Exercise Sheet 2 for Infinite Graph Theory, WS 2019/20
(to be discussed on 28. October 2019)

1.$^-$ (i) Show that for every set X there is a least ordinal α such that $|X| = |\alpha|$.

(ii) In the lectures, I said that the concatenation of any well-ordered chain of ordinals defines another ordinal. (So ordinal sums are properly defined, even for more than two summands if these are given as a well-ordered chain.) Make this statement precise, and prove it.

2. (i) Let α be an ordinal, $A \subseteq \alpha$ a subset (with the induced ordering), and let α' be the ordinal representing the o.type of A. Show that $\alpha' \leqslant \alpha$.

(ii) Show that if $A \subsetneq \alpha$ then $\alpha' < \alpha$, or find a counterexample.

3. (i) Show that every countable ordinal α embeds into $(\mathbb{R}, \leq)$.

(ii) Find a subset of $\mathbb{R}$ isomorphic to the concatenation of $\omega + 1$ many copies of ω.

(iii) Is there an uncountable ordinal which embeds into $(\mathbb{R}, \leq)$?

4. Prove in detail that, in the second (correct) proof of the well-ordering theorem from Zorn's Lemma, the partial order $\mathcal{P}$ does indeed contain an upper bound for any chain $\mathcal{C} \subseteq \mathcal{P}$.

5. Show that the Spanning Tree Theorem implies the Axiom of Choice.

6. Show that every countable, infinitely edge-connected graph G contains infinitely many edge-disjoint spanning trees.

Optional:

7.$^+$ Show, using Zorn's Lemma or otherwise, that every infinitely edge-connected graph contains infinitely many edge-disjoint spanning trees.

8.$^+$ For every $k \in \mathbb{N}$, construct a k-connected locally finite graph such that the deletion of the edge set of any cycle disconnects that graph. Deduce that the tree-packing theorem (2.4.1) of Nash-Williams and Tutte fails for infinite graphs.

Hinweise

1.$^-$ Find a suitable **set** of ordinals, from which you are then allowed to choose a minimal element. For (ii), given a well-ordered collection of disjoint well-orders $((X_\beta, \leq_\beta) \colon \beta < \alpha)$, how should the linear order on $X = \bigcup X_\beta$ look like and why is it a well-order?

2. —

3. For (i), pick an enumeration of $\alpha = \{x_n \colon n \in \mathbb{N}\}$ and embed into $\mathbb{R}$ one element after another. For (iii), count intervals between elements x and its successor x^+.

4. Zeige (2) vor (1).

5. Given a family $\{A_i \colon i \in I\}$ of disjoint sets, we need to find a choice function f, that is a formula of the form $f(i) = a \Leftrightarrow \cdots$. Can you find one using a spanning tree T for the graph G with vertex set $\{x\} \cup \bigcup_{i \in I}(A_i \cup \{y_i, z_i\})$ and edge set $xy_i, y_i a, az_i$ for all $a \in A_i$ and all i?

6. List $\mathbb{N} \times V = \{(a_n, v_n) \colon n \in \mathbb{N}\}$ and make sure that in the nth step of the construction, the a_n's spanning tree contains the vertex v_n.

7.$^+$ Consider the poset of families $(T_1, T_2, \ldots)$ of edge-disjoint subtrees of G with $V(T_i) = V(T_j)$, ordered by the coordinate-wise subgraph relation.

8.$^+$ Fix a k-connected finite graph H of girth at least k^2 (Corollary 11.2.3), so that H contains a set of k vertices of pairwise distance at least k (why?).
Now build k-connected graphs $G_0, G_1, G_2, \ldots$ all of girth at least k as follows: Start with $G_0 = H$. If G_0 has a bad cycle C (one such that deleting $E(C)$ does not disconnect G_0), subdivide some k edges of C once, and glue several (how many?) new copies of H onto those subdividing vertices. Continue until all bad cycles of G_1 are killed, and call the resulting graph G_1. But G_1 will contain new bad cycles; continue so that after countably many steps all bad cycles are killed. For the tree-packing consequence, consider a fundamental cycle of one of the spanning trees.