Exercise Sheet 4 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 18. May 2020)

Let $G = (V, E, \Omega)$ be a locally finite connected graph.

1. Let T be an end-faithful spanning tree of G. Is $\overline{T}$ a topological spanning tree of $|G|$?
2. Let F be a set of edges in G.
   (i) Show that F is a circuit if and only if F is not contained in the edge set of any topological spanning tree of G and is minimal with this property.
   (ii) Show that F is a finite bond if and only if F meets the edge set of every topological spanning tree of G and is minimal with this property.
3. Let X be a standard subspace of $|G|$ spanned by a set of at least two edges. Show that X is a circle if and only if, for every two distinct edges $e, e' \in E(X)$, the subspace $X \setminus \mathring{e}$ is connected but $X \setminus (\mathring{e} \cup \mathring{e}')$ is disconnected.
4. Let $X \subseteq |G|$ be a connected standard subspace that has an even number of edges (possibly none) in any finite cut of G. Show that X is the closure of the union of edge-disjoint circles.
5. Let T be a locally finite tree. Construct a continuous map $\sigma\colon [0,1] \to |T|$ that maps 0 and 1 to the root and traverses every edge exactly twice, once in each direction. (Formally: define σ so that every inner point of an edge is the image of exactly two points in $[0,1]$.)

   (Hint. Define σ as a limit of similar maps π_n for finite subtrees T_n.)

Bonus:

6. Consider the last question, but now for arbitrary locally finite graphs G: Is there a continuous map $\sigma\colon [0,1] \to |G|$ that maps 0 and 1 to the same vertex and traverses every edge exactly twice, once in each direction?

Hinweise

1. Fundamental cuts.
2. Treat the backwards implications in (i) and (ii) first. You may use the easy fact that S^1 does not properly contain another copy of S^1.
3. You may use that deleting an open interval from the unit circle leaves a connected rest, but that deleting two disjoint open intervals does not.
4. Can you find a circle in X containing your favourite edge of X?
5. For the subtrees T_n of T consisting of the first n levels of T it is easy to define such a map σ_n: just put T_n in the plane and walk 'around' it, with your feet just next to T_n and your right hand on it. To make the σ_n compatible, it will help if they 'pause' for a while at every leaf of T_n.
6.