Exercise Sheet 1 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 27. April 2020)

1. Let G be a graph, $U \subseteq V(G)$, and $R \in \omega \in \Omega(G)$. Show that G contains a comb with spine R and teeth in U if and only if $\omega \in \overline{U}$.

2. Show that for locally finite graphs G, the three topologies VTop $\subseteq$ MTop $\subseteq$ Top on $|G|$ are in fact equal.

3. Show that $|G|$ with MTop (and hence Top) is always Hausdorff; and that $|G|$ with VTop is Hausdorff if and only if no end is dominated.

4. Given graphs $H \subseteq G$, let $\eta\colon \Omega(H) \to \Omega(G)$ assign to every end of H the unique end of G containing it as a subset (of rays). For the following questions, assume that H is connected and $V(H) = V(G)$.
   (i) Show that η need not be injective. Must it be surjective?
   (ii) Investigate how η relates the subspace $\Omega(H)$ of $|H|$ to its image in $|G|$. Is η always continuous? Is it open onto its image? Do the answers to these questions change if η is known to be injective?
   (iii) A spanning tree is called *end-faithful* if η is bijective, and *topologically end-faithful* if η is a homeomorphism. Show that every normal spanning tree is topologically end-faithful.

The *end space* of a graph G is the subspace $\Omega(G)$ of $|G|$.

5. (i) Show that if $G = IH$ with finite branch sets, then the end spaces of G and H are homeomorphic.
   (ii) Let T_n denote the n-ary tree, the rooted tree in which every vertex has exactly n successors. Show that all these trees have homeomorphic end spaces.

6. Let G be a countable connected graph that is not locally finite. Show that $|G|$ is not compact, but that $\Omega(G)$ is compact if and only if for every finite set $S \subseteq V(G)$ only finitely many components of $G - S$ contain a ray.

Bonus:

7. Let G be a graph, $U \subseteq \Omega(G)$, and $\omega \in \Omega(G) \setminus U$. Can you find a characterization when $\omega \in \overline{U}$ similar to Q1?

Hinweise

1.
2. You need to show that for every Top-basic open set U and $x \in U$ there is a VTop-basic open set V with $x \in V \subset U$.
3.
4. Your answer may depend on whether H is known to be locally finite. Remember that a continuous bijection from a compact space to Hausdorff space is a homeomorphism. For (iii), remember Lemma 1.5.5 (ii).
5. For (i), define the homeomorphism by mapping rays of H to rays of G, not the other way round.
6. Adapt the proof from the lectures that $|G|$ is compact
7.