Exercise Sheet 1 for Infinite Graph Theory, WS 2019/20
(to be discussed on 21. October 2019)

1. Show that every partially ordered set is isomorphic to a subset of a powerset, ordered by the subset relation.

2.$^-$ Show that under the assumptions of Zorn's lemma, one can even conclude that every element lies below some maximal element.

3.$^+$ Using Zorn's lemma, show that every partial order can be extended to a linear order.

4. Let G be a countable infinitely connected graph. Show that G has, for every $k \in \mathbb{N}$, an infinitely connected spanning subgraph of girth at least k.

5. Construct, for any given $k \in \mathbb{N}$, a planar k-connected graph. Can you construct one whose girth is also at least k? Can you construct an infinitely connected planar graph?

6. Theorem 8.1.3 implies that there exists an $\mathbb{N} \to \mathbb{N}$ function f_χ such that, for every $k \in \mathbb{N}$, every infinite graph of chromatic number at least $f_\chi(k)$ has a finite subgraph of chromatic number at least k. (E.g., let f_χ be the identity on $\mathbb{N}$.) Find similar functions f_δ and f_κ for the minimum degree and connectivity, or show that no such functions exist.

Optional:

7.$^+$ Show that if a graph contains infinitely many distinct cycles, then it contains infinitely many edge-disjoint cycles.

Hints

1. Given a partially ordered set $(X, \leq)$, find an order-preserving embedding of $(X, \leq)$ into $(\mathcal{P}(X), \subseteq)$. As always, it pays off to think about really easy (finite!) examples $(X, \leq)$ first.

2. –

3.$^+$ If one declares $x \leq y$ for previously incomparable elements, what else do we have to do in order to get something transitive? Next, if you can deal with one additional step, how does Zorn help?

4. For every countable set V, there exists a sequence of pairs $\{u, v\} \in [V]^2$ in which every such pair occurs infinitely often.

5. Construct the graph inductively, starting from a vertex or a cycle. To ensure that the final graph has high connectivity, join each new vertex by many edges to the infinite set of vertices yet to be defined.

6. Think of trees. And of the previous exercise. And recall (Ch. 7.2) that large average degree implies the existence of large complete minors.

7. –