MACLANE'S THEOREM FOR GRAPH-LIKE SPACES VIA INVERSE LIMITS

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Abstract. A proof of MacLane's theorem for graph-like spaces via inverse limits.

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§1. Topological versions of Kuratowski's and MacLane's theorem

Theorem 1 (Kuratowski 1930, see [4, Theorem 4.4.6])**.** *A finite connected graph G is planar if and only if G contains no topological copy of a K_5 and $K_{3,3}$.*

Figure 1.1. The forbidden minors K_5 and $K_{3,3}$

Theorem 2 (MacLane 1937, see [4, Theorem 4.5.1])**.** *A finite connected graph G is planar if and only if its cycle space $\mathcal{C}(G)$ has a simple basis.*

Sketch. MacLane can be derived from Kuratowski as follows: By considering blocks, wlog G is 2-connected. Then every edge lies on precisely two faces, and the facial boundaries generate every cycle in G: given a cycle $C \subset G$, take the sum of all boundaries of faces "inside" C.

Conversely, one shows that if G has a simple basis, then so does every subgraph $H \subseteq G$. But TK_5 and $TK_{3,3}$ don't have simple bases by a simple counting argument, see [4, Theorem 4.5.1]. □

We now want to generalise MacLane's result to $|G|$ for locally finite connected graphs G, and even to compact graph-like metrizable spaces X:

Theorem 3 (Bruhn & Stein [1])**.** *A connected locally finite graph G (equivalently: $|G|$) is planar if and only if its topological cycle space $\mathcal{C}(G)$ has a simple basis.*

Theorem 4 (Christian, Richter & Rooney [2])**.** *A 2-connected compact metrizable graph-like space X is planar if and only if its topological cycle space $\mathcal{C}(X)$ has a simple basis.*

If X is not 2-connected, there are examples that have a simple basis but fail to be planar (see Figure 1.3)

Just like MacLane is a consequence of Kuratowski's theorem, topological MacLane is a consequence of Clayor's theorem, a deep generalisation of Kuratowksi from graphs to *Peano continua*, i.e. compact metrizable connected locally connected spaces. (Recall that graph-like continua are Peano).

Theorem 5 (Claytor 1937, see [3]). *A Peano continuum X is planar if and only if X contains no subspace homeomorphic to one of the two Kuratowski graphs K_5 and $K_{3,3}$, nor a subspace homeomorphic to the two Claytor curves K_5^∞ and $K_{3,3}^\infty$.*

FIGURE 1.2. The forbidden spaces K_5^∞ and $K_{3,3}^\infty$

FIGURE 1.3. Drawings of K_5^∞ and $K_{3,3}^\infty$ as "thumbtacks" ("Reißzwecke"), as printed in [3].

The second pair of drawings can be obtained from the first by pulling the left-upper vertex from every rectangle below the horizontal line, so that the edge to its right neighbour becomes a half-circle around the right of the figure.

Using that a Peano continuum is 2-connected if and only if any two points lie on a common simple closed curve, and that in a $|G|$ any end of degree at least 2 lies on a topological circle respectively, one readily obtains:

Corollary 6. *A 2-connected Peano continuum X is planar if and only if X contains no subspace homeomorphic to one of the two Kuratowski graphs K_5 and $K_{3,3}$.*

Proof. Exercise. □

Corollary 7. *The following are equivalent for a locally finite connected graph G:*

(1) G is planar,
(2) G contains no subdivision of K_5 and $K_{3,3}$,
(3) $|G|$ contains no subspace homeomorphic to K_5 and $K_{3,3}$, and
(4) $|G|$ is planar.

Proof. Exercise. □

Given these two corollaries, it is clear that the following result implies both Theorems 3 and 4.

Theorem 8 (Christian, Richter & Rooney [2])**.** *A connected compact metrizable graph-like space X contains no copy of K_5 or $K_{3,3}$ if and only if its topological cycle space $\mathcal{C}(X)$ has a simple basis.*

Christian, Richter & Rooney's proof in [2] uses a number of non-trivial topological lemmas. Our approach circumvents these topological results and instead relies directly on a combinatorial compactness argument. Indeed, it is clear that Theorem 8 is implied by the following lemmas.

Lemma 9. *Let X be a metrizable graph-like continuum with inverse limit representation $X = \varprojlim G_n$ with edge-contraction bonding maps. Then $\mathcal{C}(X)$ has a simple basis if and only if every $\mathcal{C}(G_n)$ has a simple basis.*

Proof. $\Rightarrow$: Let $\mathcal{B}$ be a simple basis for $\mathcal{C}(X)$.[1] Let $\pi_n \colon X \to G_n$ denote the contraction map onto the factor G_n.

Claim that $\mathcal{B}_n := \pi_n(\mathcal{B}) = \{\pi_n(C) \colon C \in \mathcal{B}\}$ is a simple basis for $\mathcal{C}(G_n)$. It is clear that every element of $\mathcal{B}_n$ is a cycle space element of G_n, and that every edge of G_n is used at most twice. Hence, it remains to show that $\mathcal{B}_n$ generates $\mathcal{C}(G_n)$. To this end, let C be an arbitrary cycle of G_n. By arc-connectedness of the fibres $\pi_n^{-1}(v)$ for $v \in V(C)$, the element C extends to a cycle $\hat{C}$ of X with $\pi_n(\hat{C}) = C$. Since $\hat{C}$ lies in the span of $\mathcal{B}$, it follows readily that C is spanned by $\mathcal{B}_n$.

$\Leftarrow$: Conversely, assume that every $\mathcal{C}(G_n)$ has a simple basis. Since every G_n is a contraction minor of G_{n+1}, it follows as above that every simple basis of $\mathcal{C}(G_{n+1})$ restricts to a simple basis $\mathcal{C}(G_n)$. Use the infinity lemma to pick a compatible sequence $\mathcal{B}_n$ of simple bases for $\mathcal{C}(G_n)$.

Claim that the collection $\mathcal{B}$ of unions of maximal chains in $(\bigcup \mathcal{B}_n, \subseteq)$ is a simple basis for $\mathcal{C}(X)$. Every element of $\mathcal{B}$ clearly projects to an element of $\mathcal{B}_n$ for each n, so meets every finite cut evenly, so is a cycle space element of X. Moreover, every edge of G_n and hence every edge of X is used at most twice. Hence, it remains to show that $\mathcal{B}$ generates $\mathcal{C}(X)$. Let $C \in \mathcal{C}(X)$ be arbitrary. Since $\mathcal{B}_n$ is a basis, there is $\mathcal{A}_n \subseteq \mathcal{B}_n$ with $\pi_n(C) = \sum \mathcal{A}_n$, and this linear combination induces on for $\mathcal{B}_{n-1}$ to generate $\pi_{n-1}(C)$. By the infinity lemma, we may select compatible linear combinations $\mathcal{A}_n$ for $n \in \mathbb{N}$. Then the collection $\mathcal{A} \subseteq \mathcal{B}$ of unions of maximal chains in $(\bigcup \mathcal{A}_n, \subseteq)$ satisfies that $C = \sum \mathcal{A}$, as both $C \subseteq \sum \mathcal{A}$ and $C \supseteq \sum \mathcal{A}$ can be checked edge-wise on all large enough G_n. Finally, this sum is automatically thin, as $\mathcal{B}$ is simple. □

Lemma 10. *Let X be a metrizable graph-like continuum with inverse limit representation $X = \varprojlim G_n$ with edge-contraction bonding maps. Then X contains no topological copy of $K_{3,3}$ or K_5 if and only if no G_n contains a subdivided $K_{3,3}$ or K_5.*

Proof. $\Rightarrow$: Let $\pi_n \colon X \to G_n$ denote the contraction map onto the factor G_n. Proving the contrapositive, assume that some G_n contains a subdivided $K_{3,3}$ or K_5 with branch vertices S say. Since $\pi_n^{-1}(v)$ are arc-connected in X for $v \in S$, it is straightforward to construct a topological copy of $K_{3,3}$ or K_5 in X by adding suitable arcs inside the fibres $\pi_n^{-1}(v)$.[2]

[1] Since $E(X)$ is countable, and every edge is contained in at most two elements of $\mathcal{B}$, also $\mathcal{B}$ is countable.

[2] This is as in Wagner's proof that the existence of a $K_{3,3}$ or K_5 minor implies the existence of a subdivided $K_{3,3}$ or K_5; note that a K_5 minor might give an inflated, so subdivided $K_{3,3}$ though. Cf. [4, Lemma 4.4.2].

$\Leftarrow$: *We prove more generally that if H is any finite graph topologically contained in X, then some G_n contains an IH.* Assume that $f\colon H \hookrightarrow X$ is the embedding. Let $V(H) = \{h_1, \dots, h_k\}$ and write $x_i = f(h_i)$. Moreover, for each $e \in E(H)$ pick an edge $e' \in E(X)$ with $e' \subset f(e)$. Write $H_i \subset X$ for the connected component of $f(H) - \{e' : e \in E(H)\}$ containing x_i.

Using the property that if A, B are disjoint closed sets of vertices of X, there is $n \in \mathbb{N}$ such that $\pi_n(A) \cap \pi_n(B) = \emptyset$ (Sheet7Q1), there is some $n \in \mathbb{N}$ such that $\pi_n(H_i) \cap \pi_n(H_j) = \emptyset$ for all $i \neq j \in [k]$.

Then $H \preccurlyeq G_n$ as witnessed by the branch sets $\pi_n(H_i)$ for $i \in [k]$ and edges $\{e' : e \in E(H)\}$.

Hence, if X contains a topological $K_{3,3}$ or K_5, then G_n contains an $IK_{3,3}$ or IK_5, but then G_n also contains a topological $K_{3,3}$ or a topological K_5 by Wagner's Lemma [4, Lemma 4.4.2]. □

Proof of Theorem 8. Let X be a compact metrizable graph-like space. Choose an inverse limit representation $X = \varprojlim G_n$ with edge-contraction bonding maps (by the main result of [5]). Then:

$$\begin{aligned} & X \text{ contains no } K_5 \text{ or } K_{3,3} \\ \Leftrightarrow\; & \text{no } G_n \text{ contains a subdivided } K_{3,3} \text{ or } K_5 \text{ (by Lemma 10)} \\ \Leftrightarrow\; & \text{every } \mathcal{C}(G_n) \text{ has a simple basis (by Theorem 2)} \\ \Leftrightarrow\; & \mathcal{C}(X) \text{ has a simple basis (by Lemma 9).} \end{aligned}$$

□

References

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