question. Is $HC(\aleph_{\omega+\omega})$ consistently wrong?

End of talk – Thanks!