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SEMILINEAR PARABOLIC EQUATIONS IN THE HYPERBOLIC SPACE

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Introduction

Nature uses as little as possible of anything.

KEPLER, JOHANNES
(1571-1630)

In this thesis we address two well-known problems concerning semilinear parabolic equations yet in a new framework: the hyperbolic space $\mathbb{H}^n$.
Both problems are of the type

$$\begin{cases} u_t = \Delta u + f(u) & \text{in} \quad \mathbb{H}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in} \quad \mathbb{H}^n \times \{0\} \end{cases} \tag{1}$$

considered under different assumptions on the function f.
The first problem is the blow-up of solutions while the second deals with front propagation.
The main reason of interest to address the above problems is, roughly speaking, to study how *geometry affects diffusion properties.* Expectedly, these two aspects are related by the *spectral properties of the Laplace-Beltrami operator in* $\mathbb{H}^n$. These properties (specifically the fact that the infimum of the L^2-spectrum of this operator in $\mathbb{H}^n$ is strictly positive) give rise to estimates of the heat kernel in $\mathbb{H}^n$, different from those valid in $\mathbb{R}^n$ with interesting consequences on the qualitative properties of solutions to problem (1).

Chapter 1 of the thesis is devoted to preliminaries. We review a number of models of $\mathbb{H}^n$ which play a role in our study.
In particular properties and estimates of the heat kernel in $\mathbb{H}^n$ are given (see Section (1.4)).

In Chapter 2 we study the following Cauchy problem:

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + h(t)|u|^{p-1}u & \text{in} \quad \mathbb{H}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in} \quad \mathbb{H}^n \times \{0\}. \end{cases} \tag{P1}$$

The weight h is a positive, continuous and locally integrable function in $\mathbb{R}_+$, $p > 1$ and $u_0 \geq 0$.

This problem is studied considering the *Poincaré disk* $\mathbb{D}^n$ as a model of the hyperbolic space (compare section 1.3).
The companion problem in $\mathbb{R}^n$, namely

$$\begin{cases} u_t = \Delta u + h(t)|u|^{p-1}u & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\} \end{cases} \tag{2}$$

has been widely investigated.
Let $u_0 \in L^\infty(\mathbb{R}^n)$ be a positive continuous function in $\mathbb{R}^n$ and $p > 1$.
Assuming $h = 1$, in [11] H.Fujita proved that if $1 < p \leq p^\star = 1 + \frac{2}{n}$ the problem (2) does not have any nontrivial, non-negative global solutions in $\mathbb{R}^n \times [0, \infty)$. On the other hand if $p > p^\star$ and u_0 is suitably small there exist global, positive solutions.
Fujita observed that intuitively, if p is large and the data u_0 is small, then diffusion suppresses the tendency of the solution to blow up. Therefore the "size" of the nonlinearity plays a big part in determining whether of not the blow-up occurs.
If $1 < p < 1 + \frac{2}{n}$ we say that the exponent p belongs to the "blow-up case" whereas if $p > 1 + \frac{2}{n}$ it belongs to the "global case". The exponent $p^\star := 1 + \frac{2}{n}$ is called the *Fujita exponent* and the existence of $p^\star \in (1, \infty)$ is often referred to as the *Fujita phenomenon*.
The results of Fujita stimulated a great deal of investigations. In particular, in [14] P. Meier showed that in the case $h(t) = t^q$ (for large t and $q > -1$) the Fujita exponent of (2) becomes $p^\star = 1 + \frac{2(q+1)}{n}$.
He also addressed the Dirichlet initial-boundary value problem on a *bounded* domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} u_t = \Delta u + h(t)|u|^{p-1}u & \text{in} \quad \Omega \times (0, T), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, T), \\ u = u_0 \geq 0 & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{3}$$

Here something very interesting happens: if $h(t) = e^{\beta t}$ $(\beta > 0)$ then the critical exponent becomes $p^\star = 1 + \frac{\beta}{\lambda_0}$ where λ_0 is the first eigenvalue of the Laplacian in Ω with homogeneous Dirichlet boundary conditions . On the other hand, if $h(t) = 1$ or $h(t) = t^q$ (t sufficiently large and $q > -1$), and u_0 is sufficiently small then global solutions always exist.
This means that a "very large" weight function is needed to produce the Fujita phenomenon.
Observe that the heat kernel of problem (3) behaves like $e^{-\lambda_0 t}$ for large time. Hence, a large weight function is needed to balance the effect of diffusion, which is stronger than for the Cauchy problem (2).
As proven in [15] and shown below a similar situation holds for the Cauchy problem (P1) in $\mathbb{H}^n$.

Here the role of the principal eigenvalue λ_0 of problem (3) is played by the strictly positive infimum λ_1 of the L^2-spectrum of the Laplace-Beltrami operator, namely $\lambda_1 = \frac{(n-1)^2}{4}$.
The techniques used for the proof are usual comparison and monoticity results for parabolic equations.

Chapter 3 deals with the following Cauchy problem

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + f(u) & \text{in} \quad \mathbb{H}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in} \quad \mathbb{H}^n \times \{0\}. \end{cases} \tag{P2}$$

The initial data u_0 is chosen to fulfill the following assumptions:

$$u_0 \text{ is continuous} \quad , \quad 0 \leq u_0 \leq 1 \quad \text{for any} \quad x \in \mathbb{H}^n. \tag{H}$$

The following hypotheses are made on the forcing term :

$$f \in C^1([0,1]), \quad f(0) = f(1) = 0. \tag{H_0}$$

Two type of functions f, which are suggested by some applications in population genetics, are treated:

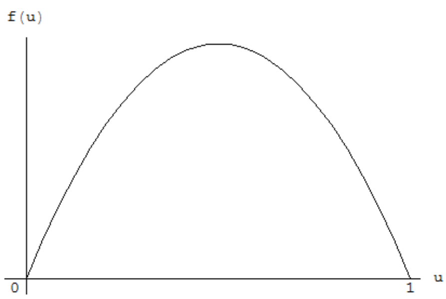
- **KPP type**[1]

$$f'(0) > 0, \quad f(u) > 0 \text{ for any } u \in (0,1), \tag{H_1}$$

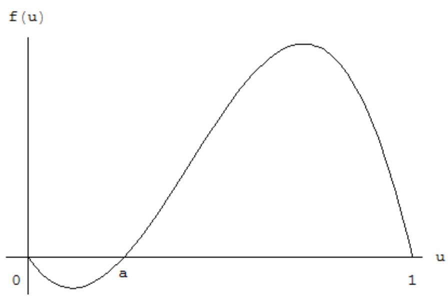
  or

- **Allen-Cahn type**

$$\begin{cases} (i) \text{ there exists } a \in (0,1) \text{ such that} \\ \qquad f(u) < 0 \text{ for any } u \in (0,a), \ f(u) > 0 \text{ for any } u \in (a,1); \\ (ii) \ f'(0) < 0, \quad \displaystyle\int_0^1 f(u)du > 0. \end{cases} \tag{H_2}$$



KPP type function



Allen-Cahn type function

[1]KPP stands for Kolmogorov,Petrovsky and Piskunov who first addressed problem (4) under these assumptions.

Under assumptions (H) and (H_0) it is known that existence and uniqueness are ensured; moreover every solution u of problem (P2) satisfies

$$0 \leq u \leq 1 \quad \text{in } \mathbb{H}^n \times \mathbb{R}_+.$$

Solution are always meant in the classical sense and we adopt the following notation: a solution $u(x,t)$ of (P2) is said to *propagate* when

$$\lim_{t\to\infty} u(x,t) = 1, \text{ uniformly on compact subsets of } \mathbb{H}^n,$$

while it is said to *get extinct* when

$$\lim_{t\to\infty} u(x,t) = 0, \text{ uniformly in } \mathbb{H}^n.$$

The companion problem of (P2) in $\mathbb{R}^n$, namely

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\} \end{cases} \tag{4}$$

was studied by D.G.Aronson and H.F Weinberger in the seminal paper [1], with f satisfying assumption (H_0).
As explained in [1], problem (4) is suggested by the following problem of population genetics.
Suppose we are given a population of diploid individuals split in three classes: the *heterozigote intermediate* (genotype Aa), the *heterozygote superior* (genotype AA) and *heterozygote inferior* (genotype aa).
The forcing term f is chosen according to the genetic model under study: the KPP-type function describes the model of *heterozygote intermediate case* whereas the Allen-Cahn-type function mathematically describes the *heterozygote inferior case.*
Assuming birth rate, death rate and diffusion coefficient to be constant in time, the paper addressed the following questions:

- How does a given initial distribution of the allele A evolve in time?
- Is the allele A wiped out, or does it persist for large time?
- If the allele A does persist, is the allele a eliminated? Otherwise, do they coexist in an equilibrium distribution?

In mathematical terms the problem is to investigate the stability of the equilibrium states $u = 0$ and $u = 1$ of problem (4).
The results established by Aronson and Weinberger can be summarized as follows.
(a) If the forcing term f is KPP type, then propagation always occurs for every solution $u \neq 0$ of problem (4). This follows from the *hair trigger effect*

which concerns the instability of the rest state $u = 0$ with respect to any nontrivial perturbation. More specifically,

(i) if there exists an $a \in (0, 1]$ such that $f(u) > 0$ for any $u \in (0, a)$

and

$$(ii) \quad \text{if } \liminf_{t \to 0^+} u^{-(1+\frac{2}{n})} f(u) > 0,$$

then for any solution $u \neq 0$ of problem (4)

$$\lim_{t \to \infty} \inf u(x, t) \geq a \quad \text{uniformly on compact subsets of } \mathbb{R}^n \tag{5}$$

[[1] Theorem 3.1 pg 41].
Clearly, hypothesis (H_1) implies (i) and (ii) with $a = 1$, thus (5) follows.
If $f(u) = O(u^p)$ as $u \to 0^+$ and if $1 < p < 1 + \frac{2}{n}$ then (ii) is satisfied while if $p > 1 + \frac{2}{n}$ then

$$\lim_{t \to \infty} u(x, t) = 0 \quad \text{uniformly in } \mathbb{R}^n$$

[see [1], Theorem 3.2].
Observe that $1 + \frac{2}{n}$ is the Fujita exponent of problem (2) with $h = 1$.
(b) If f is Allen-Cahn type then a *threshold effect* occurs. In fact extinction prevails when the initial data function u_0 is sufficiently small, while we have propagation when u_0 is large enough.[[1] Proposition 6.1 and Theorem 6.2]
(c) The existence of *plane wave solutions* of problem (4), namely solutions of the form

$$u(x, t) = q(x \cdot \nu - ct)$$

is investigated.
It is proven that for both the KPP and the Allen-Cahn case there exists an *asymptotic speed of propagation* $c^\star > 0$ which is uniquely determined by the following properties

1. no solution with compact support of problem (4) can propagate with speed greater than $c^\star$. In fact, for any $c > c^\star$ and $y \in \mathbb{R}^n$

$$\lim_{t \to \infty} \sup_{|x-y|>ct} u(x, t) = 0;$$

2. if a solution of problem (4) propagates, then its speed is no smaller than $c^\star$. In fact, if

$$\liminf_{t \to \infty} u(x, t) \geq a \quad \text{uniformly on compact subsets of } \mathbb{R}^n$$

for some $a \in (0, 1]$, then for any $c < c^\star$ and $y \in \mathbb{R}^n$

$$\liminf_{t \to \infty} \inf_{|x-y|<ct} u(x, t) \geq a.$$

Under the additional assumption

$$\sup_{u\in(0,1]} \frac{f(u)}{u} = f'(0) \tag{H_3}$$

there holds

$$c^\star = 2\sqrt{f'(0)},$$

thus the asymptotic speed of propagation $c^\star$ only depends on the forcing term f.

As proven in [17] and discussed below, the situation in $\mathbb{H}^n$ is the following.

(a') If the function f is KPP-type then, under the additional assumption (H_3) a new *threshold effect* occurs: if $c^\star = 2\sqrt{f'(0)} < n-1$ and u_0 has compact support then extinction prevails, whereas there is propagation if $c^\star > n-1$. This is the content of Theorem 3.1.

It is important to notice that under assumption (H_3) the condition $c^\star > n-1$ becomes

$$f'(0) > \frac{(n-1)^2}{4} = \lambda_1(> 0).$$

Once again, like in the case of blow-up, λ_1 plays a crucial role.

No hair-trigger effect holds in $\mathbb{H}^n$; in fact this would contrast with the fact that extinction prevails for $c^\star < n-1$ when f is KPP.

Therefore, differently from the Euclidean case, *in the hyperbolic space we can have extinction even in the KPP case*, depending on the sign of the difference $c^\star - (n-1)$.

Furthermore, if $f'(0) = 0$, at variance from the Euclidean case, we cannot have propagation. However, if $f(u) = O(u^p)$ as $u \to 0$ with $p > 1$, then extinction occurs (see Theorem 3.2).

(b') If the function f is Allen-Cahn then extinction prevails when the initial data u_0 is sufficiently small and propagation occurs when u_0 is sufficiently large and if $c^\star > n-1$. So once again the sign of the difference $c^\star - (n-1)$ is important.

(c') When propagation prevails over extinction, then *the asymptotic speed of propagation in* $\mathbb{H}^n$ *is* $c^\star - (n-1)$, differently from the case of $\mathbb{R}^n$ where it is $c^\star$.

The above differences are related to the fact that in $\mathbb{H}^n$ there is a kind of "drift from infinity" that affects the propagation of disturbances. Consider the equation

$$u_t = \Delta_M u + f(u) \quad \text{in} \quad M^n \times \mathbb{R}_+$$

and the form it takes in polar coordinates (r,θ) and (ρ,θ) respectively in $\mathbb{R}^n$ and $\mathbb{H}^n$ (see Section 1.2.1).

In $M = \mathbb{R}^n$ the equation reads

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_\theta u + f(u)$$

while in $M = \mathbb{H}^n$ it reads

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + (n-1)\coth\rho\frac{\partial u}{\partial \rho} + \frac{1}{(\sinh\rho)^2}\Delta_\theta u + f(u).$$

Now, if we compare the coefficients of the first order term in the right-hand side of the above expressions then in the first case it tends to 0 as $r \to \infty$. In the second case, instead, it tends to $(n-1)$ as $\rho \to \infty$.
This heuristically explain the presence of the term $(n-1)$ in the speed of propagation in the case of $\mathbb{H}^n$.
Therefore, in order to obtain propagation, in $\mathbb{H}^n$, the drift from infinity must be overcome. Hence the condition $c^\star > n-1$ arises.
A major tool used to prove the above results is the maximum principle in $\mathbb{R}^n$, which can be applied thanks to the ellipticity of the Laplace-Beltrami operator in the half space model and in the disk model (see (1.11)). Therefore, standard comparison principles can be used as in the case of $\mathbb{R}^n$.
In order to prove propagation in the KPP case, the condition (H_3) is instrumental to construct a suitable family of lower solutions to problem (P2).
On the other hand, to prove extinction in the Allen Cahn case, heat kernel estimates from above are used to build up a family of upper solutions of problem (P2).

Chapter 1

Riemannian Geometry, Semigroups and Heat kernel

1.1 Riemannian manifolds

Let M be a Hausdorff topological space such that any point of M admits a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.
A C^p **atlas** on a Hausdorff topological space M is given by an open cover U_i, $i \in I$, of M and a family of homeomorphisms $\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n$, such that for any $i, j \in I$ the homomorphism $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^p diffeomorphism. We call *transition functions* the maps $\{\phi_j \circ \phi_i^{-1}\}$ and *differentiable structure of class C^p on M* an equivalence class of C^p atlas. The pairs (U_i, ϕ_i) are called *charts* for M.
A **differentiable manifold** M is an Hausdorff topological space together with an atlas.
Henceforth we will deal only with C^∞ connected manifolds.
The coordinates of a point $x \in \Omega$, related to ϕ, are the coordinates of the point $\phi(x) \in \mathbb{R}^n$.
A *tangent vector* at $x \in M$ is a map $X : f \mapsto X(f) \in \mathbb{R}$ defined on the set of functions which are differentiable in a neighborhood of x , where X satisfies:

1. if $\lambda, \mu \in \mathbb{R}$ then $X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)$;

2. $X(f) = 0$ if f is constant;

3. $X(fg) = f(x)X(g) + g(x)X(f)$;

The *tangent space* T_xM at $x \in M$ is the set of tangent vectors and has a natural vector-space structure: if $\{x^i\}$ is a set of local coordinates, then a basis for T_xM is $\left\{ \frac{\partial}{\partial x_i}\Big|_x \right\}$.
The *tangent bundle* $T(M)$ is the vector bundle on M having the tangent space T_xM as its fibre over the point $x \in M$. The dual of $T(M)$ is the

cotangent bundle; its fibres are the *cotangent spaces* T_x^*M. A section of the tangent bundle is called a *vector field* over M, while a section of the cotangent bundle is called a *differential form*.
The *bracket* $[X,Y]$ of two vector fields X and Y is the vector field defined by

$$[X,Y](f) = X[Y(f)] - Y[X(f)].$$

We set $\Gamma(M)$ the space of differentiable vector fields and with $\Lambda^p(M)$ the space of differential p-forms, the latter being a section of the p-th exterior power of the cotangent bundle. In a local chart a differential p-form η may be written as

$$\eta = \sum_{1\le j_1<\cdots<j_p\le n} \alpha_{j_1,\dots,j_p} dx^{j_1}\wedge\cdots\wedge dx^{j_p}$$

where each $\alpha_{j_1,\dots,j_p}$ is a C^∞ function. Its *exterior differential* is defined as

$$d\eta = \sum_{1\le j_1<\cdots<j_p\le n} d\alpha_{j_1,\dots,j_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}$$

where if f is a C^∞ function (or equivalently a 0-form) we define

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j.$$

A **connection**[1] is a map $D\colon \Gamma(M)\times\Gamma(M)\to\Gamma(M)$ such that:

- D is bilinear;
- if f is a differentiable function and X, Y are vector fields then

$$D(X, fY) = X(f)Y + fD(X,Y).$$

For fixed $X\in\Gamma(M)$, we call

$$D_X : \Gamma(M)\to\Gamma(M), \quad Y\mapsto D(X,Y)$$

the *covariant derivative along* X.
When $X = \frac{\partial}{\partial x^i}$, we denote by $\nabla_i = \frac{\partial}{\partial x^i}$. The functions Γ_{ij}^k defined by the relation $\nabla_i\left(\frac{\partial}{\partial x^i}\right) = \Gamma_{ij}^k\frac{\partial}{\partial x^k}$ are called the *Christoffel symbols* of the connection D with respect to the local coordinates system. If a collection of functions Γ_{ij}^k is given for all i,j,k and all local charts, then these specify a unique connection D having these functions as its Christoffel symbols.

[1] The meaning of this term lies in the fact that one seeks a "connection" between different tangent spaces, which are disjoint by definition.

The **torsion** of the connection is the map $T : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ defined as

$$T(X, Y) = D_X Y - D_Y(X) - [X, Y].$$

In local coordinates this reads $T^k(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \Gamma^k_{ij} - \Gamma^k_{ji}$.
The **curvature** of the connection is the 2-form with values in $End(\Gamma(M))$ defined by

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

$R(X, Y)Z$ at x depends only upon the values of X, Y and Z in x.
In local charts, if we define

$$R^l_{kij} = R^l \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}$$

then

$$R^l_{kij} Z^k = \nabla_i \nabla_j Z^l - \nabla_j \nabla_i Z^l.$$

It follows that

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}.$$

A C^∞ **Riemannian manifold** is a pair (M, g) where M is a C^∞ differentiable manifold and $g = \{g_x\}_{x \in M}$ is a section of $T^*(M) \otimes T^*(M)$ such that at each point $x \in M$ g_x is a positive definite bilinear symmetric form; namely $g_x(X, Y) = g_x(Y, X)$ (symmetry) and $g_x(X, X) > 0$ for all $X \neq 0$ (positive definiteness). The section g is called *Riemannian metric or metric tensor.*
To every Riemannian metric is associated a matrix $g_{ij} = \langle e_i, e_j \rangle$ where $\{e_i\}_{i \in I}$ is a local frame of the tangent bundle. Then g can be written

$$g_x = \sum_{ij} g_{ij} dx^i|_x \otimes dx^j|_x \tag{1.1}$$

where every coefficient $(g)_{ij}$ is a C^∞ function of x. It is worth recalling that by Whitney's Theorem[2] we can always find a C^∞ Riemannian metric on a paracompact C^∞ differentiable manifold. As a matter of fact if M is a submanifold of $\mathbb{R}^n$ we can define a Riemmannian metric on M which is induced by the standard Euclidean metric on $\mathbb{R}^n$ (compare the following Example).
A Riemannian metric g defines at every point x and inner product on $T_x M$ and this determines the notion of angles and length between tangent vectors at $x \in M$. The norm of the vector is $\|x\| = \sqrt{g(X, X)}$ and the angle β between X and Y is uniquely determined by the formula

$$g(X, Y) = \|X\| \|Y\| \cos \beta$$

[2]Every differentiable manifold M_n has an immersion in $\mathbb{R}^{2n}$ and an embedding in $\mathbb{R}^{2n+1}$.

Furthermore we recall(see [10]) that g defines a map

$$g(x) : T_x(M) \to T_x^\star(M)$$

which is injective and also bijective. The inverse map

$$g^{-1}(x) : T_x^\star(M) \to T_x(M)$$

has components $g^{ij} = g_{ij}^{-1}$.

Example 1.1.
- The most trivial example of a Riemannian manifold is $\mathbb{R}^n$ with the canonical Euclidean metric g_0 such that $(g_0)_{ij} = \delta_{ij}$ is the identity matrix. In the standard chart of $\mathbb{R}^n$ $x^1, ..., x^n$ we can write

$$g_{\mathbb{R}^n} = (dx^1)^2 + ... + (dx^n)^2$$

- $\mathbb{S}^n = \{X \in \mathbb{R}^{n+1} | \langle X, X\rangle = \sum_i x_i^2 = 1\}$ with the metric inherited by the stereographic projection is a Riemannian manifold.

Let us recall the concept of Riemannian connection, also called *Levi Civita connection* that is a particular case of covariant derivative. Here though we require *compatibility with the Riemannian metric.*
A Riemannian connection ∇ is a connection on $T(M)$ satisfying *compatibility with the metric*, namely

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \text{for all } X, Y, Z \in \Gamma(M),$$

and which has null torsion, i.e.

$$\nabla_X Y + \nabla_Y X - [X, Y] = 0 \quad \text{for all } X, Y \in \Gamma(M).$$

Example 1.2.
- In the Euclidean space $(\mathbb{R}^n, g)$ where g is the standard metric we can set $\nabla = D$ where D is the directional derivative [3]. This means that the directional derivative is a Riemannian connection.

- In $\mathbb{R}^3$ set $\nabla_X Y := D_X Y + \frac{1}{2}(X \times Y)$ where $X \times Y$ is the usual cross product of vectors. This ∇ is not a Riemannian connection, as a matter of fact it satisfies all the properties except that of having vanishing torsion since it results

$$\nabla_X Y - \nabla_Y X = D_X Y - D_Y X + X \times Y = [X, Y] + X \times Y.$$

Theorem 1.1. *[13] On every Riemannian manifold (M, g) there exists a uniquely determined Riemannian connection ∇.*

[3] Let Y be a differentiable vector field defined on an open set of $\mathbb{R}^{n+1}$ and let X be a fixed directional vector at some fixed point x of this open set. The expression $D_X Y|_x = DY|_x(X) = \lim_{t\to 0} \frac{1}{t}(Y(x + tX) - Y(x))$ is called the *directional derivative* of Y into the direction X

In local coordinates we get that the Christoffel symbols of the Levi-Civita connection are

$$\Gamma_{ij}^m = \sum_k \Gamma_{ij,k} g^{km}$$

where

$$g^{km} = (g_{km})^{-1}$$

and

$$\Gamma_{ij,k} = \frac{1}{2}(-\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{jk}).$$

Notice that

$$\left\langle \nabla_i \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle = \Gamma_{ij,k},$$

or equivalently

$$\nabla_i \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

If $X = \sum_i \xi_i \frac{\partial}{\partial x^i}$ and $Y = \sum_j \eta_j \frac{\partial}{\partial x^j}$ then an easy computation shows that

$$\nabla_X Y = \nabla_{\sum_i \xi^i \frac{\partial}{\partial x^i}} \left(\sum_j \eta^j \frac{\partial}{\partial x^j} \right) = \sum_k \left(\sum_i \xi_i \frac{\partial \eta^k}{\partial x^i} + \sum_{ij} \Gamma_{ij}^k \xi^i \eta^j \right) \frac{\partial}{\partial x^k}$$

and for $X = \frac{\partial}{\partial x_i}$ we have

$$\nabla_X Y = \nabla_{\frac{\partial}{\partial x^i}} \left(\sum_j \eta^j \frac{\partial}{\partial x^j} \right) = \sum_k \left(\frac{\partial \eta^k}{\partial x^i} + \sum_{ij} \Gamma_{ij}^k \eta^j \right) \frac{\partial}{\partial x^k}.$$

If we consider, instead of vector fields on the manifold itself, vector fields along a curve γ, then the coordinate functions η^i are not to be viewed as functions of $x_1, \cdots, x_n$, but rather as functions of the curve parameter t. In this case, the following equation may be taken as a definition, where $\gamma_1(t), \cdots, \gamma_n(t)$ are the coordinates of γ:

$$\nabla_{\dot\gamma} Y = \sum_k \left(\frac{d\eta^k(t)}{dt} + \sum_{ij} \dot\gamma^i(t) \eta^j(t) \Gamma_{ij}^k(\gamma(t)) \frac{\partial}{\partial x^k} \right) =$$

$$= \sum_k \left(\sum_i \dot\gamma^i(t) \frac{d\eta^k(t)}{dx^i} + \sum_{ij} \dot\gamma^i(t) \eta^j(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k}$$

We say that a vector field Y is *parallel* if $\nabla_X Y = 0$ for every X.
A vector field Y along a regular curve γ is said *parallel* along the curve if $\nabla_{\dot\gamma} Y = 0$, independently from the parametrization of the curve itself.

A regular curve γ is called **geodesic** if $\nabla_{\dot\gamma}\dot\gamma = 0$, where γ is parametrized by arc length.
The equation that $Y(t) = \sum_j \eta^j(t)\frac{\partial}{\partial x^j}$ satisfies if it is parallel along γ is

$$\frac{d\eta^k}{dt} + \sum_{ij} \dot x^i(t)\cdot \eta^j(t)\cdot \Gamma^k_{ij}(\gamma(t)) = 0 \quad k = 1, \cdots, n.$$

On the other hand the system of equations which γ satisfies if it is a geodesic is

$$\frac{d^2x^k}{dt^2} + \sum_{ij} \dot x^i(t)\cdot \dot x^j(t)\cdot \Gamma^k_{ij}(\gamma(t)) = 0 \quad k = 1, \cdots, n. \tag{1.2}$$

A Riemannian manifold M is said to be *(geodesically) complete*, if every geodesic which is parametrized by arc length is defined on all of $\mathbb{R}$ as a map $\gamma : \mathbb{R} \to M$.

Example 1.3.

- In $\mathbb{R}^n$ the geodesics are the straight lines parametrized with constant velocity;

$$D_{\dot\gamma}\dot\gamma = 0 \iff \ddot\gamma = 0 \iff \gamma(t) = x_0 + tv$$

- In $\mathbb{S}^2$ the circle passing through the North and the South pole is a geodesic; then acting with the group of isometries of $\mathbb{S}^2$, $SO(3)$, we have that all the maximal circles are geodesics. This can be easily generalized to $\mathbb{S}^n$.

We have already defined the *curvature tensor* associated to the connection ∇ as a map $R^\nabla = R : \Gamma(TM)\times\Gamma(TM)\times\Gamma(E) \to \Gamma(E) : R_{XY} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. The curvature tensor gives a measure how much the manifold differs from T_xM in a neighbor of x.
The tensor R has the following properties: it is C^∞ linear in X, Y and C^∞ linear in s , namely $R(X,Y)(fs) = fR(X,Y)(s)$; $R(X,Y)(s) = -R(Y,X)(s)$; $R(fX,Y)(s) = fR(X,Y)(s)$.
If $E = TM$ then we have some more properties:

- if $T^\nabla = 0$ then $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ (Bianchi's identity),
- if $\nabla = \nabla^{LC}$ then $g(R(X,Y)Z,T) = -g(R(X,Y)T,Z)$.

With respect to a given Riemannian metric $\langle,\rangle$ [4], the *standard curvature tensor* R_1 is defined by the relation

$$R_1(X,Y)Z := \langle Y,Z\rangle X - \langle X,Z\rangle Y.$$

[4]
$$\langle X,Y\rangle = \sum_{i,j=1}^n g_{ij}X^iY^j.$$

We then can set

$$\kappa_1(X,Y) := \langle R_1(X,Y)Y, X\rangle = \langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2 ,$$

$$\kappa(X,Y) := \langle R(X,Y)Y, X\rangle .$$

Let $\sigma \subset T_xM$ be a 2-dimensional subspace, spanned by X, Y. Then the quantity

$$K_\sigma = \frac{\kappa(X,Y)}{\kappa_1(X,Y)}$$

is called the *sectional curvature* of the Riemannian manifold with respect to the plane σ.
It is interesting to notice that the knowledge of the sectional curvature and of the metric allow to reconstruct the curvature tensor.

Theorem 1.2. *Any two Riemannian metrics with the same constant sectional curvature (and the same dimension) are locally isometric to one another.*

If M is a surface in $\mathbb{R}^3$ then K_σ coincides with the *Gaussian curvature* [13]sec.4E.
Fixing a mobile normal unite vector N then we define the symmetric *Weingarten operator* $LX = -D_X N$ then the second fundamental form $II(X) = \langle LX, X\rangle$ and eventually if we call $I(X) = ||X||^2$ then $K_N(X) = \frac{II(X)}{||X||^2} = \frac{II(X)}{I(X)}$ is the *normal curvature.*
If on a Riemannian manifold K is a constant or, equivalently, if $R = KR_1$ where R_1 denotes the curvature of the unit space and K is a constant, the manifold is called a *space of constant curvature.*
We conclude this section recalling that the Ricci Tensor of the metric g is defined by

$$Ric(X,Y) := \sum_{i=1}^{n} \langle R(E_i, X)Y, E_i\rangle$$

where $\{E_i\}_{i=1,\dots,n}$ is any orthonormal frame for g. The Ricci tensor is symmetric, and we can also define the scalar curvature as

$$S = \sum_{j=1}^{n} Ric(E_j, E_j) = \sum_{i,j=1}^{n} \langle R(E_i, E_j)E_j, E_i\rangle .$$

A Riemannian manifold (M, g) is called an *Einstein space* if the Ricci Tensor is a multiple of g, namely

$$Ric(X,Y) = \lambda g(X,Y)$$

for all X, Y, where λ is a function $\lambda : M \to \mathbb{R}$. In this case we say that g is an *Einstein metric.* The expression

$$ric(X) = \frac{Ric(X,X)}{g(X,X)}$$

is called the **Ricci curvature** in the direction X.

Example 1.4.
- The curvature tensor $R(X,Y)Z$ of $\mathbb{R}^n$ vanishes identically. Metrics for which this holds are called *flat.*
- For a sphere of radius r $S^n(r) = \{X \in \mathbb{R}^{n+1} | \sum_i x_i^2 = r^2\}$ the curvature tensor is $R = \frac{1}{r^2}R_1$ and $K_\sigma = \frac{1}{r^2}$ for every plane $\sigma \subset T_x\mathbb{S}^n$.

1.2 Laplace-Beltrami operator on manifolds

Let us recall that any Riemannian manifold M features a canonical measure V, defined on the σ-algebra of all measurable sets in M, which we denote by $\Delta(M)$. This canonical measure is called the *Riemannian measure (or volume)* and it is defined by the following Theorem.

Theorem 1.3. *For any Riemannian manifold M, there exists a unique measure V on $\Delta(M)$ such that, in any chart U,*

$$dV = \sqrt{\det g} d\lambda,$$

where $g = (g_{ij})_{ij}$ is the matrix of the Riemannian metric g in U, and λ is the Lebesgue measure in U.

Let us record the following simple property of the measure V, which will be used in the next Theorem.

Lemma 1.1. *If $f \in C(M)$ and*

$$\int_M f\phi dV = 0$$

for all $\phi \in C_0^\infty$, then $f = 0$.

For any smooth f on M we define its gradient $\nabla f(x)$ at a point $x \in M$ as follows

$$\nabla f(x) = g^{-1}(x)df(x).$$

It is easy to check that the gradient in local coordinates $x^1, \cdots, x^n$ has the form

$$(\nabla f)^i = g^{ij}\frac{\partial f}{\partial x^j}.$$

For any smooth vector field X on a Riemannian manifold M its divergence $\operatorname{div} X$ is a smooth function on M defined by means of the following statement

Theorem 1.4. *For any C^∞ vector field X on a Riemannian manifold M, there exists a unique smooth function on M, denoted by* $\operatorname{div} X$*, such that the following identity holds*

$$\int_M (\operatorname{div} X) u dV = -\int_M \langle X, \nabla u \rangle_g dV \tag{1.3}$$

for all $u \in C_0^\infty$.

Proof. **Uniqueness:** if $(\operatorname{div} X)'$ and $(\operatorname{div} X)''$ are two candidates then , for all $u \in C_0^\infty$ we have

$$\int_M (\operatorname{div} X)' u dV = \int_M (\operatorname{div} X)'' u dV.$$

By Lemma 1.1 we conclude that $(\operatorname{div} X)' = (\operatorname{div} X)''$.
Existence: Firstly, we show that $\operatorname{div} X$ exists in any chart.
If U is a chart of M with coordinates $x^1, \cdots, x^n$, $X = X^i \partial_i \in TM$ [5] and $u \in C_0^\infty(U)$ then we have

$$\begin{aligned}
\langle X, \partial u \rangle &= \int_M \langle X, \nabla u \rangle dx^1 \cdots dx^n = \\
&= \int_M \left\langle X^i \partial_i, g^{kj} \partial_k u \partial_j \right\rangle dx^1 \cdots dx^n = \\
&= \int_U X^i (\partial_k u) \left\langle \partial_i, g^{kj} \partial_j \right\rangle dx^1 \cdots dx^n = \\
&= \int_U X^i (\partial_k u) g^{kj} \langle \partial_i, \partial_j \rangle dx^1 \cdots dx^n = \\
&= \int_U X^i (\partial_k u) g^{kj} g_{ij} \sqrt{\det g} \langle \partial_i, \partial_j \rangle dx^1 \cdots dx^n = \\
&= \int_U \frac{1}{\sqrt{\det g}} u \cdot \partial_i \left(X^i \sqrt{\det g} \right) \sqrt{\det g} dx^1 \cdots dx^n = \\
&= \left\langle u, -\frac{1}{\sqrt{\det g}} \partial_i \left(X^i \sqrt{\det g} \right) \right\rangle.
\end{aligned}$$

Comparing with (1.3) we see that the divergence in U can be defined as

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \partial_j (X^i \sqrt{\det g}). \tag{1.4}$$

Now, if U and V are two charts then (1.4) defines the divergence in U and in V, which agree in $U \cap V$ by the uniqueness statement. Hence, (1.4) defines $\operatorname{div} X$ as a function on the entire manifold M satisfying (1.3) for all test functions u compactly supported in one of the charts. In order to

[5] $\partial_i = \frac{\partial}{\partial x^i}$

extend (1.3) to all functions $u \in C_0^\infty(M)$ we consider a family of charts $\{\Omega_\alpha\}$ covering M. It can be shown (see [10], Corollary 3.6 p.52) that $u \in C_0^\infty$ can be represented as a sum $u_1 + \cdots + u_k$, where each u_i is compactly supported in some Ω_α. So (1.3) holds for each u_i and thus, adding up all such identities, we obtain that $u \in C_0^\infty$ satisfies (1.3). □

It follows by (1.4) that

$$\operatorname{div} X = \frac{\partial X^k}{\partial x^k} + X^k \frac{\partial}{\partial x^k} \log \sqrt{\det g}.$$

In particular, if $\det g = 1$ then we obtain the same formula as in $\mathbb{R}^n$:

$$\operatorname{div} X = \frac{\partial X^k}{\partial x^k}.$$

Corollary 1.1. *The identity* (1.3) *holds also if u is any smooth function on M and X is a compactly supported smooth vector field on M.*

Proof. Let $K = \operatorname{supp} X$. There exists a cutoff function on K (see [10], Theorem 3.5,pg 51), that is, a function $\phi \in C_0^\infty(M)$ such that $\phi = 1$ in a neighborhood of K. Then $u\phi \in C_0^\infty(M)$ and applying Theorem (1.4) we have

$$\int_M \operatorname{div} X u dV = \int_M \operatorname{div} X(u\phi) dV = -\int_M \langle X, \nabla(u\phi)\rangle\, dV = -\int_M \langle X, \nabla u\rangle\, dV$$

□

We can now introduce the *Laplace-Beltrami operator* on any Riemannian manifold M. It is defined as follows

$$\Delta = \operatorname{div} \circ \nabla. \tag{1.5}$$

For any smooth function f on M

$$\Delta f = \operatorname{div}(\nabla f); \tag{1.6}$$

therefore Δf is a smooth function on M.
In local coordinates the Laplace-Beltrami operator reads

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \frac{\partial}{\partial x^i} \left(\sqrt{\det g} \sum_{j=1}^n g^{ij} \frac{\partial u}{\partial x^j} \right).$$

Lastly, it is important to recall the Green formula

Theorem 1.5. *If u and v are smooth functions on a Riemannian manifold M and one of them has a compact support then*

$$\int_M u \Delta v dV = -\int_M \langle \nabla u, \nabla v\rangle\, dV = \int_M v \Delta u dV.$$

Proof. Consider the vector field ∇v. Since $\operatorname{supp}\nabla v \subset \operatorname{supp} v$ then either $\operatorname{supp}\nabla v$ or $\operatorname{supp} u$ is compact. Thus, by definition (1.5) and Corollary (1.1) we have

$$\int_M u\Delta v dV = \int_M u\,\mathrm{div}(\nabla v) dV = -\int_M \langle \nabla u, \nabla v\rangle\, dV.$$

□

1.2.1 Form of the Laplace-Beltrami operator on model manifolds

Firstly, we want to define *polar coordinates* on a Riemannian manifold .
Let x be a point on M and denote by γ_v^x the uniquely determined geodesic, parametrized by arc length t, passing through x and having the unit vector v as its tangent vector at time $t = 0$. It may be proved that for a suitable neighborhood U of the origin $0 \in T_x(M)$ there is a well-defined map

$$\begin{aligned}\exp_x :&U \subseteq T_x(M) \to M\\ &tv \mapsto \gamma_v^x(t).\end{aligned}$$

Note that, by definition, $\exp_x(0) = x$. The above map is called the *exponential mapping* and it is a diffeomorphism on its image.
One of its main properties is stated in the next Lemma (see [13][Lemma 7.13]).

Lemma 1.2 (Gauss). *Let $\exp_x : U \to \exp_x(U)$ be a diffeomorphism. Let $w \in T_x(M)$ be an arbitrary vector which is orthogonal to the line $t \mapsto tv$ in some fixed direction $v \in T_x(M)$, $\|v\| = 1$. Then $d\exp_x(w)$ is orthogonal to the geodesic γ_v^x.*

If we introduce *polar coordinates* on $T_x(M)$[6] then, under the exponential mapping $\exp_x$, these yield coordinates on M around x. These are called *geodesic polar coordinates* on M and we denote them with $r, \theta_1, \theta_2, \cdots$.
In this coordinates we have

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1$$

and

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_j} \right\rangle = 0$$

whence

$$g_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & * & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ 0 & * & \cdots & * \end{pmatrix}$$

[6] namely, the usual polar coordinates on $\mathbb{R}^n$.

where the submatrix, denoted by *'s, is of order r^2 for $r \to 0$.
For $n = 2$ we thus have

$$g_{ij}(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \eta(r, \theta) \end{pmatrix}$$

where η is a bounded, positive function on $\mathbb{R}_+$.
In polar coordinates r denotes the distance of a point from the origin and there are $n-1$ coordinates which are orthogonal to this.
In polar coordinates we have the following metric representation:

- $g_{\mathbb{R}^n} = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$ where the parameter r runs through the interval $(0, \infty)$. In this case polar coordinates are not well defined at $r = 0$.
- $g_{\mathbb{S}^n} = dr^2 + \sin^2 r g_{\mathbb{S}^{n-1}}$ where the parameter r runs through the interval $(0, \pi)$. In this case polar coordinates are not well defined in $r = 0$ (north pole) and in $r = \pi$ (south pole).

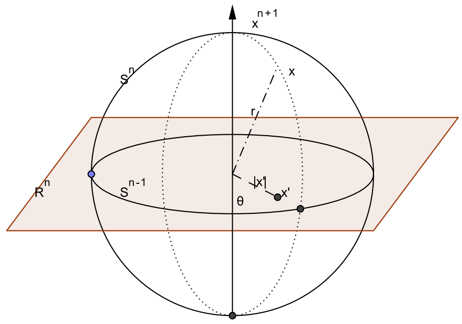


Figure 1.1: Polar coordinates on $\mathbb{S}^n$

An n-dimensional Riemannian manifold (M, g) is called a *Riemannian model* if the following two conditions are satisfied:

(i) There is a chart on M that covers all M and the image of this chart in $\mathbb{R}^n$ is a ball $B_{r_0} = \{x \in \mathbb{R}^n : |x| < r_0\}$ of radius $r_0 \in [0, \infty)$ (if $r_0 = \infty$ then $B_{r_0} = \mathbb{R}^n$).

(ii) The metric g in polar coordinates (r, θ) in the above chart has the form

$$g = dr^2 + \psi^2(r) g_{\mathbb{S}^{n-1}},$$

where $\psi(r)$ is a smooth positive function on $(0, r_0)$ [7] such that $\psi(0) = 0$ and $\psi'(0) = 1$.

[7] r_0 is called the *radius of the model M*

Example 1.5. (a) $\mathbb{R}^n$ is a model with radius $r_0 = \infty$ and $\psi(r) = r$,

(b) $\mathbb{S}^n$ without the pole is a model with radius $r_0 = \pi$ and $\psi(r) = \sin r$.

On a Riemannian manifold (M, g) with metric $g = dr^2 + \psi^2(r) g_{\mathbb{S}^{n-1}}$ the Riemannian measure $d\nu = \sqrt{\det g} d\lambda$ is given in polar coordinates by

$$d\nu = \psi(r)^{n-1} dr d\theta, \tag{1.7}$$

where $d\theta$ stands for the Riemannian measure on $\mathbb{S}^{n-1}$ and the Laplace operator on (M, g) has the form

$$\boxed{\Delta = \tfrac{\partial^2}{\partial r^2} + \left(\tfrac{d}{dr} \log \psi^{n-1}\right) \tfrac{\partial}{\partial r} + \tfrac{1}{\psi^2(r)} \Delta_{\mathbb{S}^{n-1}}.}$$

Example 1.6. In $\mathbb{R}^n$ we have $\psi(r) = r$ thus $d\nu = r^{n-1} dr d\theta$ and the Laplacian takes the form

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \left(\frac{n-1}{r}\right) \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}}.$$

In $\mathbb{S}^n$ we have $\psi(r) = \sin r$ thus $d\nu = \sin^{n-1} r dr d\theta$, and the Laplacian takes the form

$$\Delta_{\mathbb{S}^n} = \frac{\partial^2}{\partial r^2} + (n-1) \cot r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \Delta_{\mathbb{S}^{n-1}}.$$

1.3 The hyperbolic space

The hyperbolic space, denoted by $\mathbb{H}^n$, is the unique, simply connected, noncompact $n-$dimensional Riemannian manifold with sectional curvature -1. Like $\mathbb{R}^n$ and $\mathbb{S}^n$ also $\mathbb{H}^n$ is a *Riemannian model* with the radius $r_0 = \infty$ and $\psi(r) = \sinh r$. The hyperbolic metric in polar coordinate has the following form

$$g_{\mathbb{H}^n} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where the variable r runs through the interval $(0, \infty)$.
Although being diffeomorphic to $\mathbb{R}^n$, the hyperbolic space has very different properties. We denote the *distance* between two point x, y in $\mathbb{H}^n$ by $d(x, y)$ and by $\partial \mathbb{H}^n$ *the boundary of the Hyperbolic space*, whose points can be regarded as the points at infinity of $\mathbb{H}^n$. In $\mathbb{H}^n$ the geodesics are defined on the whole real line and there exists exactly one geodesic passing through any two of its points.
Thanks to the notion of geodesics in $\mathbb{H}^n$ we can give other definitions that will be fundamental for the development of future arguments.
A subset $A \subset \mathbb{H}^n$ is a *hyperbolic subspace* if it contains the entire geodesic passing through any two of its points. A hyperbolic hyperplane is a hyperbolic subspace of codimension 1.

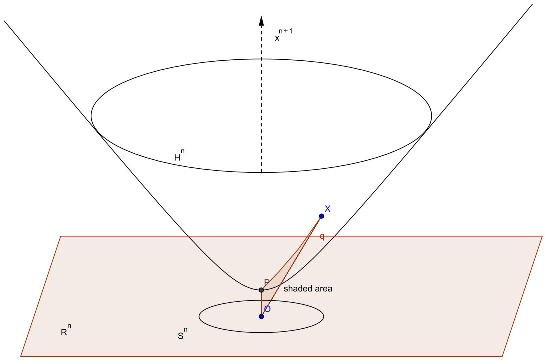


Figure 1.2: Polar coordinates on $\mathbb{H}^n$

Firstly we observe that for any $x \in \mathbb{H}^n$ there exists a $\tilde{x} \in \pi$ that realizes the minimal distance, namely $d(x, z) = \min_{y \in \pi} d(x, y)$.
A subset $K \in \mathbb{H}^n$ is said to be *convex* if for any $x, y \in K$ the geodesic arc joining x to y lies in K.
A *convex hull* of $A \in \mathbb{H}^n$ is the smallest convex set of $\mathbb{H}^n$ containing A.
A subset $\pi \subset \mathbb{D}^n$ is an *hyperbolic hyperplane* if and only if it is the intersection of $\mathbb{D}^n$ either with a hyperplane of $\mathbb{R}^n$ or with an $(n-1)$-dimensional sphere orthogonal to $\partial\mathbb{D}^n$.
Moreover let $\pi \subset \mathbb{H}^n$ be an hyperplane and $\gamma : \mathbb{R} \to \mathbb{H}^n$ be the unique entire geodesic joining x and $\tilde{x}$ such that $\gamma(0) = \tilde{x}$, $\gamma(t_0) = x$ for some $t_0 > 0$, then the *reflection through a hyperplane* $\pi \subset \mathbb{H}^n$ is a function

$$R_\pi : \mathbb{H}^n \to \mathbb{H}^n \quad \text{such that} \quad R_\pi(x) := \gamma(-t_0) \;\; \text{with} \quad x \in \mathbb{H}^n \tag{1.8}$$

where $g = \det(g_{ij})$, $g^{ij} := (g_{ij})^{-1}$.
The Hyperbolic space can be represented by using different models, all isometrically diffeomorphic to each other.

The hyperboloid model
In $\mathbb{R}^{n+1}$ we can consider the following bilinear form:

$$q(x, y) = \sum_{i=1}^{n} x_i y_i - x_0 y_0.$$

Then

$$I_n = \{x = (x_0, ..., x_n) \in \mathbb{R}^{n+1} : q(x, x) = -1 \text{ with } x_0 > 0\}$$

is the upper fold of a two-sheeted hyperboloid (which can formally be interpreted as a sphere with imaginary radius i in $\mathbb{R}^{n+1}$).
Since it is the pre-image of a regular value of a differentiable function, by Dini's Theorem we can state that I_n is a differentiable oriented hypersurface

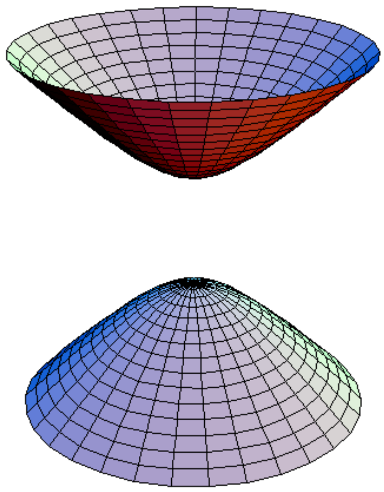


Figure 1.3: The Hyperboloid model

in $\mathbb{R}^{n+1}$ and, in particular, it is endowed with a differentiable structure which makes it a manifold of dimension n. For each point x in this upper fold there is a naturally defined metric[8] on the tangent space to each point of I_n

$$T_x I_n = \{y \in \mathbb{R}^{n+1} : q(x,y) = 0\} = \{x\}^{\perp}.$$

It is not difficult to verify that this metric is globally differentiable and therefore I_n is endowed with a Riemannian structure. We will call $\mathbb{I}^n$ the manifold I_n endowed with this structure.
In the Hyperboloid model $\mathbb{I}^n$ of $\mathbb{H}^n$ the general geodesic starting from $x \in \mathbb{I}^n$ with tangent vector y $(\in T_x\mathbb{I}^n)$ is of the form

$$\gamma(t) = xcosh(t) + ysinh(t) \text{ with } t \in \mathbb{R}.$$

The disk model
Let $\pi(x_1, \ldots, x_n) = \frac{(x_1, \cdots, x_n)}{1+x_{n+1}}$ be the restriction to $\mathbb{I}^n$ of the stereographic projection with respect to $(0, 0, \cdots, 0, -1)$ of $\{x \in \mathbb{R}^n : x_0 > 0\}$ onto $\mathbb{R}^n \times \{0\}$.
The map π is a bijection of $\mathbb{I}^n$ onto the Euclidean unit ball $B_1 = \{x \in \mathbb{R}^n ||x| < 1\}$ [9] of $\mathbb{R}^n$.
In Cartesian coordinates $x^1, \cdots, x^n$ in B_1, the canonical hyperbolic metric has the form

$$g_{\mathbb{H}^n} = \frac{4}{(1-|x|^2)^2} g_{\mathbb{R}^n}$$

[8] Since $q(x,x) = -1$, the restriction of $q(\cdot,\cdot)$ is a scalar product on $\{x\}^{\perp}$

[9] $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$

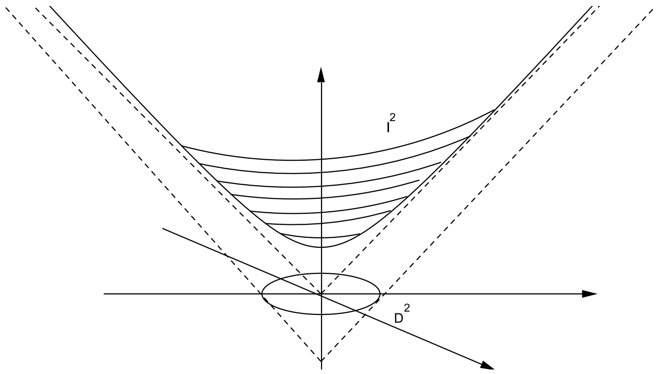


Figure 1.4: Two dimensional model of the hyperbolic space: the hyperboloid and its projection into the disk

where $|x|^2 = \sum_i (x^i)^2$ and $g_{\mathbb{R}^n} = (dx^1)^2+(dx^2)^2+\cdots+(dx^n)^2$ is the canonical Euclidean metric that we can also express as

$$ds = \frac{2}{1-|x|^2}|dx| =: \psi(r)|dx|,$$

where $r = |x|$.
The ball B_1 with this metric is called the *disk model* of the hyperbolic space and is denoted by $\mathbb{D}^n$.
We denote by $d(x,y)$ the hyperbolic distance between two points x and y in B_1. The hyperbolic distance between $x \in \mathbb{H}^n$ and the origin $O \in \mathbb{H}^n$ in the geodesic coordinates (ρ, θ) can be written as

$$\rho(x) = \int_0^r \frac{2}{1-s^2} ds = \log\left(\frac{1+r}{1-r}\right).$$

We have

$$\frac{\rho}{2} = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right) = \log\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}$$

and

$$-\frac{\rho}{2} = -\frac{1}{2}\log\frac{1+r}{1-r} = \log\left(\frac{1+r}{1-r}\right)^{-\frac{1}{2}}.$$

Thus,

$$e^{\pm\frac{\rho}{2}} = \left(\frac{1+r}{1-r}\right)^{\pm\frac{1}{2}}$$

and

$$\tanh\frac{\rho}{2} = \frac{e^{\frac{\rho}{2}} - e^{-\frac{\rho}{2}}}{e^{\frac{\rho}{2}} + e^{-\frac{\rho}{2}}} = \frac{\frac{1+r-1+r}{1-r}}{\frac{1+r+1-r}{1-r}} = \frac{2r}{2} = r.$$

Moreover

$$\psi(r) := \frac{2}{1-r^2} = \frac{2}{1-(\tanh\frac{\rho}{2})^2} = \frac{2\cosh^2\frac{\rho}{2}}{1} = 2\left(\cosh\frac{\rho}{2}\right)^2$$

Recalling (see (1.7)) that the volume element of $\mathbb{H}^n$ has the following form:

$$dV = [\psi(r)]^n dx = [\psi(r)]^n r^{n-1} dr d\theta.$$

then we can compute the term $[\psi(r)]^n r^{n-1}$ in geodesic polar coordinates. We have

$$\begin{aligned}
[\psi(r)]^n r^{n-1} dr d\theta &= 2\cosh\left(\frac{\rho}{2}\right)^2 (\sinh\rho)^{n-1} \frac{d}{d\rho}\left(\frac{\sinh\frac{\rho}{2}}{\cosh\frac{\rho}{2}}\right) d\rho d\theta = \\
&= 2\cosh\left(\frac{\rho}{2}\right)^2 (\sinh\rho)^{n-1}\left(\frac{\cosh^2\frac{\rho}{2}-\sinh^2\frac{\rho}{2}}{\cosh^2\frac{\rho}{2}}\right) d\rho d\theta = \\
&= 2\cosh\left(\frac{\rho}{2}\right)^2 (\sinh\rho)^{n-1}\left(\frac{1}{\cosh^2\frac{\rho}{2}}\right) d\rho d\theta = 2\sinh^{n-1}\rho
\end{aligned}$$

If we define the ball of radius $r > 0$

$$\mathcal{B}_r = \{x \in \mathbb{H}^n | \rho(x) < r\} \tag{1.9}$$

in $\mathbb{D}^n$ then for any $r \in (0,1)$ the relation

$$B_r = \mathcal{B}_{\log(\frac{1+r}{1-r})}$$

holds.
It is important for us to recall (compare Lamma((2.1))) that in $\mathbb{H}^n$ there is an isometry $\tau_x : B_1 \to B_1$ of the form

$$\tau_y(x) = \frac{(1-|y|^2)(x-y) - |x-y|^2 y}{\left(1+|x|^2|y|^2 - 2\langle x, y\rangle\right)^2} \tag{1.10}$$

with $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$.
It can be proved that $\tau_{-y} = \tau_y^{-1}$; moreover,

$$|\tau_y(x)| = \frac{|x-y|}{(1+|x|^2|y|^2 - 2\langle x, y\rangle)} \quad (x, y \in \mathbb{D}^n).$$

For any $x, y \in \mathbb{D}^n$ we have

$$d(x,y) = d(\tau_x(x), \tau_x(y)) = d(0, \tau_x(y)) = \log\frac{1+|\tau_x(y)|}{1-|\tau_x(y)|}.$$

The Laplace-Beltrami operator in $\mathbb{D}^n$ reads

$$\Delta_{\mathbb{H}^n} u = \frac{1}{4}(1-|x|^2)^2 \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + \frac{n-2}{2}(1-|x|^2)\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i} \tag{1.11}$$

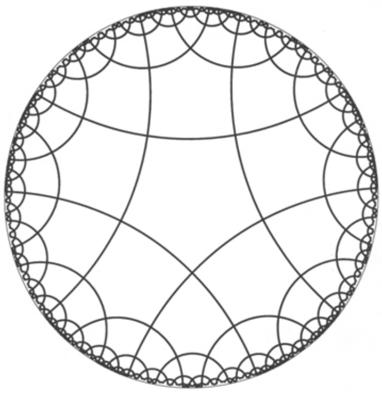


Figure 1.5: The disk model

and it can be regarded as a linear *elliptic operator* [10]on B_1 with bounded coefficient *degenerating* at the boundary in ∂B_1.
In polar coordinates (ρ, θ) [11] the Laplace-Beltrami operator in $\mathbb{D}^n$ takes the form

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}. \tag{1.12}$$

denoting with Δ_θ the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$.
Geodesics in $\mathbb{D}^n$ are either diameters of $\mathbb{D}^n$ or circles orthogonal to $\partial\mathbb{D}^n$.

The half space model

Consider the half plane $\{(x,y)|y>0\} \subset \mathbb{R}^2$ with the metric $g = \frac{1}{y^2}(dx^2+dy^2)$ ($g_{ij}(x,y) = \frac{1}{y^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$). This is a Riemannian manifold called the *half space* model of $\mathbb{H}^2$ and we denote it by $\mathbb{U}^2$. In this metric the length is given by $\frac{1}{y}$ and there holds

$$\int_\eta^1 \frac{1}{t}dt = -\log(\eta) \xrightarrow[\eta \to 0]{} \infty$$

[10]The operator

$$\mathcal{H} = \sum_{i,j=1}^n a_{ij}(x_1 \cdots x_n)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}$$

with $a_{ij} = a_{ji}$ is called *elliptic at a point* $x = (x_1, \cdots, x_n)$ if there exists a positive quantity μ such that

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq \mu(x)\sum_{i=1}^n \xi_i^2$$

for all n-tuples of real numbers $(\xi_1, \xi_2, \cdots, \xi_n)$. The operator $\mathcal{H}$ is said to be *elliptic* in a domain D if it is elliptic at each point of D. In our case the operator $\Delta_{\mathbb{H}^n}$ is elliptic with $\mu(x) = \frac{1}{4}(1-|x|^2)^2$.

[11]Here we have $\psi(r) = \sinh r$ and thus $d\nu = \sinh^{n-1} r dr d\theta$ (compare example 1.6).

$$\int_1^\eta \frac{1}{t}dt = \log(\eta) \xrightarrow[\eta\to\infty]{} \infty$$

Bearing in mind the equation of geodesics (1.2), we can compute the geodesics equation with respect to the Levi Civita connection associated to the metric g of the half space model:

$$g_{11} = g_{22} = \frac{1}{y^2}, \quad g_{12} = g_{21} = 0 \quad \text{then} \quad g^{11} = g^{22} = y^2, \quad g^{12} = g^{21} = 0,$$

thus the Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2}g^{22}\left(\partial_1 g^{21} + \partial_1 g^{12} - \partial_2 g^{11}\right) = -\frac{1}{2}y^2(-\frac{2}{y^3}) = \frac{1}{y}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y} \quad \text{and} \quad \Gamma^1_{22} = \Gamma^2_{21} = \Gamma^2_{12} = \Gamma^1_{11}.$$

Hence the equations of geodesics are

$$\begin{cases} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0, \\ \ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) = 0. \end{cases}$$

By definition of geodesics we know that the length of a velocity vector is preserved .Therefore the value

$$I_1 = \frac{\dot{x}^2 + \dot{y}^2}{y^2}$$

(the square of this length) is a first integral.
As we can easily verify, the quantity

$$I_2 = x + \frac{\dot{y}}{\dot{x}}y$$

is another first integral.
Suppose that $x = x(t)$ is constant along the geodesic $\gamma(t) = (x(t), y(t))$; then the equation of geodesics reduce to

$$\ddot{y} = \frac{\dot{y}^2}{y}$$

By substitution we find that the solution is $y = ce^t$ with $c > 0$. These considerations suggest that, if the vertical lines are parametrized in this way, we obtain geodesics for the Levi-Civita connection of the half space model.
Suppose that γ is a geodesic and its velocity vector $v(t) = \dot{\gamma}(t)$ at a point is not vertical. We draw the staight line $l = (\xi, \eta)$ through the point $\gamma(t) = (x(t), y(t))$, in the Euclidean metric of $\mathbb{R}^2$, which is orthogonal to the vector

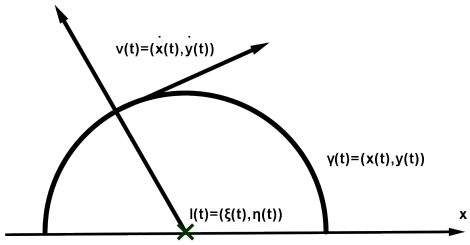


Figure 1.6: Construction of geodesics in $\mathbb{U}^2$

$v = (\dot{x}, \dot{y})$. Then the $x-$coordinate of the intersection point of l and the $x-$axis is equal to I_2. In fact if we solve the system

$$\begin{cases} \eta - y(t) = -\frac{\dot{x}}{\dot{y}}(\xi - x(t)), \\ \eta = 0 \end{cases} \tag{1.13}$$

we find

$$\xi = \frac{\dot{y}(t)}{\dot{x}(t)} y(t) + x(t)$$

which is a conserved quantity since the right hand side of the equality is the first integral I_2.
We found that, in term of the Euclidean metric on $\mathbb{R}^2$, the geodesics of the half space are

(1) rays orthogonal to the x-axis.

(2) half circles lying in the upper half space.

It follows that arbitrary pairs of points of $\mathbb{H}^2$ are joined by a unique geodesic. The half space model $\mathbb{U}^2$ of $\mathbb{H}^2$ is a complete manifold because, as we observed before, every geodesic is defined for all $t \in \mathbb{R}$.
We can easily generalize the previous construction to $\mathbb{R}^n$ by defining $\mathbb{U}^n$ as the upper half-space $\{x \in \mathbb{R}^n | x_n > 0\}$ endowed with the metric

$$ds^2 = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2.$$

The Laplace-Beltrami operator on $\mathbb{U}^n$ has the following form

$$\Delta_{\mathbb{H}} u = x_n^2 \Delta u + (2 - n) x_n \frac{\partial u}{\partial x_n}, \tag{1.14}$$

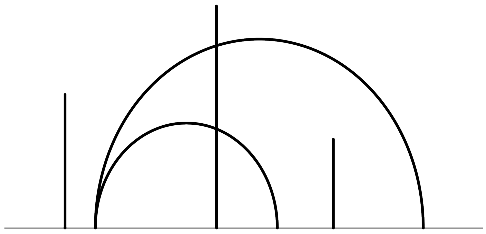


Figure 1.7: Geodesics in the half space model

where Δ stands for the usual Laplacian in $\mathbb{R}^n$.

For the sake of completeness we introduce another model of the hyperbolic space.

The Klein model

The *Klein model* $\mathbb{K}^n$ of $\mathbb{H}^n$ is the unit ball B_1 equipped with the metric obtained by carrying the hyperbolic metric of $\mathbb{I}^n$ along the following bijective map

$$\psi : B_1 \subseteq \mathbb{R}^n \to \mathbb{I}^n, \quad \psi(x) := \frac{(x,1)}{\sqrt{1-|x|^2}}.$$

In the Klein model $\mathbb{K}^n$ of $\mathbb{H}^n$ the geodesics are the traces of ordinary affine lines in $\mathbb{K}^n$.
It is worth observing two more features of $\mathbb{K}^n$:

(a) for any convex subset A of $\mathbb{K}^n$ and any point $c \in \mathbb{K}^n \setminus A$ there exists a hyperplane π such that $x \in \pi$ and $\pi \cap A = \emptyset$;

(b) for any convex subset A of $\mathbb{K}^n$ and any entire geodesic $\gamma \subseteq \mathbb{K}^n$ such that $\gamma \cap A = \emptyset$, there exists a hyperplane $\pi \subseteq \mathbb{K}^n$ such that $\gamma \subseteq \pi$ and $\pi \cap A = \emptyset$.

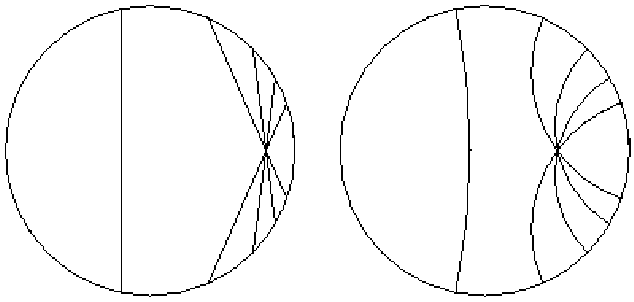
Finally, we consider the bijective and isometric map

$$\phi : \mathbb{D}^n \to \mathbb{I}^n, \quad \phi(x) := \frac{(2x, 1+|x|^2)}{1-|x|^2}$$

and the composition

$$\eta = \phi^{-1} \circ \psi : \mathbb{K}^n \to \mathbb{D}^n.$$

It is easy to prove that η is a bijective map which transforms geodesics of $\mathbb{K}^n$ on geodesics of $\mathbb{D}^n$, and the properties $(a) - (b)$ are still valid.



1.4 Heat kernel on Riemannian manifolds

Let us first recall for further reference some well-known facts and definitions of semigroup theory (*e.g*, see [19])
Let X be a Banach space. We denote with $\mathcal{B}(X)$ the Banach space of bounded linear operators from X to X, endowed with the uniform norm.

Definition 1.1. A family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is called a *strongly continuous semigroup* on X if it satisfies the following properties:

(a) $T(s+t) = T(s)T(t) = T(t)T(s)$ for every $s, t \geq 0$.

(b) $T(0) = I$;

(c) for all $x \in X$ the map $t \to T(t)x$ is continuous.

Definition 1.2. The operator A defined as follows

$$\begin{cases} D(A) := \{x \in X : \text{ there exists } \lim_{h\to 0^+} h^{-1}[T(h)x - x]\}, \\ Ax := \lim_{h\to 0^+} h^{-1}[T(h)x - x] \quad (x \in D(A)) \end{cases}$$

is called the *infinitesimal generator* of the semigroup $\{T(t)\}_{t\geq 0}$.

Some relevant properties of the infinitesimal generator of strongly continuous semigroups are contained in the following Theorem.

Theorem 1.6. *Let A be the infinitesimal generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Then*

i) $D(A)$ is dense in X;

ii) for all $x \in D(A)$ and $t \in \mathbb{R}_+$, $T(t)x \in D(A)$. Moreover $t \to T(t)x$ is in $C^1(\mathbb{R}_+, X)$ and there holds

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax;$$

iii) A is closed.

Necessary and sufficient conditions for the generation of strongly continuous semigroups are given by the Hille-Yosida Theorem (*e.g*, see [19]).

Definition 1.3. A semigroup $\{T(t)\}_{t\geq 0}$ is said to be *contractive* if there hold

$$||T(t)|| \leq 1 \text{ for all } \quad t \geq 0.$$

Let M be a smooth connected non-compact (geodesically) complete Riemannian manifold, of dimension $n \geq 2$.
The *heat kernel* or, alternatively, the *fundamental solution of the heat equation* is a function $p(x, y, t)$:$M \times M \times \mathbb{R}_+ \to M$, which satisfies the following conditions:

(a) p is C^2 with respect to x and y and C^1 with respect to t,

(b)
$$p_t - \Delta_M p = 0,$$
where Δ_M is the Laplace-Beltrami operator with respect to x;

(c)
$$\lim_{t\to 0^+} \int_M p(x, y, t) f(y) dV_y = f(x)$$
for any compactly supported function on M.

Its importance stems from the fact that it is the *smallest positive* solution of the heat equation

$$u_t = \Delta_M u \quad \text{ in } M. \tag{1.15}$$

The heat kernel exists and is unique for a compact Riemannian manifold (see [3]) . However, in [6] , existence and uniqueness are proven also for non-compact Riemannian manifold.
The heat kernel $p(x, y, t)$ possesses the following general properties:

(i) *positivity*
$$p(x, y, t) \geq 0.$$

(ii) *total mass inequality*:
$$\int_M p(x, y, t) dV_y \leq 1;$$

(iii) *semigroup property*:
$$p(x, y, t) = \int_M p(x, z, t) p(z, y, t - \tau) dV_z \quad (x, y \in M,\ 0 < \tau < t).$$

(iv) *symmetry:*
$$p(x,y,t) = p(y,x,t);$$

(v) *approximation of identity*: for any $u_0 \in L^2(M)$
$$u(\cdot,t) := \int_M p(t,\cdot,y)u_0(y)dV_y \xrightarrow[t\to 0^+]{} u_0 \quad \text{in } L^2(M). \tag{1.16}$$

By (1.15) and (1.16) above and the regularity of p, the function defined in (1.16) is C^∞ smooth in $(x,t) \in M \times \mathbb{R}_+$ and it is the unique solution of the Cauchy problem
$$\begin{cases} u_t = \Delta_M u & \text{in } M \times \mathbb{R}_+ \\ u = u_0 & \text{in } M \times \{0\} \end{cases} \tag{1.17}$$
(the initial condition being satisfied as in (1.16)).
The solution if (1.17) is given by
$$u(x,t) = \int_M p(x,y,t)f(y)dV_y.$$
Let us prove property (iii), referring the reader to [3] for the proofs of other properties.
On the one hand we have
$$u(x,t+t_1) = \int_M p(x,y,t+t_1)u_0(y)dV_y.$$
On the other hand,
$$\begin{aligned} u(x,t+t_1) &= \int_M p(x,z,t_1)u(x,t)dV_z = \\ &= \int_M p(x,y,t_1)\left(\int_M p(x,y,t)u_0(y)dV_y\right)dV_z, \end{aligned}$$
Thanks to property (i) we can apply Tonelli's Theorem and write
$$u(x,t+t_1) = \int_M \left(\int_M p(x,z,t_1)p(x,y,t)dV_z\right)u_0(y)dV_y.$$
Hence, by uniqueness of the solution of (1.17) we have
$$\int_M p(x,y,t+t_1)u_0(y)dV_y = \int_M \left(\int_M p(x,z,t_1)p(x,y,t)dV_z\right)u_0(y)dV_y$$
whence the conclusion
$$p(x,z,t+t_1) = \int_M p(x,y,t)p(y,z,t_1)dV_y.$$

follows by the arbitrariness of u_0.
In view of the above remarks, any heat kernel gives rise to the strongly continuous semigroup $\{\mathcal{K}_t\}_{t\geq 0}$ on $L^2(M)$ defined by

$$\mathcal{K}_t u_0(x) = \int_M p(x,y,t) u_0(y) dV_y \quad (x \in M)$$

for any $u_0 \in L^2(M)$.
Properties $i) - ii)$ ensure that $\{\mathcal{K}_t\}_{t\geq 0}$ is a *Markov semigroup*, namely

$$0 \leq u_0 \leq 1 \quad \Rightarrow \quad 0 \leq \mathcal{K}_t u_0 \leq 1 \quad \text{a.e in } M.$$

The symmetry property (iv) implies that the operator $\mathcal{K}_t$ is symmetric and hence self-adjoint.
It can be proved that the infinitesimal generator $\mathcal{L}$ of the semigroup $\{\mathcal{K}_t\}_{t\geq 0}$ is the minimal self-adjoint extension of $-\Delta$ in $L^2(M)$, Δ_M denoting the Laplace-Beltrami operator on M (*e.g.*, see [5])

Example 1.7. It is well known that in $\mathbb{R}^n$

$$p(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x-y|^2}{4t}}.$$

Example 1.8. In $\mathbb{H}^3$, the heat kernel is (see [9])

$$p(x,y,t) = \frac{1}{(4\pi t)^{3/2}} \frac{d(x,y)}{\sinh(d(x,y))} e^{-\frac{d^2(x,y)}{4t} - t}.$$

In general the heat kernel in $\mathbb{H}^n$ is computed by using a recursive formula.

In $\mathbb{H}^n$ we have a bilateral estimate of the heat kernel which plays an important role in the following (see Chapter 2).

Theorem 1.7. *[5]For all $n \geq 1$ there exists a positive constant c_n such that*

$$\frac{1}{c_n} h_n(d(x,y),t) \leq p(x,y,t) \leq c_n h_n(d(x,y),t) \tag{1.18}$$

for all $t > 0$ and $d > 0$, where

$$h_n(d,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\lambda_1 t - \frac{n-1}{2} d - \frac{d^2}{4t}} (1 + d + t)^{\frac{n-3}{2}} (1 + d).$$

If we fix $d > 0$ then Theorem 1.7 implies that

$$p(x,y,t) \approx \begin{cases} t^{\frac{n}{2}} & \text{as} \quad t \to 0, \\ t^{\frac{3}{2}} e^{-\lambda_1 t} & \text{as} \quad t \to \infty. \end{cases}$$

1.5 The principal eigenvalue

Consider $\Omega \subset M$ a compact open set and the eigenvalue problem

$$\begin{cases} \Delta_M u + \lambda u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega \end{cases} \tag{1.19}$$

In [3] it is proved that there exists a complete orthonormal basis $\{\phi_1, \phi_2, \cdots\}$ of $L^2(\Omega)$ consisting of Dirichlet eigenfunctions of Δ_M, with ϕ_j having eigenvalue λ_j satisfying

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \uparrow +\infty.$$

In particular, each eigenvalue has finite multiplicity and

$$\phi_j \in C^\infty(\Omega) \cap \bar{C}^1(\Omega).$$

If p_Ω denotes the heat kernel in Ω subject to the Dirichlet boundary conditions then the eigenvalue expansion is

$$p_\Omega(x, y, t) = \sum_{k=1}^{n} e^{-\lambda_k(\Omega)t} \phi_k(x)\phi_k(y) \tag{1.20}$$

where λ_k is the $k-$th Dirichlet eigenvalue of Ω and ϕ_k is the corresponding eigenfunction.
The eigenvalue $\lambda_1 = \lambda_1(\Omega)$ is called *the first Dirichlet eigenvalue* of problem (1.19).
Equality (1.20) derives by the application of the spectral Theorem [19] to the semigroup $\{\mathcal{K}_t\}_{t\geq 0}$, defined previously.
The same results can be generalized to precompact open sets of Ω (see [3] for details).
In a general manifold M the quantity $\lambda_1 = \lambda_1(M)$ is defined by means of the Rayleigh variational principle

$$\lambda_1(M) := \inf_{\phi \in C_c^\infty(M), \phi \neq 0} \frac{\int_M |\nabla\phi|^2 d\mu}{\int_M \phi^2 d\mu}.$$

Equivalently, $\lambda_1(M)$ can be defined as the infimum of the spectrum of the operator $-\Delta$ in $L^2(M)$ with domain $C_c^\infty(M)$.
If $\{\Omega_\alpha\}$ is a family of precompact open sets which exhaust[12] M then $p_\Omega \to p$ and $\lambda_1(\Omega) \to \lambda_1(M)$ as Ω exhausts M (see [8]).

[12] The collection of precompact open set $\{\Omega_i\}_{i\in\mathbb{I}} \subseteq M$ is said to *exhaust* M if

$$\bar{\Omega}_i \subset \Omega_{i+1} \quad \text{for all} \quad i = 1, 2...$$

and

$$\cup_{i=1}^{\infty} \Omega_i = M$$

In particular, if $\mathcal{B}_r$ is the geodesic ball of $\mathbb{H}^n$ defined in (1.9) then

$$\lambda_1(\mathcal{B}_r) \searrow \lambda_1 \quad \text{as} \quad r \to \infty \tag{1.21}$$

where $\lambda_1(\mathcal{B}_r)$ is the first eigenvalue for $-\Delta_{\mathbb{H}}$ in $\mathcal{B}_R$ with Dirichlet zero boundary conditions (*e.g.* (1.19) with $\Omega = \mathcal{B}_r$).
If $\lambda_1(M) > 0$ as for the case of the hyperbolic space then

$$p \approx e^{-\lambda_1(M)t}$$

as $t \to \infty$. For more details the reader is referred to Chavel [3].

Theorem 1.8 (McKean). *([8]) Let M be a geodesically complete, simply connected manifold of dimension n which has non-positive sectional curvature; if its sectional curvature is bounded from above by $-\kappa^2$, then*

$$\lambda_1 \geq \frac{1}{4}(n-1)^2\kappa^2. \tag{1.22}$$

There is relation between $\lambda_1(M)$ and the volume growth of a geodesically complete manifold [13]: if

$$\nu := \lim_{r\to\infty} \sup \frac{\log V(x,r)}{r} \tag{1.23}$$

is the volume of the ball of radius r [14] then

$$\lambda_1(M) \leq \frac{\nu^2}{4}. \tag{1.24}$$

In particular, $\lambda_1(M) = 0$ for manifold with subexponential volume growth.

Example 1.9. $\mathbb{R}^n$ and $\mathbb{H}^n$ both satisfy the hypotheses of Mc Kean's Theorem so that (1.22) and (1.24) holds true for them. Besides, for them, both (1.22) and (1.24) become equalities.
We can find easily:

- $\lambda_1(\mathbb{R}^n) = 0$. In fact on one hand $\kappa = 0$ and on the other hand
$$V(x,r) = \int_{\mathbb{S}_{n-1}} \int_0^r \xi^{n-1} d\xi = c_{n-1}\frac{r^n}{n}.$$

- $\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4}$. In fact on one side $\kappa = -1$ so that $\lambda_1 \geq \frac{1}{4}(n-1)^2$ by (1.22) and on the other side on $\mathbb{H}^n$ the volume has the form (see (1.7))
$$V(x,r) = \int_{\mathbb{S}^{n-1}} \int_0^r \sinh^{n-1}\xi d\xi d\theta.$$

[13] See R.Brook in "A relation between growth and the spectrum of the Laplacian", Mathematische Zeitschrift Volume 178, Number 4, 501 – 508.
[14] Notice that the volume is independent of the center of the ball

$$
\begin{aligned}
\nu &= \lim_{r\to\infty} \sup \frac{\log \int_{\mathbb{S}^{n-1}} \int_0^r \sinh^{n-1} \xi d\xi d\theta}{r} = \\
&= \lim_{r\to\infty} \sup \frac{\log \Omega_{n-1} \int_0^r \sinh^{n-1} \xi d\xi d\theta}{r} = \\
&= \lim_{r\to\infty} \sup \frac{\log \Omega_{n-1} + \log \int_0^r \sinh^{n-1} \xi d\xi d\theta}{r} = \\
&= \lim_{r\to\infty} \sup \left\{ \frac{\log \Omega_{n-1}}{r} + \frac{\log \int_0^r \sinh^{n-1} \xi d\xi d\theta}{r}. \right\}
\end{aligned}
$$

Applying De L'Hospital's rule twice we obtain

$$
\begin{aligned}
\nu &= \lim_{r\to\infty} \sup \frac{\frac{\sinh^{n-1} r}{\int_0^r \sinh^{n-1} \xi d\xi d\theta}}{1} = \\
&= \lim_{r\to\infty} \sup \frac{(n-1)\cosh^{n-1} r \sinh r}{(\sinh r)^{n-1}} = \\
&= \lim_{r\to\infty} \sup \frac{(n-1)\cosh^{n-2} r}{(\sinh r)^{n-2}} = \\
&= \lim_{r\to\infty} (n-1) \operatorname{cotgh}^{n-2} r = (n-1).
\end{aligned}
$$

Remark 1.1. *In $\mathbb{R}^n$, the L^2-spectrum of $-\Delta$ is $(0,\infty)$; therefore $\lambda_1 = 0$. Instead, in $\mathbb{H}^n$, the L^2-spectrum of $-\Delta_{\mathbb{H}^n}$ is $\left[\frac{(n-1)^2}{4}, \infty\right)$; thus $\lambda_1 = \frac{(n-1)^2}{4}$ (see[5]).*

Chapter 2

Blow-Up for the Cauchy Problem in $\mathbb{H}^n$

As explained in Chapter 1 the heat kernel in $\mathbb{H}^n$ has a different behavior for large times compared to the one in $\mathbb{R}^n$.
In this chapter we will investigate the effect of the heat kernel in $\mathbb{H}^n$ on the positive solutions of the following Cauchy problem with power nonlinearities.

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + h(t)|u|^{p-1}u & \text{in} \quad \mathbb{H}^n \times \mathbb{R}_+, \\ u = u_0 & \text{in} \quad \mathbb{H}^n \times \{0\} \end{cases}$$

where h is a positive, continuous and locally integrable function in $\mathbb{R}_+$, $p > 1$ and $u_0 \geq 0$.
A local (weak or classical) solution of (P1) is a solution that exists in $\mathbb{R}^n \times [0, T)$, $T < \infty$, and the maximal time of existence T_{max} is the supremum of all such $T's$ for which a solution exists. If $T_{max} = \infty$ the solution is global and if $T_{max} < \infty$ then the solution is said to blow up in finite time. The blow-up phenomenon is connected to the time in which the solution (or its derivative) become unbounded in (some) norm.
We will now face three problems:

1. **Local existence**: does the solution exist in $\mathbb{H}^n \times [0, T)$ for some $T \in (0, \infty)$?

2. **Finite time blow-up**: if we have local existence then is the maximal time of existence finite? In this case we will say that the solution blows up in finite time.

3. **Global existence**: if we have local existence, when does the maximal time of existence become infinite ?

2.1 Classical, mild and weak solutions

Since no growth conditions are imposed at infinity, a precise definition of solution is required. We now compare different type of solutions of problem (P1) under the following hypotheses

$$\begin{cases} (i)\ h : \mathbb{R}_+ \to \mathbb{R}_+ & \text{is continuous and } h \in L^1_{loc}(\bar{\mathbb{R}}_+); \\ (ii)\ p > 1; \\ (iii)\ u_0 \in C(\mathbb{H}^n), n \geq 2. \end{cases} \tag{2.1}$$

For any $\tau > 0$ we set $Q_\tau := \mathbb{H}^n \times (0, \tau]$.

Definition 2.1. A function $u \in C^{2,1}(Q_\tau) \cap C(\bar{Q}_\tau)$ is called a *classical* solution of problem (P1) in $[0, \tau]$ if

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + h(t)|u|^{p-1}u & \text{in} \quad Q_\tau \\ u(x, 0) = u_0(x) \geq 0 & \text{for any} \quad x \in \mathbb{H}^n. \end{cases} \tag{2.2}$$

Making use of the following notation :

$$\left(e^{t\Delta_{\mathbb{H}}}\phi\right)(x) := \int_{\mathbb{H}^n} p(x, y, t)\phi(y)dV_y$$

where $\phi \in C(\mathbb{H}^n)$, we can define a weaker concept of solution, based on the heat kernel (compare section 1.4) in $\mathbb{H}^n$.

Definition 2.2. A function $u \in C(\bar{Q}_\tau)$ is called a *mild solution* of the problem (P1) for $t \in [0, \tau]$ if

$$u(x, t) = \left(e^{t\Delta_{\mathbb{H}}}u_0\right)(x) + \int_0^t \left(e^{(t-s)\Delta_{\mathbb{H}}}h(s)|u|^{p-1}u\right)(x)ds \tag{2.3}$$

for any $t \in [0, \tau]$.
Since

$$\left(e^{t\Delta_{\mathbb{H}}}u_0\right)(x) = \int_{\mathbb{H}^n} p(x, y, t)u_0(y)dV_y$$

and

$$\left(e^{(t-s)\Delta_{\mathbb{H}}}h(s)|u|^{p-1}u\right)(x) = \int\int_{Q_t} p(x, y, t-s)h(s)|u|^{p-1}u dV_y ds,$$

we can express the solution as

$$u(x, t) = \int_{\mathbb{H}^n} p(x, y, t)u_0(y)dV_y + \int\int_{Q_t} p(x, y, t-s)h(s)|u|^{p-1}u(y)dV_y ds.$$

Using classical regularity results and the estimate on the heat kernel (1.18) it can be proved that

Proposition 2.1. *A mild solution is a classical solution.*

The concept of weak solution is expressed in the next definition:

Definition 2.3. A function $u \in C(\bar{Q}_\tau)$ is called a continuous *weak solution* of problem (P1) in $[0,\tau]$ if for any $\tau_1 \in (0,\tau]$

$$-\int\int_{Q_{\tau_1}} u\{\Delta_{\mathbb{H}}\psi + \psi_t\}dVdt = \int_{\mathbb{H}^n} u_0\psi(\cdot,0)dV + \int\int_{Q_{\tau_1}} h(t)|u|^{p-1}u\psi dVdt \tag{2.4}$$

where $\psi \in C^{2,1}(\bar{Q}_{\tau_1})$ is an arbitrary function such that for any $t \in [0,\tau_1)$ $\operatorname{supp}\psi(\cdot,t) \subset\subset \mathbb{H}^n$ and $\psi(\cdot,\tau_1) = 0$.

Moreover,

Definition 2.4. The function $\bar{u}$ is called an *upper solution* of (P1) if

$$-\int\int_{Q_{\tau_1}} u\{\Delta_{\mathbb{H}}\psi + \psi_t\}dVdt \geq \int_{\mathbb{H}^n} u_0\psi(\cdot,0)dV + \int\int_{Q_{\tau_1}} h(t)|u|^{p-1}u\psi dVdt \tag{2.5}$$

holds for positive ψ. Reversing the sign in (2.5) we have the definition of *lower solution* and we denote it with $\underline{u}$.

Lastly, we define the blow-up for a solution of (P1).

Definition 2.5. Let u be a continuous weak solution of (P1) for $t \in [0,T)$ where T is the maximal time of existence.
If $T < \infty$ and

$$\lim_{t\to T^-} ||u(\cdot,t)||_\infty = \infty$$

then u is said to *blow up in finite time.* T is the *blow-up time.*
If $T = \infty$ then the solution u is said to be global.

Since any classical solution is a continuous weak solution, Proposition 2.1 implies that a mild solution is a continuous weak solution as well. The opposite implication is clarified in the next Lemma.

Lemma 2.1. *Let u be a continuous weak solution of problem* (P1) *in* $[0,\tau]$, *satisfying the growth condition*

$$0 \leq u(x,t) \leq Ae^{cd(x,0)^\beta} \tag{2.6}$$

for some positive A, c and $0 < \beta < 2$. Then u is a mild solution of the problem (P1) *in* Q_τ.

Proof. Firstly, we consider the problem

$$\begin{cases} I^0_t = \Delta_{\mathbb{H}} I^0 & \text{in} \quad Q_\tau \\ I^0 = u_0 & \text{in} \quad \mathbb{H}^n \times \{0\} \end{cases}$$

whose classical solution is already known to be (compare 1.4)

$$I^0(x,t) = \int_{\mathbb{H}^n} q(x,y,t)u_0(y)dV_y.$$

Our claim is that the function

$$I(x,t) := \int\int_{Q_\tau} q(x,y,t-s)h(s)|u|^{p-1}u(y,s)dV_y ds \tag{2.7}$$

is a weak solution of the problem

$$\begin{cases} I_t = \Delta_{\mathbb{H}} I + h(t)|u|^{p-1}u & \text{in } Q_\tau, \\ I = 0 & \text{in } \mathbb{H}^n \times \{0\}. \end{cases}$$

We prove this claim by considering the function $f_a(\cdot,t) := \chi_a |u|^{p-1}u(\cdot,t)$, where $\chi_a = \chi_{\mathcal{B}_a^{\mathbb{H}}}$ is the characteristic function of the geodesic ball $\mathcal{B}_a^{\mathbb{H}}$ of radius a, and its mollification $f_{a,\epsilon}(\cdot,t)$. Then by classical results

$$I_{\alpha,\epsilon}(x,t) := \int\int_{Q_\tau} q(x,y,t-s)h(s)f_{\alpha,\epsilon}(y,s)dV_y ds ds \tag{2.8}$$

is a classical solution of the problem

$$\begin{cases} I_t = \Delta_{\mathbb{H}} I + h(t)f_{\alpha,\epsilon} & \text{in} \quad Q_\tau, \\ I = 0 & \text{in} \quad \mathbb{H}^n \times \{0\}. \end{cases}$$

This last solution, being classical, is also a weak solution satisfying

$$-\int\int_{Q_{\tau_1}} I_{\alpha,\epsilon}\{\Delta_{\mathbb{H}}\psi + \psi_t\}dV_y dt = \int\int_{Q_{\tau_1}} h(t)f_{\alpha,\epsilon}\psi dV_y dt$$

for any $\tau_1 \in [0,\tau]$ and for any test function ψ[1]. We now observe that for any $q > 1$, as $\epsilon \to 0$, we have $f_{\alpha,\epsilon} \to f_a$ in $L^q(Q_\tau)$. It follows from L^q estimates for parabolic equation and embedding results that

$$I_a(x,t) := \int\int_{Q_t} p(x,y,t-s)h(s)|u|^{p-1}u dV(y)ds$$

is Höelder continuous in Q_τ and satisfies

$$-\int\int_{Q_{\tau_1}} I_a\{\Delta_{\mathbb{H}}\psi + \psi_t\}dV(y)dt = \int\int_{Q_{\tau_1}} h(t)\chi_a|u|^{p-1}u\psi dV(y)dt.$$

Letting $a \to \infty$ and passing the limit under the integral sign applying the dominate convergence Theorem

$$-\int\int_{Q_{\tau_1}} \lim_{a\to\infty} I_a\{\Delta_{\mathbb{H}}\psi+\psi_t\}dV(y)dt = \int\int_{Q_{\tau_1}} \lim_{a\to\infty} h(t)\chi_a|u|^{p-1}u\psi dV(y)dt,$$

[1] see the definition of mild solution in the previous chapter

we have

$$-\int\int_{Q_{\tau_1}} I\{\Delta_{\mathbb{H}}\psi+\psi_t\}dV(y)dt=\int\int_{Q_{\tau_1}} h(t)|u|^{p-1}u\psi dV(y)dt$$

so $I(x,t)$ is a weak solution of the problem (2.1).
Thus we deduce that the function

$$\begin{aligned}v(x,t)=I^0(x)+I(x,t)=\int_{\mathbb{H}^n} q(x,y,t)u_0(y)dV_y+\\ +\int\int_{Q_\tau} q(x,y,t-s)h(s)|u|^{p-1}u(y,s)dV_y ds.\end{aligned}$$

is a continuous weak solution of problem

$$\begin{cases} v_t=\Delta_{\mathbb{H}}v+h(t)|u|^{p-1}u & \text{in} \quad \mathbb{H}^n\times[0,\tau],\\ v=u_0 & \text{in} \quad \mathbb{H}^n\times\{0\}\end{cases}$$

so that the difference $w=u-v$ is a continuous weak solution of the problem

$$\begin{cases} w_t=\Delta_{\mathbb{H}}w & \text{in} \quad Q_\tau,\\ w(x,0)=0. \end{cases}$$

We would like to conclude that u is a mild solution of problem (P1) so that we need to show that the hypothesis (2.6) is satisfied. Thanks to inequality (1.18) we deduce that

$$I^0(x,t)=\int_{\mathbb{H}^n} q(x,y,t)u_0dV_y\le c_n\int_{\mathbb{H}^n} h_n(d(x,y),t)u_0(y)dV_y \tag{2.9}$$

and inserting the definition of $h(d(x,y),t)$ we have

$$\begin{aligned}I^0(x,t)=c_n(4\pi t)^{-\frac{n}{2}}e^{-\lambda_1 t}\int_{\mathbb{H}^n}(1+d(x,y))\times\\ \times(1+d(x,y)+\tau)^{\frac{n-3}{2}}e^{-\frac{(n-1)d(x,y)}{2}-\frac{d^2(x,y)}{4t}}u_0(y)dV_y.\end{aligned} \tag{2.10}$$

If τ_x is the isometry defined in (1.10) and we set $z:=\tau_x y$ ($y=\tau_{-x}(z)$) then

$$d(x,y)=d(\tau_x,\tau_x(y))=d(\tau_x(x),\tau_x(\tau_{-x}z))=d(0,z)$$

and applying the property that $\tau_{-x}=\tau_x^{-1}$ and the triangular inequality we have

$$\begin{aligned}d(y,0)=d(\tau_{-x}(z),0)=d(\tau_{-x}(z),\tau_{-x}(-x))=d(\tau_x^{-1}(z),\tau_x^{-1}(-x))\le\\ \le d(x,0)+d(z,0).\end{aligned}$$

We thus can write

$$u_0(y) \leq Ae^{cd^\beta(y,0)} \leq Ae^{2^{\beta-1}c[d^\beta(x,0)+d^\beta(z,0)]}$$

and inserting this last estimate in (2.10) we obtain

$$\begin{aligned}
I^0(x,t) \leq& \\
&\leq c_n(4\pi)^{-\frac{n}{2}}At^{-\frac{n}{2}}\int_{\mathbb{H}^n}(1+d(z,0))(1+d(z,0)+\tau)^{\frac{n-3}{2}}\times \\
&\times e^{-\frac{(n-1)d(z,0)}{2}-\frac{d^2(z,0)}{4t}}e^{2^{\beta-1}c[d^\beta(x,0)+d^\beta(z,0)]}dV_y \leq \\
&\leq c_1e^{c_2d^\beta(x,0)}t^{-\frac{n}{2}}\int_0^\infty(1+\rho+\tau)^{\frac{n-3}{2}}e^{-\frac{\rho^2}{4t}+c_2\rho^\beta}\sinh^{n-1}\rho d\rho = \\
&= c_1e^{c_2d^\beta(x,0)}t^{-\frac{n}{2}}\left\{\int_0^1(1+\rho+\tau)^{\frac{n-3}{2}}e^{-\frac{\rho^2}{4t}+c_2\rho^\beta}\sinh^{n-1}\rho d\rho+\right. \\
&\left.+\int_1^\infty(1+\rho+\tau)^{\frac{n-1}{2}}e^{-\frac{\rho^2}{4t}+c_2\rho^\beta}\sinh^{n-1}\rho d\rho\right\}.
\end{aligned}$$

Now we estimate the two integral separately:
if $\rho \geq 1$ then $\sinh\rho \leq c_3\rho$; setting $\rho = 2\sqrt{t}\hat{\rho}$ we get

$$\begin{aligned}
t^{-\frac{n}{2}}\int_0^1(1+\rho+\tau)^{\frac{n-3}{2}}e^{-\frac{\rho^2}{4t}+c_2\rho^\beta}\sinh^{n-1}\rho d\rho \leq& \\
&\leq c_4\int_0^1 e^{-\hat{\rho}^2+c_5\hat{\rho}^\beta}\hat{\rho}^{n-1}d\hat{\rho} \leq c_6,
\end{aligned}$$

where c_4, c_5 and c_6 depend only on τ.
Observing that if $t \leq \tau$ then a coefficient c_7 depending on τ exists and satisfies $t^{-\frac{n}{2}} \leq c_6e^{\frac{1}{8t}}$ and that $\sinh\rho < \frac{e^\rho}{2}$ we can estimate the other integral

$$\begin{aligned}
\int_1^\infty(1+\rho+\tau)^{\frac{n-1}{2}}e^{-\frac{\rho^2}{4t}+c_2\rho^\beta}\sinh^{n-1}\rho d\rho \leq& \\
&\leq c_7\int_1^\infty e^{\frac{1}{8t}-\frac{\rho^2}{4t}+c_2\rho^\beta+(n-1)\rho}d\rho \leq c_8.
\end{aligned}$$

Thus we obtain

$$I^0(x,t) \leq c_1e^{c_2d^\beta(x,0)}(c_6+c_8) = c_9e^{c_2d^\beta(x,0)}.$$

We can repeat the same estimates for I and therefore for $v = I^0 + I$. Thus we have shown that v and therefore w satisfy (1.18). Applying the previous Lemma we can conclude that $(0 \leq)w \leq 0$ thus $w = 0$ and so $v = u$.

$\square$

As a consequence we have the following

Corollary 2.1.
A classical solution u satisfying

$$0 \leq u(x,t) \leq Ae^{cd(x,0)^\beta}$$

for some positive A, c and $0 < \beta < 2$ is a mild solution

Remark 2.1. *If a classical solution satisfies the estimate (2.6) for $\beta = 2$ then it is a mild solution for $t \in [0, \frac{1}{4cp})$*

The following Lemma is an important tool for the application of comparison techniques.

Lemma 2.2. *Le w be a continuous function satisfying*

$$\int_{Q_\tau} w(\phi_t + \Delta_{\mathbb{H}}\phi) dV dt \geq 0 \tag{2.11}$$

for all positive $\phi \in C_0^\infty(Q_\tau)$ and $w(x,0) = 0$. Suppose in addition that w satisfies the growth condition

$$w(x,t) \leq Ae^{cd(x,0)^2} \tag{2.12}$$

for some positive A, c.
Then $w \leq 0$ in Q_τ.

Proof. We set $\eta = e^{\frac{\alpha\rho^2}{\gamma - t} + \beta t}$ where α, β, γ will be chosen later and $v := w/\eta$. If $w = \eta v$ is **regular** then it is a lower solution of the heat problem and satisfies in Q_τ

$$\begin{aligned} 0 \leq \Delta_{\mathbb{H}} w - w_t &= \Delta(\eta v) - (\eta v)_t = \\ &= \eta(\Delta_{\mathbb{H}} v - v_t) + v(\Delta_{\mathbb{H}}\eta - \eta_t) + 2p^{-2}\sum_{i=1}^n v_{x_i}\eta_{x_i} = \\ &= \eta\left[(\Delta_{\mathbb{H}} v - v_t) + \eta^{-1}v(\Delta_{\mathbb{H}}\eta - \eta_t) + 2\eta^{-1}p^{-2}\sum_{i=1}^n v_{x_i}\eta_{x_i}\right] \end{aligned}$$

so that

$$(\Delta_{\mathbb{H}} v - v_t) + \eta^{-1}v(\Delta_{\mathbb{H}}\eta - \eta_t) + 2\eta^{-1}p^{-2}\sum_{i=1}^n v_{x_i}\eta_{x_i} \geq 0. \tag{2.13}$$

Deriving we have

$$\eta_\rho = \frac{2\alpha\rho}{\gamma - t}\eta, \quad \eta_{\rho\rho} = \eta\left[\frac{2\alpha\rho}{\gamma - t} + \left(\frac{2\alpha\rho}{\gamma - t}\right)^2\right], \quad \eta_t = \eta\left[\frac{\alpha\rho^2}{(\gamma - t)^2} + \beta\right].$$

If $a := \frac{\alpha}{\gamma - t}$ then

$$\begin{aligned}
\Delta_{\mathbb{H}}\eta &= \frac{1}{\sinh^{n-1}}\frac{\partial}{\partial\rho}\left((\sinh\rho)^{n-1}\left(\frac{2\alpha\rho}{\gamma-t}\eta\right)\right) = \\
&= \frac{1}{\sinh^{n-1}}\{(n-1)(\sinh\rho)^{n-2}\cosh\rho\frac{2\alpha\rho}{\gamma-t}\eta + \\
&\quad + (\sinh\rho)^{n-1}\left[\frac{2\alpha}{\gamma-t} + \left(\frac{2\alpha\rho}{\gamma-t}^2\right)\right]\eta\} = \\
&= \frac{1}{\sinh^{n-1}}\{(\sinh\rho)^{n-2}2a\eta\left[(n-1)\rho\cosh\rho + 1 + 2a\rho^2\right]\} = \\
&= \{2a\eta[(n-1)\rho\coth\rho + 1 + 2a\rho^2]\} = \\
&= \eta[2a\rho(n-1)\coth\rho + 2a + 4a^2\rho^2],
\end{aligned}$$

so that

$$\begin{aligned}
\Delta_{\mathbb{H}}\eta - \eta_t &= \eta\left(4a^2\rho^2 + 2a + 2a\rho(n-1)\coth\rho - \frac{\rho^2 a}{\gamma-t} - \beta\right) = \\
&= \eta\left[\rho^2\left(4a^2 - \frac{1}{\alpha}\right) - \beta + 2a\rho(n-1)\coth\rho + 2a\right] = \\
&:= \eta C(\rho, t).
\end{aligned}$$

Since $\rho\coth\rho \leq 1 + \rho$ it follows that

$$\begin{aligned}
C(\rho,t) &\leq \rho^2 a^2\left(4 - \frac{1}{a}\right) + 2a(n-1)(1+\rho) + 2a - \beta = \\
&= \rho^2 a^2\left(4 - \frac{1}{\alpha}\right) + 2an(1+\rho) - 2a(1+\rho) + 2a - \beta = \\
&= \rho^2 a^2\left(4 - \frac{1}{\alpha}\right) + 2an1 + 2an\rho - 2a - 2a\rho + 2a - \beta = \\
&= \rho^2 a^2\left(4 - \frac{1}{\alpha}\right) + 2a(n-1)\rho + 2an - \beta.
\end{aligned}$$

If we take $\alpha < \frac{1}{4}$ then we have that the function

$$f(\rho) := \rho^2 a^2\left(4 - \frac{1}{\alpha}\right) + 2a(n-1)\rho + 2an - \beta$$

achieves its maximum at the point $\rho^* = -\frac{n-1}{a\left(4-\frac{1}{\alpha}\right)}$ where it takes the value of

$$f(\rho^*) = -\frac{(n-1)^2}{4-\frac{1}{\alpha}} + 2na - \beta = \frac{(n-1)^2}{\frac{1}{\alpha} - 4} + 2n\frac{\alpha}{\gamma - t} - \beta.$$

Hence $C(\rho, t)$ is bounded from above and it is clear then

$$C(\rho, t) \leq \frac{(n-1)^2}{\frac{1}{\alpha} - 4} + 2n\frac{\alpha}{\gamma - t} - \beta.$$

Now, let $\gamma < \frac{\alpha}{c}$ and $\tau_1 := min\{\gamma/2, \tau\}$ where c is the constant met in the assumption (2.12). As $d(x,0) \to \infty \quad v = w/\eta \to 0 \quad (\eta \to \infty)$ for $t \in [0, \tau_1]$. Finally we take $\beta = \frac{(n-1)^2}{\frac{1}{\alpha}-4} + 2n\frac{\alpha}{\gamma - t}$ in order to have $C(\rho, t) \leq 0$ in $\mathbb{R}_+ \times (0, \tau_1]$. Applying the parabolic maximum principle to v in the region

$$E_R = \{(x,t) : d(x,0) < R, \quad t \in (0, \tau_1)\}$$

then it follows that v achieves its maximum on the parabolic boundary of E_R. By assumption $w(x,0) = 0$ so that $v(x,0) = 0$ and, as we observed before, v converge to 0 as the distance from the origin tends to infinity. Thus, letting $R \to \infty$ we can conclude that $v \leq 0$.

We have proved that $w \leq 0$ when $t \in [0, \tau_1]$. If $\tau_1 < \tau$ we repeat the same argument starting at $t = \tau_1$ and we obtain the assertion in a finite number of iterations. □

Remark 2.2. *We observe that at the beginning of the proof we had made the assumption that w is regular. If we weaken this hypothesis and we take w only continuous then v satisfies* (2.13) *in the weack sense. It is a weakly subparabolic function in the sense of Friedman (see [7]). In this case a strong maximum principle for subparabolic functions has to be applied and the conclusion easily follows.*

2.2 Local existence

Theorem 2.1 (Existence and uniqueness of a L^∞ solution)**.** *Let u_0 be a positive function in $L^\infty(\mathbb{H}^n) \cap C(\mathbb{H}^n)$. Then there exists $\tau > 0$ such that the problem* (P1) *has a unique continuous weak solution $u \in L^\infty(Q_\tau)$. Either the solution is global or there is a maximal existence interval* $[0, T)(0 < T < \infty)$ *such that $||u||_{L^\infty(Q_\tau)} \to \infty$ as $\tau \to T^-$.*

Proof. • **Existence**: We set

$$H(t) := \int_0^t h(s)ds \qquad (t \in [0, \infty)) \tag{2.14}$$

and let τ be uniquely determined implicitly by the following relation

$$H(\tau) = \frac{1}{(p-1)||u_0||_\infty^{p-1}}.$$

We set the following problem:

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + h(t)u^p & \text{in} \quad B_{1-\frac{1}{n}} \times [0,\infty), \\ u = 0 & \text{in} \quad B_{1-\frac{1}{n}} \times (0,\tau), \\ u = u_0 & \text{in} \quad B_{1-\frac{1}{n}} \times \{0\} \end{cases} \tag{2.15}$$

where

$$B_{1-\frac{1}{n}} := \{x \in \mathbb{R}^n ; |x| < 1 - \frac{1}{n}\}$$

$(n \in \mathbb{N}, n \geq 2)$.

It is very easy to check that

$$\bar{u}(t) := ||u_0||_\infty [1 - (p-1)||u_0||_\infty^{p-1} H(t)]^{-\frac{1}{p-1}}$$

is a classical upper solution of both problem (P1) and (2.15).

In fact, since

$$\Delta_{\mathbb{H}} \bar{u}(t) = 0$$

then

$$\bar{u}_t = ||u_0||_\infty \left(-\frac{1}{p-1}\right) \left[1 - (p-1)||u_0||_\infty^{p-1} H(t)\right]^{-\frac{1}{p-1}} H'(t)(p-1)||u_0||_\infty^{p-1}$$

and so

$$-||u_0|| \left(-\frac{1}{p-1}\right) \left[1 - (p-1)||u_0||_\infty^{p-1} H(t)\right]^{-\frac{p}{p-1}} h(t)(p-1)||u_0||_\infty^{p-1} \geq$$
$$\geq \left[1 - (p-1)||u_0||_\infty^{p-1} H(t)\right]^{-\frac{p}{p-1}} h(t)||u_0||_\infty^p$$

that is

$$\left[1 - (p-1)||u_0||_\infty^{p-1} H(t)\right]^{-\frac{p}{p-1}} h(t)||u_0||_\infty^p \geq$$
$$\geq \left[1 - (p-1)||u_0||_\infty^{p-1} H(t)\right]^{-\frac{p}{p-1}} h(t)||u_0||_\infty^p.$$

Instead, $\underline{u} = 0$ is a classical lower solution of both problem (P1) and (2.15).

So by standard monoticity results there exists a function $u_n \in C(B_{1-\frac{1}{n}} \times [0,\tau))$ such that

$$0 \leq u_n \leq \bar{u}$$

which solves (2.15).

For any fixed integer $n_0 \geq 2$ the sequence $\{u_n\}_{n>n_0}$ is uniformly bounded and equicontinuous in the cylinder $B_{1-\frac{1}{n_0-1}} \times [0, \tau - \frac{1}{n_0}]$. Applying well known compactness results we find that there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ which converges to a function u which is a weak solution of the problem (P1). By Lemma 2.1 u is also a mild solution and hence by Proposition 2.1 it is a classical solution.

- **Uniqueness**: Now we suppose that u_1 and u_2 are two weak bounded solution (see definition 2.3) of the problem (P1) and we define $w := u_1 - u_2$ then

$$-\int\int_{Q_{\tau_1}} \{u_1 - u_2\}\{\Delta_{\mathbb{H}}\psi + \psi_t\} dV dt =$$
$$= \int_{\mathbb{H}^n} \{u_{01} - u_{02}\}\psi(\cdot, 0) dV + \int\int_{Q_{\tau_1}} h(t)(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2)\psi dV dt$$

and since $u_{01} = u_{02} = u_0$ we have

$$\int\int_{Q_{\tau_1}} \{u_1 - u_2\}\{\Delta_{\mathbb{H}}\psi + \psi_t\} dV dt + \int\int_{Q_{\tau_1}} h(t)(u_1^p - u_2^p)\psi dV dt = 0$$

where ψ is defined in (2.3), hence w satisfies

$$\int\int_{Q_{\tau_1}} \{w\,[\Delta_{\mathbb{H}}\psi + \psi_t] + h(t)(u_1^p - u_2^p)\psi\}\, dV dt = 0$$

and $w(x, 0) = 0$.
Set $w_+ = max\{0, w\}$ and $D := \{z \in Q_\tau : w_+(z) > 0\}$, then

$$u_1^p - u_2^p = \int_0^1 d\lambda \frac{d}{d\lambda}(u_1 + \lambda w)^p = p \int_0^1 d\lambda (u_1 + \lambda w)^{p-1} w.$$

This implies

$$\int\int_Q (u_1^p - u_2^p) dV dt \leq p \int_D w_+ \int_0^1 ||u_1 + \lambda w||_\infty^{p-1} d\lambda dV dt \leq$$
$$\leq p \max\{||u_1||_\infty^{p-1}, ||u_2||_\infty^{p-1}\} \int_D w_+ dV dt,$$

hence for any positive ψ we have

$$0 \leq \int_D w_+[\psi_t + \Delta_{\mathbb{H}}\psi + \psi h p ||u_1||^{p-1}_{L^\infty(Q_\tau)}] dV dt.$$

If we define $\xi := e^{At}\psi$ ($\psi = e^{-At}\xi$, $\psi_t = -Ae^{-At}\xi + \xi_t$, $\Delta_{\mathbb{H}}\psi = e^{-At}\Delta_{\mathbb{H}}\xi$) and we get

$$\begin{aligned} w_+[\psi_t + \Delta_{\mathbb{H}}\psi + \psi hp||u_1||^{p-1}_{L^\infty(Q_\tau)}] = \\ = e^{-At}w_+[-A\xi + \xi_t + \Delta\xi - \xi hp||u_1||^{p-1}_{L^\infty(Q_\tau)}] \end{aligned}$$

If $A > \max_{0,\tau} h(t)p||u_1||^{p-1}_{L^\infty(Q_\tau)}$ then

$$\begin{aligned} 0 \le \int_D (\psi_t + \Delta_{\mathbb{H}}\psi + A\psi)\, dV dt = \\ = \int_D e^{-At}w_+ \left[(\psi e^{At}_t + \Delta_{\mathbb{H}}(\psi e^{At}))\right] dV dt. \end{aligned}$$

On the parabolic boundary of D we have $w_+ = 0$ as a consequence of the fact that $w_+(x, 0) = 0$. Applying Lemma 2.2 in D to $e^{-At}w_+$ we can conclude that $w_+ \le 0$ and so that $u_1 \le u_2$.

Exchanging the role of the solutions u_1 and u_2 the result follows.

□

2.3 Blow-up

2.3.1 Instantaneous blow-up

In this subsection we want to show that if no growth conditions are imposed on u_0 (at infinity) then the blow-up time is $T = 0$. This implies that the problem (P1) may not even possess a local solution. We may refer to this as the case of **instantaneous blow-up**.

2.3.1.1 Principal eigenvalue in annuli

We define the annulus $\mathcal{A} := B^{\mathbb{H}}_{2a} \setminus B^{\mathbb{H}}_a$ ($a \ge 1$) and the differential problem

$$\begin{cases} \Delta_{\mathbb{H}}\phi_a + \lambda_a\phi_a = 0 & \text{in} \quad \mathcal{A}, \\ \phi_a = 0 & \text{in} \quad \partial\mathcal{A}. \end{cases} \tag{2.16}$$

where λ_a denotes the principal eigenvalue and $\phi_a > 0$ is the first Dirichlet eigenfunction of $\Delta_{\mathbb{H}}$ in $\mathcal{A}$.
In order to develop some arguments in the next chapter (see 2.2) we need to recall the following

Lemma 2.3. *For any $a \ge 1$ we have $\lambda_0 \le \lambda_a$. Moreover for large a, $\lambda_a \le \lambda_1 + o(a^{-2})$. Hence $\lim_{a\to\infty} \lambda_a = \lambda_1$.*

Proof. Searching for radial solutions of the form $\phi(\rho,\theta) = z(\rho)\alpha(\theta)$ of problem (2.16) we deduce that $z = z(\rho)$ must satisfy the problem

$$\begin{cases} z'' + (n-1)\coth\rho z' + \lambda_a z = 0 \quad \text{in} \quad (a, 2a) \\ z(a) = z(2a) = 0. \end{cases} \tag{2.17}$$

Now, observing that

$$\Delta_{\mathbb{H}} z = \frac{1}{\sinh^{n-1}(\rho)} \{\sinh^{n-1}(\rho) z'\}'$$

and multiplying all the equation of (2.17) by $(\sinh\rho)^{n-1} w$ where w is a ground state (compare definition 2.6) we get

$$\int_a^{2a} (\Delta_{\mathbb{H}} z + \lambda_a z)(\sinh\rho)^{n-1} w d\rho = 0,$$

$$\int_a^{2a} \Delta_{\mathbb{H}} z (\sinh\rho)^{n-1} w d\rho + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho = 0,$$

$$\int_a^{2a} \left\{(\sinh\rho)^{n-1} z'\right\}' w d\rho + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho = 0.$$

Then, integrating two times by parts the first term in the last equation we get

$$\begin{aligned} &\left[w(\sinh\rho)^{n-1} z'\right]_a^{2a} - \left[z(\sinh\rho)^{n-1} w'\right]_a^{2a} + \\ &\qquad + \int_a^{2a} \left[(\sinh\rho)^{n-1} w'\right]' z d\rho + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho = 0 \end{aligned}$$

and imposing that $z(a) = z(2a) = 0$ (see (2.17))

$$\int_a^{2a} \frac{(\sinh\rho)^{n-1}}{(\sinh\rho)^{n-1}} \left[(\sinh\rho)^{n-1} w'\right] z + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho \geq 0$$

which is

$$\int_a^{2a} (\sinh\rho)^{n-1} \Delta_{\mathbb{H}} wz d\rho + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz \geq 0,$$

and thus reminding that w is a ground state that solve the equation $\Delta_{\mathbb{H}} w + \lambda w = 0$ we have

$$-\lambda \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho + \lambda_a \int_a^{2a} (\sinh\rho)^{n-1} wz \geq 0,$$

$$(-\lambda + \lambda_a) \int_a^{2a} (\sinh\rho)^{n-1} wz d\rho \geq 0.$$

Fixing a, we can apply the Rayleigh principle and deduce that

$$\lambda_a = \inf_{v(a)=v(2a)} \frac{\int_a^{2a} (\sinh\rho)^{n-1} v'^2 d\rho}{\int_a^{2a} (\sinh\rho)^{n-1} v^2 d\rho}.$$

Now, assuming a to be very large we are led to approximate the equation (2.26) with

$$\begin{cases} v'' + (n-1)v' + \tilde{\lambda} v = 0 \quad \text{in} \quad (a, 2a) \\ v(a) = v(2a) = 0, \end{cases}$$

which solution is

$$v(\rho) = \sin\left[\sqrt{\tilde{\lambda} - \lambda_0}(\rho - a)\right] e^{-\frac{n-1}{2}(\rho - a)}$$

with

$$\tilde{\lambda} = \lambda_0 + \left(\frac{\pi}{a}\right)^2.$$

Since $(\sinh\rho)^{n-1} = \frac{e^{n-1}\rho}{2^{n-1}}[1 + O(e^{-2\rho})]$ for large ρ and

$$\tilde{\lambda} = \frac{\int_a^{2a} e^{(n-1)\rho} v'^2}{\int_a^{2a} e^{(n-1)\rho} v^2 d\rho}.$$

then, for large values of a we get

$$\lambda_a \leq \frac{\int_a^{2a} e^{(n-1)\rho}[1 + O(e^{-2\rho})]v'^2}{\int_a^{2a} e^{(n-1)\rho}[1 + O(e^{-2\rho})]v^2 d\rho} = \tilde{\lambda} + O(e^{-2a}).$$

Thus

$$\lambda_a \leq \lambda_0 + \left(\frac{\pi}{a}\right)^2 + O(e^{-2a})$$

and by letting $a \to \infty$ we have $\lambda_a \to \lambda_0$.

$\square$

Remark 2.3. *Observe that this Lemma provides us of an estimate of λ_a depending only on a.*

Theorem 2.2. *If*

$$\inf u_0(x) \to \infty \quad \text{as} \quad d(x, 0) \to \infty \tag{2.18}$$

then there is no solution for the Cauchy problem (P1) *for any $T > 0$.*

Proof. Let $u(x, t)$ be a classical solution of problem (P1) in Q_T satisfying

$$\lim_{d(x,0)\to\infty} u_0(x) = \infty.$$

Let $k(t)$ be a continuous positive function such that $k(0) = 1$ and $k(T) = 0$. Consider the annulus

$$\mathcal{A}(\rho_0) := B^{\mathbb{H}}_{2\rho_0} \setminus B^{\mathbb{H}}_{\rho_0}, \quad \rho_0 >> 1,$$

and the first Dirichlet eigenfunction in $\mathcal{A}(\rho_0)$ namely a function $\psi > 0$ which satisfies the following eigenvalue problem

$$\begin{cases} \Delta_{\mathbb{H}}\psi = -\lambda\psi & \text{in} \quad \mathcal{A}(\rho_0), \\ \psi = 0 & \text{on} \quad \partial\mathcal{A}(\rho_0) \end{cases}$$

where $\lambda = \lambda(\rho_0)$.
We have already shown (see Lemma (2.3)) that $\lambda = \lambda(\rho_0) \to \lambda_1 = \frac{(n-1)^2}{4}$ as $\rho_0 \to \infty$.
If we multiply the equation in (P1) with $\phi = k(t)\psi(x)$ and we integrate by parts then we get

$$-\int\int_{Q_{\tau_1}} u\{\Delta_{\mathbb{H}}(k\psi)+(k\psi)_t\}dVdt = \int_{\mathbb{H}^n} u_0 k(0)\psi(x)dV + \int\int_{Q_{\tau_1}} h(t)|u|^{p-1}u(k\psi)dVdt,$$

$$-\int_0^T\int_{Q_{\tau_1}} u\{k\Delta_{\mathbb{H}}\psi+k'\psi\}dVdt = \int_{\mathbb{H}^n} u_0\psi(x)dV + \int\int_{Q_{\tau_1}} h(t)|u|^{p-1}uk\psi dVdt,$$

$$\int_0^T\int_{\mathcal{A}_{\rho_0}} -u\{-k\lambda + k' + |u|^{p-1}kh\}\psi dVdt \geq \int_{\mathcal{A}(\rho_0)} u_0\psi(x)dV,$$

$$\int_0^T\int_{\mathcal{A}_{\rho_0}} \{\frac{k\lambda - k'}{kh}u - u^p\}kh\psi dVdt \geq \int_{\mathcal{A}(\rho_0)} u_0\psi(x)dV.$$

We now set $\omega := \frac{-k'+\lambda k}{kh}$. As a function of u, $\omega u - u^p$ assume its maximum at $u = \left(\frac{\omega}{p}\right)^{\frac{1}{p-1}}$ and the value of the function in that point is $\frac{p-1}{p^{\frac{p}{p-1}}}\omega_+^{\frac{p}{p-1}}$.
Calling $p' = \frac{p}{p-1}$ $\quad (\frac{1}{p} + \frac{1}{p'} = 1)$ we have

$$\sup_{u>0}(\omega u - u^p) = \frac{p-1}{p^{p'}}\omega_+^{p'}.$$

A suitable choice of the function $k(t)$ has to be made. So let

$$k(t) = \begin{cases} e^{\lambda t - \beta H(t)} & \text{if} \quad 0 < t < T/2, \\ \alpha(T-t)^{p'} & \text{if} \quad T/2 \leq t \leq T. \end{cases}$$

The parameters α and β are chosen such that $k(t)$ and $k'(t)$ are continuous at $\frac{T}{2}$ so

- $e^{\lambda\frac{T}{2} - \beta H(\frac{T}{2})} = \alpha(\frac{T}{2})^{p'} \Rightarrow \alpha = \left(\frac{2}{T}\right)^{p'} e^{\lambda\frac{T}{2} - \beta H(\frac{T}{2})}$

- $[\lambda - \beta H'(\frac{T}{2})]^{\lambda\frac{T}{2}} = -\alpha p'\left(\frac{T}{2}\right)^{p'-1} \Rightarrow \beta = \frac{\lambda + \frac{2p'}{T}}{h(\frac{T}{2})}$.

Now we compute the function $\omega_+^{p'} hk$: in the interval $(\frac{T}{2}, T]$ we have

$$\begin{aligned}
\omega_+^{p'} hk = \left(\frac{-k' + \lambda k}{kh}\right)_+^{p'} hk = \\
&= \left[\frac{-\alpha p'(T-t)^{p'-1} + \lambda\alpha(T-t)^{p'}}{kh}\right]_+^{p'} hk = \\
&= \frac{\alpha^{p'}}{h^{p'}}\left[\frac{\left(-p'(T-t)^{p'-1} + \lambda\alpha(T-t)^{p'}\right)}{(T-t)^{p'}}\right]_+^{p'} hk = \\
&= \frac{\alpha[-p' + \lambda(T-t)]^{p'}}{h^{p'-1}}.
\end{aligned}$$

In the interval $(0, \frac{T}{2})$ instead

$$\begin{aligned}
\omega_+^{p'} hk = \left(\frac{-k' + \lambda k}{kh}\right)_+^{p'} hk = \\
&= \left[\frac{(\lambda - \beta H'(t) + \lambda)e^{\lambda t - \beta H(t)}}{kh}\right]_+^{p'} hk \\
&= \beta^{p'} e^{\lambda t - \beta H(t)} h.
\end{aligned}$$

So finally we have

$$\omega_+^{p'} hk = \begin{cases} \beta^{p'} e^{\lambda t - \beta H} h & \text{in} \quad (0, \frac{T}{2}], \\ \frac{\alpha[-p' + \lambda(T-t)]_+^{p'}}{h^{p'-1}} & \text{in} \quad (\frac{T}{2}, T] \end{cases}$$

and by this we deduce that $\omega_+^{p'} hk$ is a continuous non-negative function $t \in (0, T]$ and it is integrable in $(0, T)$ (see 2.1).
Putting together all these results we can conclude that

$$\inf_{\mathcal{A}(\rho_0)} u_0 \leq \frac{p-1}{p^{p'}} \int_0^T \omega_+^{p'} hkds. \tag{2.19}$$

The right-hand side of the above inequality can be bounded by a constant which depends only on T, p' but not on ρ_0. Letting $\rho_0 \to \infty$, on the one side, by assumption, the limit of u_0 diverges and on the other side we found the the $\inf_{\mathcal{A}(\rho_0)} u_0$ remains bounded. This is a clear contradiction that proves the nonexistence of a local solution of problem (P1) under the assumption (2.18). □

Remark 2.4. *It is possible to weaken the hypothesis* (2.18). *We can only require that* $\lim_{d(x,0)\to\infty} u_0(x) = \infty$ *in a given cone centered in* $x = 0$. *In fact the limit in* (2.18) *need not to be uniform as* $d(x, 0) \to \infty$.

2.3.2 Finite time blow-up

We now want to give a blow-up criterion. Before this we need two preliminary Lemmas.

Lemma 2.4. *Let* $u_0 \geq 0$, $u_0 \neq 0$. *Then for any* $\epsilon > 0$ *there exists a function* $f \in C(\mathbb{H}^n)$, $f(x) > 0$ *for any* $x \in \mathbb{H}^n$ *(f depending only on* ϵ,n *and* u_0*) such that*

$$(e^{t\Delta_{\mathbb{H}}} u_0)(x) \geq f(x) t^{-\frac{3}{2}} e^{-\lambda_1 t} \tag{2.20}$$

for any $t \in [\epsilon, \infty)$ *and* $x \in \mathbb{H}^n$.

Proof. By assumptions $u_0 \neq 0$ and u_0 is positive we can infer that there exists $z \in \mathbb{H}^n$ and $\delta > 0$ such that $u_0(y) \geq \delta$ in some ball $\{d(z, y) \leq \gamma\}$. Without loss of generality we can suppose $z = 0$ and thus $u_0(y) \geq \delta$ in the geodesic ball $\mathcal{B}^{\mathbb{H}}_{\gamma} = \{d(z, y) \leq \gamma\}$ and $\gamma < 1$.
Using estimate (1.18) we can write

$$\begin{aligned}
\left(e^{t\Delta_{\mathbb{H}}} u_0\right)(x) \geq \delta c_n^{-1} (4\pi t)^{-\frac{n}{2}} e^{\lambda_1 t} \int_{B^{\mathbb{H}}_{\gamma}} [1 + d(x, y)][1 + d(x, y) + t]^{\frac{n-3}{2}} \times \\
\times e^{-\frac{d(x,y)^2}{4t} - \frac{n-1}{2} d(x,y)} dV_y \geq \\
\geq \delta c_n^{-1} (4\pi)^{-\frac{n}{2}} t^{-\frac{3}{2}} e^{\lambda_1 t} \times \\
\times \int_{B^{\mathbb{H}}_{\gamma}} [1 + d(x, y)] g(x, y) e^{-\frac{d(x,y)^2}{4t} - \frac{n-1}{2} d(x,y)} dV_y
\end{aligned}$$

for any $t \in [\epsilon, \infty)$ and $x \in \mathbb{H}^n$, where

$$g(x, y) = g_{\epsilon,n}(x, y) := \begin{cases} 1 & \text{if} \quad n \geq 3, \\ \left[1 + \frac{1+d(x,y)}{\epsilon}\right]^{-\frac{1}{2}} & \text{if} \quad n = 2. \end{cases}$$

Setting

$$f(x) := \delta c_n^{-1} (4\pi)^{-\frac{n}{2}} \int_{\mathcal{B}^{\mathbb{H}}_{\gamma}} [1 + d(x, y)] g(x, y) e^{-\frac{d(x,y)^2}{4\epsilon} - \frac{n-1}{2} d(x,y)} dV_y$$

(notice that $f(x) > 0$) we have

$$\left(e^{t\Delta_{\mathbb{H}}} u_0\right)(x) \geq f(x) t^{-\frac{3}{2}} e^{-\lambda_1 t}$$

that is what we were looking for. □

Lemma 2.5. *Let u be a mild solution of problem* (P1) *in* $[0,T)$ *and set*

$$\phi_x(t) := \int_{\mathbb{H}^n} q(x,z,T-t)u(z,t)dV_z. \tag{2.21}$$

Note that $\phi_x(0) = (e^{T\Delta_{\mathbb{H}}}u_0)(x)$.
Then

$$[\phi_x(0)]^{p-1} \le \frac{1}{(p-1)H(T)} \quad \text{for any } x \in \mathbb{H}^n \tag{2.22}$$

where H is defined in (2.14).

Proof. Let us recall that, by definition, a mild solution of problem (P1) is of the form

$$u(x,t) = \int_{\mathbb{H}^n} q(x,y,t)u_0 dV_y + \int\int_{Q_t} q(x,y,t-s)h(s)|u|^{p-1}u dV(y)ds.$$

If we multiply this expression by $q(x,z,T-t)$ and we integrate over $\mathbb{H}^n$ we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} q(x,z,T-t)u(x,t)dV_z &= \\ &= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} q(x,z,T-t)q(z,y,t)u_0 dV(y)dV_z + \\ &+ \int_{\mathbb{H}^n}\int\int_{Q_t} q(x,z,T-t)q(z,y,t-s)h(s)|u|^{p-1}u dV_y dV_z ds.\end{aligned}$$

Now applying the semigroup property ((*iii*) of Definition (1.1)) we get

$$\begin{aligned}\int_{\mathbb{H}^n} q(x,z,T-t)u(x,t)dV_z &= \\ &= \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} q(x,z,T-t)q(z,y,t)dV_z\right)u_0 dV_y + \\ &+ \int\int_{Q_t}\left(\int_{\mathbb{H}^n} q(x,z,T-t)q(z,y,t-s)\right)h(s)|u|^{p-1}u dV_y ds,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{H}^n} q(x,z,T-t)u(x,t)dV_z &= \\ &= \int_{\mathbb{H}^n} q(x,y,T)u_0 dV_y + \\ &+ \int\int_{Q_t} q(x,y,T-s)h(s)u^p(y,s)dV(y)ds,\end{aligned}$$

that is equivalent to write

$$\phi_x(t) = \phi_x(0) + \int \int_{Q_t} q(x,y,T-s)h(s)u^p(y,s)dV(y)ds. \tag{2.23}$$

Using Jensen's inequality to (2.21) we get

$$[\phi_x(s)]^p \le \int_{\mathbb{H}^n} q(x,y,T-s)u^p(y,s)$$

and thus combining it with (2.23) we get

$$\int_0^t h(s)[\phi_x(s)]^p ds \le \phi_x(t) - \phi_x(0).$$

Now, deriving this inequality

$$\frac{d}{dt}\int_0^t h(s)[\phi_x(s)]^p ds \le \frac{d}{dt}\phi_x(t) - \phi_x(0)$$

we get

$$[\phi_x(t)]^p h(t) \le \frac{d\phi_x(t)}{dt}$$

and thus integrating the latter inequality

$$h(t)dt \le \frac{d\phi_x(t)}{[\phi_x(t)]^p}$$

we get

$$H(t) + k \le \frac{[\phi_x(t)]^{-p+1}}{-p+1}$$

that is

$$(-p+1)(H(t)+k) \le \phi_x(t)^{-p+1}.$$

Imposing initial data we find $k = \frac{-[\phi_x(0)]^{-p+1}}{p-1}$.
Finally we have

$$H(t) + \frac{-[\phi_x(0)]^{-p+1}}{p-1} \le \frac{[\phi_x(t)]^{-p+1}}{-p+1}$$

$$H(t) \le \frac{\phi_x(t)^{-p+1}}{-p+1} + \frac{\phi_x(t)^{p-1}}{p-1}$$

and so

$$(p-1)H(t) \le \frac{1}{[\phi_x(0)]^{p-1}} - \frac{1}{[\phi_x(t)]^{p-1}} \le \frac{1}{[\phi_x(0)]^{p-1}}.$$

This allows us to conclude that

$$[\phi_x(0)]^{p-1} \geq \frac{1}{(p-1)H(t)}.$$

□

Now we are ready to give the blow-up criterion announced previously

Theorem 2.3 (Blow-up criterion)**.** *Let*

$$\lim_{T\to\infty} \frac{[H(T)]^{\frac{1}{p-1}}}{T^{\frac{3}{2}}e^{\lambda_1 T}} = \infty. \tag{2.24}$$

Then, every nontrivial weak solution of problem (P1) *blows-up in finite time.*

Proof. Combining together (2.20),(2.18) and (2.22) we obtain

$$f(x)T^{-\frac{3}{2}}e^{-\lambda_1 t} \leq \phi_x(0) \leq \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} [H(t)]^{-\frac{1}{p-1}}.$$

Therefore we have

$$\frac{[H(t)]^{\frac{1}{p-1}}}{T^{\frac{3}{2}}e^{\lambda_1 T}} \leq \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \frac{1}{f(x)} \quad c \in \mathbb{H}^n.$$

If u is a global solution of (P1) this inequality holds for any $T > 0$. For $T \to \infty$, however, it contrasts with assumption (2.30). Therefore the solution cannot exists for all $t \in (0, \infty)$ and it blows-up in finite time. □

Remark 2.5. *In $\mathbb{R}^n$ the counterpart of estimate* (2.18) *is*

$$(e^{t\Delta}u_0)(x) \geq \bar{f}(x)t^{-\frac{N}{2}}$$

for any $t \in [\epsilon, \infty)$, $x \in \mathbb{R}^n$ where

$$\bar{f}(x) := \delta c_n^{-1}(4\pi)^{-\frac{n}{2}} \int_{B_\gamma^{\mathbb{R}}} r^{-\frac{|x-y|}{4\epsilon}}\, dy.$$

Besides, the necessary condition of the blow-up criterion in $\mathbb{R}^n$ is

$$\lim_{T\to\infty} \frac{[H(T)]^{\frac{1}{p-1}}}{T^{\frac{N}{2}}} = \infty.$$

If $h(t) = 1$ then $H(T) = \int_0^T h(s)ds = \int_0^T 1ds = T$. Thus

$$\lim_{T\to\infty} \frac{[T]^{\frac{1}{p-1}}}{T^{\frac{N}{2}}} = \infty$$

and this happened if $\frac{1}{p-1} - \frac{N}{2} > 0$ whence when $p > 1 + \frac{2}{N}$, in accordance with the Fujita's result.

2.4 Global existence

Before deriving sufficient conditions for the existence of global solutions of problem (P1) we need to introduce the concept of *ground state*.

2.4.1 Ground states

Definition 2.6. A **ground state** in $\mathbb{H}^n$ is a positive classical solution of the equation

$$\Delta_{\mathbb{H}^n}\phi + \lambda\phi = 0 \quad \text{in} \quad \mathbb{H}^n. \tag{2.25}$$

We seek a solution of the form

$$\phi(r,\theta) = w(r)\alpha(\theta).$$

This means that

$$\Delta_{\mathbb{H}^n}\phi + \lambda\phi = \frac{\partial^2}{\partial r^2}\phi + ((n-1)\coth r)\frac{d}{dr}\phi + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}\phi + \lambda\phi =$$
$$= w''(r)\alpha(\theta) + (n-1)\coth r w'(r)\alpha(\theta) + \frac{1}{\sinh^2 r}w(r)\Delta_\theta\alpha(\theta) + \lambda w(r)\alpha(\theta)$$

If $-\zeta_k := -(k-1)(k+n-3)$ is an eigenvalue of Δ_θ on $\mathbb{S}^{n-1}$ then α satisfies $\Delta_\theta\alpha + \zeta_k\alpha = 0$ and

$$\begin{cases} w''(r) + (n-1)\coth r w'(r) - \frac{1}{\sinh^2 r}w(r)\zeta_k\alpha(\theta) + \lambda w(r) = 0, \\ \Delta_\theta\alpha + \zeta_\kappa\alpha = 0. \end{cases}$$

Since we are interested in radial solutions, then the equation to study, for $k = 1$, is

$$w''(r) + (n-1)\coth r w'(r) + \lambda w(r) = 0 \quad \text{in} \quad [0,\infty). \tag{2.26}$$

If we choose $w(r) := (\sinh r)^{-\frac{n-1}{2}}u(r)$ we obtain

$$\begin{aligned}
&w''(r) + (n-1)\coth r w'(r) + \lambda w(r) = \\
&-\left(\frac{n-1}{2}\right)\left(-\frac{n-1}{2}-1\right)(\sinh r)^{-\frac{n-1}{2}-2}(\cosh r)^2 u(r) - \\
&-\left(\frac{n-1}{2}\right)(\sinh r)^{-\frac{n-1}{2}-1}\sinh r u(r) - \left(\frac{n-1}{2}\right)(\sinh r)^{-\frac{n-1}{2}-1}\cosh r u'(r) + \\
&+\left(\frac{n-1}{2}\right)(\sinh r)^{-\frac{n-1}{2}-1}\cosh r u'(r) + (\sinh r)^{-\frac{n-1}{2}}u''(r) - \\
&-\frac{(n-1)^2}{2}(\coth r)(\sinh r)^{-\frac{n-1}{2}-1}\cosh r u(r) + \\
&+(n-1)\coth r(\sinh r)^{-\frac{n-1}{2}-1}u'(r) + \lambda(\sinh r)^{-\frac{n-1}{2}}u(r) = 0
\end{aligned}$$

$$
\begin{aligned}
(\sinh r)^{-\frac{n-1}{2}}u''(r) = &-\left(\frac{n-1}{2}\right)\left(-\frac{n-1}{2}-1\right)(\sinh r)^{-\frac{n-1}{2}-2}(\cosh r)^2 u(r)+\\
&+\left(\frac{n-1}{2}\right)(\sinh r)^{-\frac{n-1}{2}-1}\sinh r u(r)+\\
&+\frac{(n-1)^2}{2}(\coth r)(\sinh r)^{-\frac{n-1}{2}-1}\cosh r u(r)-\\
&-\lambda(\sinh r)^{-\frac{n-1}{2}}u(r)
\end{aligned}
$$

so that

$$
\begin{aligned}
u''(r) = &\Big[-\left(\frac{n-1}{2}\right)\left(-\frac{n-1}{2}-1\right)(\sinh r)^{-2}(\cosh r)^2 - (\frac{n-1}{2})+\\
&+\frac{(n-1)^2}{2}(\coth r)(\sinh r)^{-1}\cosh r+\lambda\Big]u(r) =\\
&=\left\{\left[-\left(\frac{n-1}{2}\right)\left(-\frac{n-1}{2}-1\right)+\frac{(n-1)^2}{2}\right]\frac{\cosh^2 r}{\sinh^2 r}-\lambda+\left(\frac{n-1}{2}\right)\right\}u(r) =\\
&=\left[\left(\frac{n^2-4n+3}{4}\right)\left(\frac{1+\sinh^2 r}{\sinh^2 r}\right)-\lambda+\left(\frac{n-1}{2}\right)\right]u(r)
\end{aligned}
$$

and thus

$$
u''(r) = \left[\frac{(n-2)^2-1}{4}\frac{1}{\sinh^2 r}+\lambda_1-\lambda\right]u(r) =: -q(r)u. \tag{2.27}
$$

Theorem 2.4. *A necessary condition for the existence of the ground state is $\lambda \leq \lambda_1$.*

Proof. If u is a positive solution for equation (2.27) for $\lambda > \lambda_1$ then we can choose a so large that for $\rho > a$ we have

$$
q(r) > \frac{\lambda-\lambda_1}{2}.
$$

If we consider the ordinary differential equation

$$
u'' + \frac{\lambda-\lambda_1}{2}u = 0 \tag{2.28}
$$

then its solution is

$$
u(x) = c\sin\left(\sqrt{\frac{\lambda-\lambda_1}{2}}x\right)
$$

that vanishes when $\sqrt{\frac{\lambda-\lambda_1}{2}}x = m\pi$ and thus when $x = \frac{m\pi}{\sqrt{\frac{\lambda-\lambda_1}{2}}}$ with $m \in \mathbb{N}$.

Therefore we can apply the Sturm Comparison Principle ([16], Theorem 19, pg 45) to the equations (2.27) and (2.28) and conclude that u vanishes at least once in the interval $\left(\frac{m\pi}{\sqrt{\frac{\lambda-\lambda_1}{2}}}, \frac{(m+1)\pi}{\sqrt{\frac{\lambda-\lambda_1}{2}}}\right)$ where $m \in \mathbb{N}$ and $\frac{m\pi}{\sqrt{\frac{\lambda-\lambda_1}{2}}} > a$. □

Lemma 2.6. *For any $\lambda \leq \lambda_1$ and $c > 0$ there exists a unique ground state w such that $w(0) = c$. There holds*

$$\lim_{\rho\to\infty} w(\rho)e^{-\nu\rho} = k$$

for some $k > 0$, where

$$\nu := \sqrt{\lambda_1 - \lambda} - \sqrt{\lambda_1}.$$

In particular, if $\lambda > 0$ then

$$\lim_{\rho\to\infty} w(\rho) = 0.$$

Proof. By classical theory of differential equation with singular coefficients (see [4]) the equation (2.26) has one regular solution w such that $w(0) = c$ for every $c \in \mathbb{R}$. By assumption $c > 0$ so that, by the previous construction, the function $u(\rho) = (\sinh\rho)^{\frac{n-1}{2}} w(\rho)$ satisfies (2.27) and has the properties of vanishing at $\rho = 0$ and of being positive in a neighborhood of the origin, say $B_\epsilon(0)$. All this implies that $u'(\rho) > 0$ for small ρ.
Now, if $n \geq 3$ then $q(\rho) > 0$ and since, by assumption, $\lambda_0 - \lambda \geq 0$ then $u''(\rho) > 0$ and by convexity $u'(\rho) \geq 0$ for all $\rho > 0$.
Instead, if $n = 2$ we have to make different considerations. We make the change of variable. We set $t = \cosh\rho$ which is $\rho = \operatorname{acosh} t$.

$$d(\operatorname{acosh} t) = d\left(\log(t + \sqrt{t^2-1})\right)\frac{1}{\sqrt{t^2-1}}$$

and

$$\cosh(\operatorname{acosh} t) = \frac{t}{\sqrt{t^2-1}}$$

in fact

$$\begin{aligned}\sinh(\operatorname{acosh} t) &= \frac{e^{\ln(t+\sqrt{t^2-1})} - e^{-ln(t+\sqrt{t^2-1})}}{2} = \\ = \frac{(t+\sqrt{t^2-1}) - (t+\sqrt{t^2-1})}{2} &= \frac{(t+\sqrt{t^2-1})^{-1}}{2}((t+\sqrt{t^2-1})-1) = \\ = \frac{t^2+t\sqrt{t^2-1}}{t+\sqrt{t^2-1}} - \frac{1}{t+\sqrt{t^2-1}} &= t - t + \sqrt{t^2-1} = \sqrt{t^2-1}\end{aligned}$$

thus

$$\coth(\operatorname{acosh} t) = \frac{\cosh t}{\sinh t} = \frac{t}{\sqrt{t^2-1}}.$$

Therefore , we have

$$w' = \frac{dw}{d \operatorname{acosh} t} = \sqrt{t^2-1}\frac{dw}{dt},$$

$$w'' = \frac{d^2 w}{d(\operatorname{acosh} t)^2} = \sqrt{t^2-1}\frac{dw}{dt}\left(\sqrt{t^2-1}\frac{dw}{dt}\right) =$$
$$= \sqrt{t^2-1}\sqrt{t^2-1}\frac{1}{2}2t\frac{1}{\sqrt{t^2-1}}\frac{dw}{dt} + \frac{d^2w}{dt^2}(t^2-1).$$

Consequently the equation (2.26) (for $n = 2$) becomes

$$(t^2-1)\frac{d^2w}{dt^2} + t\frac{dw}{dt} + \frac{t}{\sqrt{t^2-1}}\sqrt{t^2-1}\frac{dw}{dt} + \lambda w = 0$$

that is

$$-2t\frac{dw}{dt} + \sqrt{1-t^2}\frac{d^2w}{dt^2} - \lambda w = 0.$$

The last equation is the Legendre differential equation which can be written in the form

$$\frac{d}{dt}\left((1-t^2)\frac{dw}{dt}\right) - \lambda w = 0$$

which is the so-called "associated Legendre differential equation" corresponding to the case $m = 0$.
The indicial equation associated to the Legendre function is [2]

$$-\nu(\nu+1) - \lambda = 0$$

and since $\lambda \leq \lambda_0 = \frac{1}{4}$ then ν is a real number.
The ground state solution is a polynomial function $P_\nu(x)$ where ν is a root of the indicial equation.
Applying a classical result on the distribution of zeros of Legendre functions we can conclude that $P_\nu(t)$ does not vanish in $(1, \infty)$
If $\lambda > 0$ the solution of (2.27) behaves like $e^{\pm\sqrt{\lambda_1 - \lambda + \eta}\rho}$ with $\eta \to 0$ as $\rho \to \infty$ for large ρ. Since u is increasing then $u(\rho) \approx e^{\alpha\rho}$ and thus

$$\lim_{\rho\to\infty} (\sinh)^{-\frac{n-1}{2}} e^{\alpha\rho} \approx \lim_{\rho\to\infty} e^{-\rho(n-1)} e^{\alpha\rho} = \lim_{\rho\to\infty} e^{(\alpha-(N-1))\rho} = 0.$$

[3] $\square$

[2] see [4], chap.4
[3] $\alpha - (N-1) < 0$.

Remark 2.6. *We notice that the ground state solution w found in the previous Lemma does not belong to $L^2(\mathbb{H}^n)$ in fact*

$$\int_0^\infty [w(\rho)]^2(\sinh\rho)^{n-1}d\rho = \int_0^\infty u(\rho)^2(\sinh\rho)^{-(n-1)}(\sinh\rho)^{(n-1)} \approx$$
$$\approx \int_0^\infty e^{2\sqrt{\lambda_1-\lambda}\rho}d\rho = \infty$$

We can now prove the following result:

Theorem 2.5 (Global existence criterion)**.** *Let*

$$\tilde{h}(t) := h(t)e^{-(p-1)\lambda_1 t}, \quad \tilde{H}(t) = \int_0^t \tilde{h}(s)ds$$

and set $\tilde{H}_\infty := \lim_{t\to\infty}\tilde{H}(t)$. Suppose that $\tilde{H}_\infty < \infty$ and let w be a ground state corresponding to $\lambda = \lambda_1$ such that

$$||w||_\infty < \left[\frac{1}{(p-1)\tilde{H}_\infty}\right]^{\frac{1}{p-1}}. \tag{2.29}$$

If $u_0 \leq w$ then the solution of the problem (P1) *is global.*

Proof. Let w be the ground state corresponding to the eigenvalue $\lambda = \lambda_1$ such that $w(0) = c$ ($c > 0$) (compare Lemma 2.6) and $||w|| < \infty$ [4] and ζ be the solution of the problem

$$\begin{cases} \zeta' = ||w||_\infty^{p-1}\tilde{h}(t)\zeta^p, \\ \zeta(0) = 1. \end{cases} \tag{2.30}$$

We now compute the solution of this ordinary differential equation:

$$\begin{aligned}
&\frac{d\zeta}{dt} = ||w||_\infty^{p-1}\tilde{h}(t)\zeta^p, \\
&\frac{d\zeta}{\zeta^p} = ||w||_\infty^{p-1}\tilde{h}(t)dt, \\
&\int_{\zeta(0)}^{\zeta(t)} \frac{d\eta}{\eta^p} = \int_0^t \frac{d\zeta}{\zeta^p} = ||w||_\infty^{p-1}\int_0^t \tilde{h}(s)ds, \\
&\frac{\zeta^{-(p-1)}(t) - \zeta^{-(p-1)}(0)}{-(p-1)} = ||w||_\infty^{p-1}\tilde{H}(t).
\end{aligned}$$

Hence we have

$$\zeta = \left[1 - (p-1)||\omega||_\infty^{p-1}\tilde{H}(t)\right]^{-\frac{1}{p-1}} \tag{2.31}$$

[4] It is clear $||w||_\infty < \infty$ since w is smooth and $\lim_{\rho\to\infty} w(\rho) = 0$.

Let us show that, provided of assumption (2.23), the function

$$\bar{u}(x,t) := e^{-\lambda_1 t}\zeta(t)w(x)$$

is an upper solution of (P1) for all $t > 0$.
It is very easy to check that $\bar{u}$ satisfies

$$\bar{u}_t \geq \Delta_{\mathbb{H}}\bar{u} + h(t)|u|^{p-1}\bar{u}.$$

In fact deriving we have

$$-\lambda_1 e^{-\lambda_1 t}\zeta w(x) + e^{-\lambda_1 t}\zeta'(t)w(x) \geq -\lambda_1 e^{-\lambda_1 t}\zeta(t)w(x) + h(t)|\bar{u}|^{p-1}e^{-\lambda_1 t}\zeta w(x),$$

which is

$$-\lambda_1\zeta w(x) + \zeta'(t)w(x) \geq -\lambda_1\zeta(t)w(x) + h(t)||w||^{p-1}e^{-\lambda_1(p-1)t}\zeta^p w(x).$$

Thus the condition to be satisfied is

$$\zeta'(t) \geq h(t)||w||^{p-1}e^{-\lambda_1(p-1)t}\zeta^p$$

and this last inequality is always true because ζ satisfies (2.30).
Since, by hypothesis, $u_0 \leq w = \bar{u}(0)$ then $\bar{u}$ is clearly an upper solution.
If $\tilde{H}_\infty < \infty$ and we choose $c > 0$ such that (2.29) holds, then the upper solution $\bar{u}$ exists for all $t > 0$.
From this argument we can deduce that $\bar{u}$ is always an upper solution whilst $\underline{u} = 0$ is always a lower solution so that, by standard comparison results, $0 \leq u \leq \bar{u}$.

□

2.5 A general result

Theorem 2.6. *1. Let $h(t) = 1$. Then for small initial data u_0 there exist global solutions of problem* (P1)*.*

2. Let $h(t) = t^q$ with $q > -1$. Then for small initial data u_0 there exist global solutions of problem (P1)*.*

3. Let $h(t) = e^{\alpha t}$ and set $p^\star_{\mathbb{H}} := 1 + \frac{\alpha}{\lambda_1}$ $(\alpha > 0)$.

(a) If $1 < p < p^\star_{\mathbb{H}}$, every nontrivial solution of problem (P1) *blows up in finite time.*

(b) If $p > p^\star_{\mathbb{H}}$, problem (P1) *possesses global solutions for small initial data.*

(c) If $p = p^\star_{\mathbb{H}}$ and $\alpha > \frac{2}{3}\lambda_1$ there exist global solutions.

Proof.

- If $h(t) = 1$ then

$$\tilde{H}_\infty = \int_0^\infty e^{-(p-1)\lambda_1 s} ds < \infty$$

and we can apply Theorem 2.5.

- If $h(t) = t^q$ then it is very simple to verify that we have global existence for $q > -1$ in fact

$$\tilde{h}(t) = e^{-(p-1)\lambda_1 t} t^q$$

so that

$$\tilde{H}_\infty = \lim_{t\to\infty} \int_0^t e^{-(p-1)\lambda_1 s} s^q ds = \int_0^\infty e^{-(p-1)\lambda_1 s} s^q ds =$$
$$= \lim_{t\to\infty} \int_0^1 e^{-(p-1)\lambda_1 s} s^q ds + \lim_{t\to\infty} \int_1^\infty e^{-(p-1)\lambda_1 s} s^q ds.$$

For every q we have

$$|e^{-(p-1)\lambda_1 t} t^q| \le |e^{-(p-1)\lambda_1 s} e^{(p-1)\lambda_1 + \epsilon}| \le |e^{-\epsilon}|$$

and this estimate makes the second integral always finite.

As far as the first integral is concerned we have that in a neighborhood of the origin

$$|e^{-(p-1)\lambda_1 s} s^q| \le |s^q|$$

so that the integral is bounded from below by

$$\int_0^1 s^q ds = \left[\frac{s^{q+1}}{q+1}\right].$$

We need to distinguish two cases: if $q+1 > 0$ then the integral in finite in $(0,1)$ whilst if $q+1 < 0$ then the integral is divergent.

Therefore if $q > -1$ then we can apply Theorem 2.5.

- If $h(t) = e^{\alpha t}$ and $p^\star_{\mathbb{H}} = 1 + \frac{\alpha}{\lambda_1}$ then the condition (2.24) reads as

$$\frac{\left[\frac{e^{\alpha T}}{\alpha}\right]^{\frac{1}{p-1}}}{T^{3/2} e^{\lambda_1 t}} = \frac{e^{\frac{\alpha T}{p-1}}}{T^{3/2} e^{\lambda_1 t} \alpha^{\frac{1}{p-1}}} \approx \frac{e^{\frac{\alpha T}{p-1}}}{e^{\lambda_1} T}.$$

which diverges if $\frac{\alpha T}{p-1} > \lambda_1 T$. Applying Theorem 2.3 we can infer that for $1 < p < p^\star$ the solutions to equation (P1) with $h(t) = e^{\alpha t}$ blow up in finite time.

Instead, if $p > p^\star_{\mathbb{H}}$ then the problem (P1) possesses global solutions thanks to Theorem 2.5 in fact the condition

$$\tilde{H}_\infty := \lim_{t\to\infty} \tilde{H} < \infty$$

is satisfied if

$$\lim_{t\to\infty} \frac{e^{(\alpha-(p-1)\lambda_1)t}}{\alpha-(p-1)\lambda_1} < \infty$$

and thus we need $\alpha - (p-1)\lambda_1 < 0$ that is $\frac{\alpha}{p-1} < \lambda_1$.

Finally we discuss the critical case ($p = p^\star$ so that $\alpha = \lambda_1(p-1)$):

we define the function

$$z(x,t) := \xi(t)q(x,0,t+t_0) \quad (t_0 > 0).$$

Since q is the heat kernel (i.e. satisfies $q_t = \Delta_{\mathbb{H}} q$) then the differential equation for z is

$$\begin{aligned}
&z_t(x,t) - \Delta_{\mathbb{H}} z(x,t) - e^{\alpha t} z^p(x,t) = \\
&= \dot{\xi}(t)q(x,0,t+t_0) + \xi(t)q_t(z,0,t+t_0) \\
&- \xi(t)\Delta_{\mathbb{H}} q(x,0,t+t_0) - e^{\alpha t}\xi^p(t)q^p(x,0,t+t_0) = \\
&= q(x,0,t+t_0)\left[\dot{\xi}(t) - e^{\alpha t} q^{p-1}(x,0,t+t_0)\xi^p(t)\right].
\end{aligned}$$

so

$$z_t - \Delta_{\mathbb{H}} z - e^{\alpha t} z^p = q(\dot{\xi} - e^{\alpha t} q^{p-1}\xi^p). \tag{2.32}$$

By (1.18) we get

$$\begin{aligned}
q(x,0,t+t_0) &\le c_n(4\pi(t+t_0))^{-\frac{n}{2}}(1+d(x,0))\times \\
&\times (1+d(x,0)+(t+t_0))^{\frac{n-3}{2}}\, e^{-\lambda_1(t+t_0)-\frac{n-1}{2}-\frac{d^2(x,0)}{4t}} \le \\
&\le c_n(4\pi)^{-\frac{n}{2}}(t+t_0)^{-\frac{3}{2}}e^{-\lambda_1(t+t_0)}(1+\rho(x))g_1(x)e^{-\frac{(n-1)d}{2}}
\end{aligned}$$

where

$$g_1(x) = \begin{cases} 1 & n \leq 3 \\ (1 + \frac{1+\rho}{t_0})^{-\frac{1}{2}} & n > 3 \end{cases}.$$

If we call $k_1 = c_n(4\pi)^{-\frac{n}{2}}(1+\rho)g_1(x)e^{-\frac{n-1}{2}\rho} > 0$ we have found that

$$q(x, 0, t + t_0) \leq k_1(t + t_0)^{-\frac{3}{2}} e^{-\lambda_1(t+t_0)} \tag{2.33}$$

for any $x \in \mathbb{H}^n$ and $t \in [0, \infty)$.

Inserting the estimate (2.33) in (2.32) we obtain

$$\begin{aligned} z_t - \Delta_{\mathbb{H}} z - e^{\alpha t} z^p &\geq q(\dot{\xi} - e^{\alpha t} q^{p-1} \xi^p) = \\ &= q \left[\dot{\xi} - e^{\alpha t} \left(k_1(t + t_0)^{-\frac{3}{2}} e^{-\lambda_1(t+t_0)} \right)^{p-1} \xi^p \right] = \\ &= q \left[\dot{\xi} - e^{\alpha t} k_1^{p-1} (t + t_0)^{-\frac{3}{2}(p-1)} e^{-\lambda_1(t+t_0)(p-1)} \xi^p \right] = \\ &= q \left[\dot{\xi} - e^{\alpha t - (p-1)(\lambda t + \lambda t_0)} (t + t_0)^{-\frac{3}{2}(p-1)} k_1^{p-1} \xi^p \right] = \\ &= q \left[\dot{\xi} - e^{-(p-1)\alpha t_0} (t + t_0)^{-\frac{3}{2}(p-1)} k_1^{p-1} \xi^p \right]. \end{aligned}$$

Thus, if ξ satisfies the problem

$$\dot{\xi} = \kappa(t + t_0)^{-\frac{3\alpha}{2\lambda_1}} \xi^{1+\frac{\alpha}{\lambda_1}}, \tag{2.34}$$

where $\kappa = k_1^{\frac{\alpha}{\lambda_1}} e^{-\alpha t_0}$

and if

$$z(x, 0) = \xi(0)q(x, 0, t_0) \geq u_0(x)$$

then z is clearly an upper solution of (P1).

It only remains to find a condition which ensure the existence of the function $\xi(t)$.

So we can do nothing but solving the equation (2.34) using a similar strategy to the one used for solving (2.30).

By (2.34), we write

$$d\xi = \kappa(t + t_0)^{-\frac{3\alpha}{2\lambda_1}} \xi^{1+\frac{\alpha}{\lambda_1}} dt$$

and we integrate. Hence we get

$$\int_{\zeta_0}^{\zeta(t)} \frac{d\eta}{\eta^{1+\frac{\alpha}{\lambda_1}}} = \int_0^t \frac{d\zeta}{\zeta^{1+\frac{\alpha}{\lambda_1}}} = \kappa \int_0^t (s + t_0)^{-\frac{3\alpha}{2\lambda_1}} ds \tag{2.35}$$

$$\frac{\xi^{-\frac{\alpha}{\lambda_1}}(t) - \xi^{-\frac{\alpha}{\lambda_1}}(0)}{-\frac{\alpha}{\lambda_1}} = \kappa \int_0^t (s + t_0)^{-\frac{3\alpha}{2\lambda_1}}\, ds,$$

$$\xi(t)^{-\frac{\alpha}{\lambda_1}} = -\frac{\kappa\alpha}{\lambda_1} \int_0^t (s + t_0)^{-\frac{3\alpha}{2\lambda_1}}\, ds + \xi(0)^{-\frac{\alpha}{\lambda_1}}.$$

If $\xi(0)$ is sufficiently small and $3\alpha > 2\lambda_1$, i.e. $\alpha > \frac{2}{3}\lambda_1$ then $\xi(t)$ exists for all $t > 0$.

This proves the last assertion of the Theorem.

$\square$

Chapter 3

Front Propagation of Semilinear Diffusion Equations in $\mathbb{H}^n$

3.1 Preliminary results

In this section we recall some basic results about *plane wave solutions in* $\mathbb{R}^n$ given in [1].
Consider the following ordinary differential equation

$$q'' + \kappa q' + f(q) = 0 \quad \text{in} \quad \mathbb{R} \quad \kappa \in \mathbb{R} \tag{3.1}$$

A *plane wave solution* of the equation (4) is a function of the form $q(x\cdot\nu - ct)$ where q solves (3.1) and $\nu \in \mathbb{R}^n$ is an arbitrary unit vector.
It is our interest to study the case in which the following conditions are satisfied

$$\begin{cases} q(\xi) \in [0,1],\ q(\xi) \neq 0 \text{ and} \\ \lim_{\xi\to\infty} q(\xi) = 0. \end{cases}$$

Equation (3.1) can be written as the system of two equations

$$\begin{cases} q' = p, \\ p' = -\kappa p - f(q). \end{cases}$$

whose solutions $q(\xi)$, $p(\xi)$ describe orbits on the phase space (q, p).
The trajectories that are traced out have slope

$$\frac{dp}{dq} = -\kappa - f(q)/p$$

at any point $p \neq 0$.

If f satisfies hypothesis (H_0), then the points $(0,0)$ and $(1,0)$ are critical points for (3.1), as well as all the points of the form $(a,0)$ with $f(a)=0$. A plane
Let $\kappa > 0$. If $\kappa^2 > 4f'(0)$ it can be shown (see [1]) that there exists a nontrivial trajectory from the origin. The unique trajectory in the strip S that goes from the point $(0,-\nu)$ with $\nu > 0$ cannot cross any trajectory that goes to the origin. If we consider the trajectories, making ν varying, and we take the limit $\nu \to 0$ we obtain a nontrivial extremal trajectory going to the origin, that we call T_κ.
The critical value κ for which there exist wave solutions will be defined in terms of the trajectories T_κ.
Firstly we define

$$\sigma = \sup_{u\in(0,1]} \left\{ \frac{f(u)}{u} \right\}$$

so that $f(u) \leq \sigma u$ for $u \in [0,1]$ and we observe that if $\kappa^2 > 4\sigma$ then T_κ is bounded above by the line through the origin $p = -\frac{1}{2}\left(\kappa + \sqrt{\kappa^2 - 4\sigma}\right) q$. In particular T_κ connects the origin to the point of the form $(1,-\nu)$ with $\nu > 0$. Then it is well defined the number

$$c^\star = \inf\{\kappa : \kappa > 0, \kappa^2 > 4f'(0), \text{ there exists } \nu > 0 \text{ such that } (1,-\nu) \in T_\kappa\}$$

which satisfies

$$4f'(0) \leq (c^\star)^2 \leq 4\sigma.$$

In particular, the number $c^\star$ is *the asymptotic speed of propagation* associated with the equation (P2).
If we make the additional assumption that

$$\sigma = f'(0) \tag{3.2}$$

then $c^\star = 2\sqrt{f'(0)}$. (see [1], Prop 4.2, p. 53).
We can now give the main result on the *existence of plane wave solutions* whose proof is given in details in [1][chapter 4, Theorem 4.1 and Lemma 4.3]

Proposition 3.1. *Let assume that (H_0) and either (H_1) or (H_2) holds. Then there exists $c^\star > 0$ with the following properties:*

1. *for $\kappa = c^\star$ equation (3.1) admits a decreasing solution $q^\star$ in $\mathbb{R}$ satisfying*

$$\lim_{\xi\to-\infty} q^\star(\xi) = 1, \lim_{\xi\to\infty} q^\star(\xi) = 0; \tag{3.3}$$

2. *for any $\kappa \in (0, c^\star)$ there exists $\gamma_\kappa \in (0,1)$ such that: for any $\eta \in (\gamma_\kappa, 1)$ there exist $b = b(\kappa,\eta) > 0$ and a solution q to equation (3.1) satisfying*

$$q(0) = \eta,\ q'(0) = 0,\ q(b) = 0,\ q' < 0 \text{ in } (0,b]; \tag{3.4}$$

3. *for any* $\kappa > c^\star$ *there exists a solution* q *to equation* (3.1) *in* $\mathbb{R}_+$ *such that*

$$q(0) = 1,\ q' < 0 \ in\, \mathbb{R}_+,\ q(\xi) \to 0 \ as\, \xi \to \infty. \tag{3.5}$$

3.2 Behavior of disturbances: extinction or propagation?

In this section we will compare the long time behavior of solutions of (P2), with respect to the choice of the forcing term.
Let us fix some notations: we refer to the case of **propagation** when the solution $u(x,t)$ of (P2) satisfies

$$\lim_{t\to\infty} u(x,t) = 1, \quad \text{uniformly on compact subset of } \mathbb{H}^n. \tag{3.6}$$

Instead, we refer to the case of **extinction** when the solution $u(x,t)$ satisfies

$$\lim_{t\to\infty} u(x,t) = 0, \quad \text{uniformly in } \mathbb{H}^n. \tag{3.7}$$

3.2.1 The KPP case

In this section we will investigate the behavior of the solution of (P2) if we choose a forcing term of KPP-type.
We recall that in $\mathbb{R}^n$ if f is KPP then propagation occurs in any case .
The next result shows us that in $\mathbb{H}^n$ we can have both propagation and extinction, depending on the speed of propagation.

Theorem 3.1. *Let assumption* (H_0), (H_1) *and* (H_3) *be satisfied. Let* $u_0 \neq 0$ *and let* u *be the corresponding solution of problem* (P2).

- *Suppose that* u_0 *has compact support and* $c^\star < n-1$. *Then*

$$\lim_{t\to\infty} u(x,t) = 0, \quad uniformly\ in\, \mathbb{H}^n.$$

- *Suppose that* $c^\star > n-1$. *Then*

$$\lim_{t\to\infty} u(x,t) = 1, \quad uniformly\ on\ compact\ subset\ of\ \mathbb{H}^n.$$

Remark 3.1. *It is important to notice that by the assumption* (H_3), $c^\star = 2\sqrt{f'(0)}$ *then*

$$c^\star < n-1 \Rightarrow 2\sqrt{f'(0)} < n-1 \Rightarrow f'(0) < \frac{(n-1)^2}{4} = \lambda_1.$$

We can thus re-phrase the Theorem by saying

- *if* $f'(0) < \lambda_1$ *and* u_0 *has compact support then extinction occurs.*
- *if* $f'(0) > \lambda_1$ *then propagation occurs.*

So, once again we find that the infimum of the spectrum of $\mathbb{H}^n$ *plays a big part in describing behaviors of solutions of a semilinear parabolic equation in* $\mathbb{H}^n$*.*

Proof. **First part:**
Since u_0 is assumed to be compactly supported, then, by definition there exists $R > 0$ such that $\operatorname{supp} u_0 \subseteq B_R$. There exists a C^∞ function $\tilde{u}_0 = \tilde{u}_0(\rho) : \bar{\mathbb{R}}_+ \to [0, 1]$ with

$$\tilde{u}_0' \leq 0 \ \text{ in } \ \mathbb{R}_+ \ , \ \tilde{u}_0 = 0 \quad \text{for any} \quad \rho > R$$

and such that

$$u_0(x) \leq \tilde{u}_0(\rho(x)) \ \text{ for any } \ x \in \mathbb{R}. \tag{3.8}$$

1

Let us suppose that $\tilde{u}$ is a solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + (n-1)\coth\rho \frac{\partial u}{\partial \rho} + f(u) & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \rho} = 0 & \text{in} \quad \{0\} \times \mathbb{R}_+, \\ u = \tilde{u}_0 & \text{in} \quad \mathbb{R}_+ \times \{0\}. \end{cases} \tag{3.9}$$

It is then clear that $\tilde{u}(\rho(x), t)$ satisfies (P2) with Cauchy data $\tilde{u}_0(\rho(x))$. Applying standard comparison results we can state that $\tilde{u}$ is an upper solution and 0 is a lower solution of equation (P1), so that

$$0 \leq u(x,t) \leq \tilde{u}(\rho(x), t) \text{ for any } (x,t) \in \mathbb{H}^n \times \mathbb{R}_+.$$

Now we consider the problem

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \rho^2} + (n-1)\frac{\partial w}{\partial \rho} + f(w) & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ \frac{\partial w}{\partial \rho} = 0 & \text{in} \quad \{0\} \times \mathbb{R}_+, \\ w = \tilde{u}_0 & \text{in} \quad \mathbb{R}_+ \times \{0\}. \end{cases} \tag{3.10}$$

Since by assumption $c^\star < n-1$ then by Theorem 3.1 the ordinary differential equation (3.1) with $\kappa = n-1$ in $\mathbb{R}_+$ has a solution $q = q(\rho)$ such that

$$q(0) = 1, \ q' < 0 \ \text{ in } \ \mathbb{R}_+, \ \ \lim_{\rho \to \infty} q(\rho) = 0.$$

[1] For instance if we define

$$\tilde{u}_0(\rho) = \begin{cases} 1 & \text{for} \quad \rho < R_0, \\ \max_{x \in \mathbb{H}^n : d(x,0) = \rho} u_0(x) & \text{for} \quad R_0 < \rho < R, \\ 0 & \text{for} \quad \rho \geq R \end{cases}$$

with $R_0 < R$ sufficiently large.

Let v be the solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial \rho^2} + (n-1)\frac{\partial v}{\partial \rho} + f(v) & \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \\ v = \phi & \text{in} \quad \{0\} \times \mathbb{R} \times \end{cases}$$

where

$$\phi := \begin{cases} 1 & \text{if } \rho < R, \\ q(\rho - R) & \text{if } \rho \geq R. \end{cases}$$

In [1][chapter 2, Theorem 5.1] it is shown that

$$\frac{\partial v}{\partial t} \leq 0 \ \text{ in } \ \mathbb{R} \times \mathbb{R}_+ \ \text{ and } \ \lim_{t \to \infty} v(\rho, t) = 0.$$

It is clear, by definition of ϕ, that $\tilde{u}_0 \leq \phi$; thus

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial \rho^2} - (n-1)\frac{\partial v}{\partial \rho} + f(v) =$$
$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial \rho^2} - (n-1)\frac{\partial w}{\partial \rho} + f(w) \text{ in } \quad \mathbb{R}_+ \times \mathbb{R}_+$$

and

$$v = \phi \geq \tilde{u}_0 \quad \text{in} \quad \mathbb{R}_+ \times \{0\}.$$

Hence v is an upper solution of problem (3.10) , namely

$$w \leq v \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+.$$

Now, taking the limit for $t \to \infty$ we get

$$\lim_{t \to \infty} \sup_{\rho \in \mathbb{R}_+} w(\rho, t) \leq \lim_{t \to \infty} w(\rho, t) \leq \lim_{t \to \infty} v(\rho, t) = 0$$

Thus we obtained that the solution w of problem (3.10) satisfies

$$\sup_{\rho \in \mathbb{R}_+} w(\rho, t) \to 0 \text{ as } t \to \infty \tag{3.11}$$

and

$$\frac{\partial w}{\partial \rho} \leq 0 \ \text{ in } \ \mathbb{R}_+ \times \mathbb{R}_+.$$

Again if we consider the problem

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial \rho^2} + (n-1)\frac{\partial z}{\partial \rho} + f'(w)z & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ z = 0 & \text{in} \quad \{0\} \times \mathbb{R}_+, \\ z = 0 & \text{in} \quad \mathbb{R}_+ \times \{0\}. \end{cases}$$

where w is a solution of (3.10) then the function $\bar{z} = \frac{\partial w}{\partial \rho}$ is a lower solution of (3.10) in fact we have

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial \rho}\right) = \frac{\partial}{\partial \rho}\left(\frac{\partial^2 w}{\partial \rho^2} + (n-1)\left(\frac{\partial w}{\partial \rho}\right) + f(w)\right) =$$
$$= \frac{\partial^2}{\partial \rho^2}\left(\frac{\partial w}{\partial \rho}\right) + (n-1)\frac{\partial}{\partial \rho}\left(\frac{\partial w}{\partial \rho}\right) + f'(w)\frac{\partial w}{\partial \rho}$$

and

$$\tilde{u}_0' = \frac{\partial \tilde{u}_0}{\partial \rho} = \tilde{u}_0' \leq 0 \;\text{ in }\; \mathbb{R}_+ \times \{0\}.$$

On the other hand $z = 0$ is a solution. By comparison we have

$$\frac{\partial w}{\partial \rho} \leq 0$$

in $\mathbb{R}_+ \times \mathbb{R}_+$.
Now, observing that $\coth \rho \geq 1$ and that $\frac{\partial w}{\partial \rho} \leq 0$ we have that w satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial \rho^2} - (n-1)\frac{\partial w}{\partial \rho} + f(w) \geq \\
&\quad \geq \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \rho^2} - (n-1)\coth\rho\frac{\partial u}{\partial \rho} + f(u) \text{ in } \quad \mathbb{R}_+ \times \mathbb{R}_+ \\
&\frac{\partial w}{\partial \rho} \leq 0 = \frac{\partial u}{\partial t} \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+ \\
&\quad w = u = \tilde{u}_0 \quad \text{in} \quad \mathbb{R}_+ \times \{0\}.
\end{aligned}$$

so that w is an upper solution of problem (3.9), namely

$$0 \leq \tilde{u}(x,t) \leq w(\rho,t) \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

Now, taking the supremum in (3.8) and using (3.11) we can conclude that

$$0 \leq \sup_{x\in\mathbb{H}^n} u(x,t) \leq \sup_{\rho\in\mathbb{R}_+} \tilde{u}(\rho,t) \to 0 \;\text{ as }\; \rho \to \infty$$

Second part:
The assumption $c^\star > n-1$ implies that $f'(0) > \lambda_1$ (see Remark 2.2).
In the geodesic ball $\mathcal{B}_R$ with $R > 0$ we consider the eigenvalue problem

$$\begin{cases} \Delta_{\mathbb{H}}\phi + [f'(0) + \mu]\phi = 0 & \text{in} \quad \mathcal{B}_R, \\ \phi = 0 & \text{on} \quad \partial\mathcal{B}_R. \end{cases} \tag{3.12}$$

We call $\mu_1 = \mu_1(\mathcal{B}_R)$ the first eigenvalue of (3.12) and $\phi_1 = \phi_1(\rho) > 0$ the corresponding eigenfunction.

Bearing in mind (1.19) when $\Sigma = \mathcal{B}_1$ we have

$$\Delta_{\mathbb{H}}\phi_1 + [f'(0) + \mu_1(\mathcal{B}_R)]\phi_1 = 0 \Rightarrow \left(-\lambda_1\phi + f'(0) + \mu_1(\mathcal{B}_R)\right)\phi_1 = 0 \Rightarrow$$
$$\Rightarrow \mu_1(\mathcal{B}_R) = \lambda_1\phi - f'(0).$$

Since $\lambda_1(\mathcal{B}_R) \searrow \lambda_1$ for $R \to \infty$ (see section 1.5) and $f'(0) > \lambda_1$ then there exists $R_0 > 0$

$$\mu_1(\mathcal{B}_{\mathcal{R}}) < 0$$

for any $R > R_0$.
We now set

$$w_\epsilon = \begin{cases} \epsilon\phi_1(\rho) & \text{in} \quad \mathcal{B}_R, \\ 0 & \text{otherwise} \end{cases}$$

with $R > R_0$ fixed.
Consider the elliptic problem in $\mathbb{H}^n$

$$\Delta_{\mathbb{H}} v + f(v) = 0 \;\text{ in }\; \mathbb{H}^n. \tag{3.13}$$

Since

$$\Delta_{\mathbb{H}} w_\epsilon + f(w_\epsilon) = \epsilon\Delta_{\mathbb{H}}\phi_1 + f(w_\epsilon) = -\epsilon\lambda_1\phi_1 + f(w_\epsilon) = -\lambda_1 w_\epsilon + f(w_\epsilon)$$

then by assumption (H_3) it is possible to find an $\epsilon_0 > 0$ such for any $\epsilon \in (0, \epsilon_0)$ holds $\frac{f(w_\epsilon)}{w_\epsilon} \geq \lambda_1$ and so $-\lambda_1 w_\epsilon + f(w_\epsilon) > 0$. Hence w_ϵ is a lower solution of equation (3.13).
On the other side, 0 is a lower solution of (P2), so $u(x,t) \leq 0$ and if we think of the form of the Laplace-Beltrami in the disk model (1.12) (or in the half space model (1.14)) it is easy to see that the operator is elliptic inside the ball B_1 with bounded coefficient degenerating only on the boundary. Moreover, observing assumption (H_3) then it is clear that

$$\frac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + f(u) = \frac{\partial^2 u}{\partial r^2} + (n-1)\coth r\frac{du}{dr} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}} u + f(u) \geq$$
$$\geq \frac{\partial^2 u}{\partial r^2} + (n-1)\coth r\frac{du}{dr} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}} u + c_1 u.$$

Thus applying a standard consequence of the strong maximum principle to the linear problem with bounded coefficient (see [16] Theorem 5 p.9) we can finally state that

$$u(\cdot, \bar{t}) > 0 \;\text{ in }\; \mathbb{H}^n \;\text{ for any }\; \bar{t} \in \mathbb{R}_+. \tag{3.14}$$

We choose $\epsilon \in (0, \epsilon_0)$ so small that

$$w_\epsilon(\rho(x)) \leq u(x, \bar{t})$$

for any $x \in \mathbb{H}^n$.
If u_ϵ denotes the solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + f(u) & \text{in } \mathbb{H}^n \times \mathbb{R}_+, \\ u = w_\epsilon & \text{in } \mathbb{H}^n \times \{0\}. \end{cases} \tag{3.15}$$

then by comparison arguments

$$u_\epsilon(\cdot, t) \leq u(\cdot, t + \bar{t}) \leq 1 \text{ in } \mathbb{H}^n \text{ for any } t \in \mathbb{R}_+$$

Again by the ellipticity of the operator $\Delta_{\mathbb{H}}$ in the disk model (or in the half space model) we can apply standard theorems for semilinear parabolic equation in $\mathbb{R}^n$ to (3.15). Thus there hold the following:

1. the function $t \to u_\epsilon(x, t)$ is nondecreasing in $\mathbb{R}_+$ for any $x \in \mathbb{H}^n$;
2. the pointwise limit
$$u_\infty(x) := \lim_{t \to \infty} u_\epsilon(x, t)$$
is a solution of equation (3.13);
3. the convergence $u_\epsilon(\cdot, t) \to u_\infty$ as $t \to \infty$ is uniform on compact subsets of $\mathbb{H}^n$.

The last thing to prove is that

$$u_\infty = 1 \text{ in } \mathbb{H}^n.$$

Hence we choose $\epsilon \in (0, \epsilon_0)$ so small such that

$$0 \leq w_\epsilon(x) < u_\infty(x) \text{ for any } x \in \mathbb{H}^n. \tag{3.16}$$

Then we observe that

$$B := \{y \in \mathbb{H}^n | w_\epsilon(\tau_y(x)) = w_\epsilon(0) < u_\infty(x) \text{ for any } x \in \mathbb{H}^n\} = \mathbb{H}^n$$

where τ is defined in (1.10).
In fact

1. B is nonempty. In fact $0 \in B$ since $w_\epsilon(\tau_0(x)) = w_\epsilon(x) < u_\infty(x)$ by (3.22).
2. B is open by the continuity of the map $y \to \tau_y(x)$ for every $x \in \mathbb{H}^n$.
3. B is closed because if $\{y_n\} \subseteq B$ such that $d(y_n, y) \to 0$ then by continuity of w_ϵ there holds
$$w_\epsilon(\tau_y(x)) \leq u_\infty \text{ for any } x \in \mathbb{H}^n. \tag{3.17}$$
By the strong maximum principle $u_\infty(x) > 0$ inside the ball B_1 (think of the Laplace-Beltarmi operator in the disk model) thus we cannot have equality in (3.17) otherwise we fall into contradiction.

Hence B coincides with the whole space $\mathbb{H}^n$ and this fact implies that

$$w_\epsilon(\tau_x(x)) = w_\epsilon(0) < u_\infty(x) \text{ for any } x \in \mathbb{H}^n.$$

Thus if ξ is the solution of the problem

$$\begin{cases} \xi' = f(\xi), \\ \xi(0) = w_\epsilon(0) \end{cases}$$

then it is obvious that

$$\xi(t) \leq u_\infty(x) \text{ for any } x \in \mathbb{H}^n, \ \ t \in \mathbb{R}^+.$$

Since the equilibrium point 1 is an asymptotically stable i.e. $\xi(t) \to 1$ as $t \to \infty$ then

$$1 = \lim_{t\to\infty} \xi(t) \leq \lim_{t\to\infty} u_\infty(x) < \lim_{t\to\infty} u_\epsilon(x,t) \leq 1.$$

so

$$u_\infty(x) = 1$$

and putting it together with (3.14) we conclude that

$$\lim_{t\to\infty} u(x,t) = 1.$$

□

Remark 3.2. *We observe that in the second part of the proof of Theorem 3.1 it is shown that if $f'(0) > 1$ then $v = 1$ is the unique nontrivial solution of equation* (3.13) *such that* $0 \leq v \leq 1$.

Extinction occurs also in case we remove the assumption that $f'(0) > 0$ in (H_1). In fact we can prove

Theorem 3.2. *Let assumption (H_0) be satisfied and suppose that*

$$f(u) \leq \beta u^p \qquad (0 \leq u \leq 1)$$

for some $\beta > 0$ and $p > 1$. Let ω be a ground state and define

$$w := c\omega \quad \text{ in } \mathbb{H}^n \tag{3.18}$$

where c is chosen to satisfy

$$0 < c < \frac{\lambda_1^{\frac{1}{p-1}}}{||\omega||_\infty}. \tag{3.19}$$

2

Let u be a solution of problem (P2) *with $u_0 \leq w$ in $\mathbb{H}^n$. Then*

$$\lim_{t\to\infty} u(x,t) = 0 \text{ uniformly in } \mathbb{H}^n.$$

[2] remember that ω is in $L^\infty(\mathbb{H}^n)$ but it does not belong to $L^2(\mathbb{H}^n)$

Proof. Going through the proof in (2.5) we have $h(t) = 1$ so that $\tilde{h}(t) = e^{-\lambda_1(p-1)t}h(t)$ and $\tilde{H}(t) = [1 - e^{-(p-1)\lambda_1 t}]$.
We recall that in this case we have always global existence of a solution of (P2) in $\mathbb{H}^n \times (0,\infty)$ for small initial data (see Theorem 2.6).
Consider the problem

$$\begin{cases} \zeta' = k||w||_{\infty}^{p-1} e^{-(p-1)\lambda_0 t}\zeta^p, \\ \zeta(0) = 1. \end{cases} \tag{3.20}$$

Its solution

$$\zeta(t) = \left[1 - (p-1)k||w||_{\infty}^{p-1}\tilde{H}(t)\right]^{-\frac{1}{p-1}}$$

is well-defined in $[0,\infty)$ thanks to assumption (3.18) and (3.19).
Setting

$$\bar{u} := e^{-\lambda_1 t}\zeta(t)w(x) \qquad (x,t) \in \mathbb{H}^n \times [0,\infty)$$

it is immediate to show that it is an upper solution to problem

$$\begin{cases} u_t = \Delta_{\mathbb{H}} u + u^p & \text{in} \quad \mathbb{H}^n \times (0,\infty), \\ u = u_0 \geq 0 & \text{in} \quad \mathbb{H}^n \times \{0\}. \end{cases} \tag{3.21}$$

Since, by assumption, $f(x) \leq \beta u^p$ then $\bar{u}$ is an upper solution of (P2) too. By comparison principles we have

$$0 \leq u \leq \bar{u}$$

in $\mathbb{H}^n \times [0,\infty)$ and therefore we may conclude that

$$u(x,t) \leq \bar{u}(x,t) \leq \sup_{\mathbb{H}^n} \bar{u} \leq ||w||_{\infty} e^{-\lambda_1}\zeta(t) \to 0 \quad \text{as } t \to \infty.$$

□

We note that this result is essentially due to Theorem 2.4, chapter 2.3.

3.2.2 The Allen-Cahn case

We now investigate the behavior of solutions of (P2) when we choose f of KPP-type. We note that a threshold phenomenon occurs according to the size of the initial data u_0 and the sign of the difference $c^\star - (n-1)$.

Lemma 3.1. *Assume that* (H_0) *and either* (H_1) *or* (H_2) *are satisfied. Let* $c^\star > n-1$. *Then by Theorem 3.1, for any* $c \in (0, c^\star - (n-1))$ *there exists a* $\gamma_c \in (0,1)$ *such that for any* $\eta \in (\gamma_c, 1)$ *there exists* $b = b(c,\eta) > 0$ *and a solution* q *of equation*

$$q'' + (c+n-1)q' + f(q) = 0$$

such that

$$q(0) = \eta,\ q'(0) = 0,\ q(b) = 0,\ q' < 0 \text{ in } (0, b].$$

Consider the problem

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta_{\mathbb{H}} v + f(v) & \text{in} \quad \mathbb{H}^n, \\ v = v_0(\rho) & \text{in} \quad \mathbb{H}^n \times \{0\} \end{cases} \tag{3.22}$$

where, for every fixed R, v_0 is defined as follows

$$v_0 = \begin{cases} \eta & \text{if} \quad \rho \le R, \\ q(\rho - R) & \text{if} \quad R < \rho < R + b, \\ 0 & \text{in} \quad R + b < \rho. \end{cases} \tag{3.23}$$

Then for any $R > 0$ such that

$$\tanh R > \frac{n-1}{c+n-1} \tag{3.24}$$

the solution v of the problem (3.22) *satisfies the inequality*

$$v(x,t) \ge \eta \text{ for any } (x,t) \in \mathbb{H}^n \times \mathbb{R}_+ \text{ such that } \rho(x) \le R + kt, \tag{3.25}$$

where

$$k = k(c) := c + (n-1)(1 - \coth R) > 0. \tag{3.26}$$

Therefore ,

$$\lim_{t \to \infty} v(x,t) = 1 \quad \text{uniformly in compact subsets of } \mathbb{H}^n. \tag{3.27}$$

Proof. We define

$$W(x,t) := v_0(\rho(x) - kt) \ \ (k,t) \in \mathbb{H}^n \times \mathbb{R}_+$$

where k is defined in (3.26).
Setting $s = \rho(x) - kt$ and observing that

$$\Delta_{\mathbb{H}} \rho(x) = -(n-1) \coth \rho(x)$$

we have

$$\Delta W = \frac{\partial v_0(s)}{\partial s^2} |\nabla \rho(x)|^2 + \frac{\partial v_0}{\partial s}(s) \Delta_{\mathbb{H}} \rho(x).$$

If $\rho(x) \le R + kt$ $(s \le R)$ then

$$\begin{aligned} &\frac{\partial}{\partial t} W - \Delta W - f(W) = \\ &= -k \frac{\partial v_0}{\partial s} - \frac{\partial^2 v_0(s)}{\partial s^2} |\nabla \rho(x)|^2 - \frac{\partial v_0(s)}{\partial s} \Delta_{\mathbb{H}} \rho(x) - f(w) = \\ &= -f(\eta). \end{aligned}$$

If $R+kt<\rho(x)\le R+kt+b$ ($R<s\le R+b$) then

$$
\begin{aligned}
&\frac{\partial}{\partial t}W-\Delta W-f(W)= \\
&=-kq'(s-R)-q''(s-R)|\nabla\rho(x)|^2-q'(s-R)\Delta_{\mathbb{H}}\rho-f(q(s-R)) &= \\
&=q'(s-R)[-k-\Delta_{\mathbb{H}}\rho]+q'(s-R)(c+n-1) &= \\
&=q'(s-R)[-k-\Delta_{\mathbb{H}}\rho+c+n-1] &= \\
&=q'(s-R)[-k-(n-1)\coth\rho(x)+c+n-1] &= \\
&=q'(s-R)\left[-k+(n-1)(1-\coth\rho(x))+c\right].
\end{aligned}
$$

If $\rho(x)\ge R+kt+b$ $(s>R+b)$ then

$$\frac{\partial}{\partial t}W-\Delta W-f(W)=0.$$

By the fact that (see(3.24))

$$c-k+(n-1)(1-\coth\rho(x))\ge c-k+(n-1)(1-\coth R)=0$$

and by Theorem 3.1 we can state that if $R+kt<\rho(x)\le R+b+kt$ then

$$q'(\rho(x)-kt-R)<0.$$

Moreover, by assumption (H_2) we know that $f(\eta)>0$ for $\eta>a$ and sufficiently close to 1.
Gathering the informations we deduce that W is a lower solution of problem (3.22), thus applying Comparison theorems we have $W\le v$ in $\mathbb{H}^n\times\mathbb{R}_+$.
Inequality (3.25) is due to the fact that $W(x,t)=\eta$ for $\rho(x)\le R+kt$ and (3.27) follows by (3.25) and by the fact that η is arbitrarily close to 1. □

We can now give the major result concerning the Allen-Chan case.

Theorem 3.3. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Let* u *be the solution of problem* (P2).

(i) If $\sup_{\mathbb{H}^n}u_0<a$ *then we have* (3.7).

(ii) If u_0 *is suitably large and* $c^\star>n-1$ *then we have* (3.6).

Proof. (i) Let ζ be a solution of the following problem

$$\begin{cases}\zeta'=f(\zeta),\\ \zeta(0)=\mu\end{cases}\tag{3.28}$$

where $\mu = \sup_{x\in\mathbb{H}^n} u_0(x)$. Since $\mu < a$ then $f(\zeta) < 0$ for all $t \in \mathbb{R}_+$. Thus $\zeta(t)$ is decreasing [3] and it converges asymptotically to the equilibrium point 0 as $t \to \infty$.

On the other hand, by hypothesis, $\mu = \sup_{\mathbb{H}^n} u_0 < a$. Moreover, we note that $\Delta_{\mathbb{H}}\zeta = 0$ and , in particular, $\frac{\partial\zeta}{\partial t} - f(\zeta) \geq 0$. Therefore, applying comparison results we deduce that $0 \leq u(x,t) \leq \zeta(t)$ for any $(x,t) \in \mathbb{H}^n \times \mathbb{R}_+$.

We can therefore conclude that $\lim_{t\to\infty} u(x,t) = 0$.

(ii) By assumption we choose $u_0(\rho) \geq v_0(\rho)$ in $\mathbb{H}^n$, where v_0 is defined in (3.23). By comparison principles we have $u \geq v$, v being the solution of (3.23), for all $(x,t) \in \mathbb{H}^n \times \mathbb{R}_+$. Applying Lemma 3.1 we have

$$1 \geq \lim_{t\to\infty} u(x,t) \geq \lim_{t\to\infty} v(x,t) = 1$$

uniformly on compact subsets of $\mathbb{H}^n$.

$\square$

Remark 3.3. *The conclusion of Theorem* (3.3) *holds true also in the case* (H_3) *is substitute by the following weaker assumption: there exists* $a \in (0,1)$ *such that* $f(u) < 0$ *for any* $u \in [0,a]$ *and* $f(\bar{u}) > 0$ *for some* $\bar{u} \in (a,1)$.
The proof make use of the fact that there exists $C_n \geq 0$ *such that for any* $t \in \mathbb{R}_+$

$$\sup_{x,y\in\mathbb{H}^n} p(x,y,t) \leq C_n \frac{(1+t)^{\frac{n-3}{2}}}{t^{\frac{n}{2}}} e^{-\lambda_1 t}.$$

This is a direct consequence of the bilateral estimate of the heat kernel in $\mathbb{H}^n$ (1.18).

3.3 Speed of propagation

In this section we will study in deep the case in which propagation prevails over extinction; in particular the main aim is to determine the speed of propagation of disturbances.

Theorem 3.4. *Let assumption* (H_0) *and either* (H_1)*-*(H_3) *or* (H_2) *be satisfied . Let* $u_0 \not\equiv 0$ *have compact support, and* u_0 *be suitably large if* (H_2) *holds. Moreover, assume* $c^\star > n-1$.

i) Let $c > c^\star - (n-1)$. *Then for any* $y \in \mathbb{H}^n$

$$\lim_{t\to\infty} \sup_{d(x,y)>ct} u(x,t) = 0.$$

[3] The problem (3.28) has a unique solution which can be obtained, solving with respect to ζ the equation $t = \int_\mu^\zeta \frac{d\lambda}{f(\lambda)}$. From this last expression we deduce $\zeta \to \infty$ as $t \to \infty$

ii) Let $0 < c < c^\star - (n-1)$. Then for any $y \in \mathbb{H}^n$

$$\lim_{t\to\infty} \sup_{d(x,y)<ct} u(x,t) = 1.$$

Proof. **First part:** Choose $\tilde{u}_0 = \tilde{u}_0(\rho) : \bar{\mathbb{R}}_+ \to [0,1]$

$$\tilde{u}_0' \leq 0 \text{ in } \mathbb{R}_+ , \ \tilde{u}_0 = 0 \quad \text{for any} \quad \rho > R$$

and such that

$$u_0(x) \leq \tilde{u}_0(\rho(x)) \text{ for any } x \in \mathbb{R}. \tag{3.29}$$

(as in the proof of Theorem (3.1)).
Let $\tilde{u}$ satisfy the boundary-value problem (3.9), that is

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + (n-1)\coth\rho \frac{\partial u}{\partial \rho} + f(u) & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \rho} = 0 & \text{in} \quad \{0\} \times \mathbb{R}_+, \\ u = \tilde{u}_0 & \text{in} \quad \mathbb{R}_+ \times \{0\}. \end{cases} \tag{3.30}$$

Since $c+n-1 > c^\star$, by Proposition (3.1) we know that there exists a solution $q = q(\rho)$ of the ordinary equation

$$q'' + (c+n-1)q' + f(q) = 0 \quad \text{in} \quad \mathbb{R}_+$$

such that

$$q(0) = 1,\ q' < 0 \text{ in } \mathbb{R}_+,\ q(\rho) \to \infty \text{ as } \rho \to \infty.$$

We now set

$$\phi(\xi) := \begin{cases} 1 & \text{if } \xi < R \\ q(\xi - R) & \text{if } \xi \geq R \end{cases}.$$

We notice that, by construction,

$$\text{if} \quad \xi \geq R \quad \phi(\xi) = q(\rho - R) \geq 0 = \tilde{u}_0(\rho)$$

and

$$\text{if} \quad \xi < R \quad \phi(\xi) = 1 \geq u_0(\xi)$$

so that

$$\tilde{u}_0 \leq \phi \text{ in } \mathbb{R}_+. \tag{3.31}$$

Let v solve the following problem

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2}{\partial \xi^2} + (c+n-1)\frac{\partial v}{\partial \xi} + f(v) & \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \\ v = \phi & \text{in} \quad \mathbb{R} \times \{0\}. \end{cases}$$

We have already proved in Theorem 3.1 that

$$v(\xi,t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for any fixed} \quad \xi \in \mathbb{R}$$

and

$$\frac{\partial v}{\partial \xi} \geq 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+.$$

We define

$$w(\rho,t) := v(\rho - ct, t) \quad (\rho \geq 0, t \geq 0).$$

Then

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \rho^2} + (n-1)\frac{\partial w}{\partial \rho} + f(w) & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ \frac{\partial w}{\partial \rho} \leq 0 & \text{in} \quad \{0\} \times \mathbb{R}_+, \\ w = \phi & \text{in} \quad \mathbb{R}_+ \times \{0\}. \end{cases} \tag{3.32}$$

By (3.8) and (3.31) we have

$$u_0(x) \leq \tilde{u}_0(\rho(x)) \leq \phi(\rho(x)).$$

Observing that $\coth\rho \geq 1$ then

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + (n-1)\coth\rho\frac{\partial u}{\partial \rho} + f(u) \geq \frac{\partial^2 u}{\partial \rho^2} + (c+n-1)\frac{\partial u}{\partial \rho} + f(u)$$

in $\mathbb{R}_+ \times \mathbb{R}_+$,

$$\frac{\partial w}{\partial \rho} \leq 0 = \frac{\partial \tilde{u}}{\partial \rho} \quad \text{in} \quad \{0\} \times \mathbb{R}_+,$$

$$\tilde{u}_0 \leq w = \phi \quad \text{in} \quad \mathbb{R}_+ \times \{0\}.$$

Therefore, by comparison results we have

$$u(x,t) \leq \tilde{u}(\rho(x),t) \leq w(\rho(x),t) = v(\rho(x) - ct, t) \quad \text{in} \quad \mathbb{H}^n \times \mathbb{R}_+.$$

We notice that since $d(x,y) < d(x,0) + d(y,0) = \rho(x) + \rho(y)$ then the condition $d(x,y) > ct$ implies that $-ct + \rho(x) > -\rho(y)$.
Now, since $\xi \to v(\xi,t)$ is non-increasing for any $t > 0$

$$\begin{aligned} \sup_{\{x\in\mathbb{H}^n | d(x,y)>ct\}} u(x,t) &\leq \sup_{\{x\in\mathbb{H}^n | d(x,y)>ct\}} v(\rho(x) - ct, t) \leq \\ &\leq \sup_{\{x\in\mathbb{H}^n | d(x,y)>ct\}} v(-\rho(y),t) = v(-\rho(y),t) \to 0 \quad \text{as} \quad t \to \infty. \end{aligned}$$

Second part:
We want to prove that

for any $\bar\eta \in (\gamma_{\bar c}, 1)$ **there exists** $\tau \in \mathbb{R}_+$ **such that**

$$\inf_{d(x,y)<ct} u(x,t) \geq \bar\eta \quad \textbf{for any} \quad t > \tau.$$

If $c^\star > n-1$, thanks to Theorem (3.1), we can state that

$$\lim_{t\to\infty} u(x,t) = 1 \quad \text{uniformly in every compact subset of} \quad \mathbb{H}^n$$

so that for any compact subset $K \subseteq \mathbb{H}^n$ and for any $\eta \in (0,1)$ there exists $h = h(K,\eta) > 0$ such that

$$u(x,t) \geq \eta \ \text{ for any } x \in K, \ t \geq h. \tag{3.33}$$

Let $c \in (0, c^\star - (n-1))$ and fix $\bar c \in (c, c^\star - (n-1))$, $\bar R > 0$ such that

$$\coth \bar R < 1 + \frac{\bar c - c}{n-1} < 1 + \frac{\bar c}{n-1}. \tag{3.34}$$

Let $\gamma_{\bar c} \in (0,1)$, $\bar\eta \in (\gamma_{\bar c}, 1)$, $\bar b = b(\gamma_{\bar c}, \bar\eta) > 0$. Let $\bar q$ be the solution of problem

$$q'' + (c+n-1)q' + f(q) = 0$$

such that

$$q(0) = \eta, \ q'(0) = 0, \ q(b) = 0, \ q' < 0 \text{ in } (0,b]. \tag{3.35}$$

Moreover let $\bar v$ solve the problem (3.22) and take $\bar v_0$ as in (3.23).
On the one hand, by Lemma 3.1 we deduce that

$$\bar v(x,t) \geq \bar\eta \quad \text{if} \quad \rho(x) \leq \bar R + \bar k t, \tag{3.36}$$

where $\bar k = k(\bar c) = \bar c + (n-1)(1 - \coth R) > 0$.
On the other hand, by definition of $\bar v_0$ and the fact that q is decreasing in $(0, \bar b]$ (see (3.35)) we have

$$\bar v_0(\rho(x)) = \begin{cases} \bar\eta & \text{if} \quad \rho \leq \bar R, \\ 0 & \text{in} \quad \rho > \bar R + \bar b, \\ q(\rho - R) \leq \bar\eta & \text{in} \quad \bar R < \rho \leq R + \bar b. \end{cases}$$

so that

$$\begin{cases} \bar v_0(\rho(x)) \leq \eta & \text{if} \quad \rho(x) \leq \bar R + \bar b, \\ \bar v_0(\rho(x)) = 0 & \text{otherwise}. \end{cases} \tag{3.37}$$

The estimate (3.33) in a compact $K = \{x \in \mathbb{H}^n | \rho(x) \leq \bar R + \bar b\}$ implies that

$$u(x,h) \geq \bar\eta \geq \bar v_0(\rho(x)) \quad \text{for any} \quad (x,t) \in \mathbb{H}^n \times \mathbb{R}_+$$

and by comparison results

$$u(x,t+h) \geq \bar{v}(x,t) \quad \text{for any} \quad (x,t) \in \mathbb{H}^n \times \mathbb{R}_+. \tag{3.38}$$

From (3.36) and (3.38) it follows that

$$u(x,t+h) \geq \bar{v}_0(x,t) \geq \bar{\eta} \ \text{ for all } \ (x,t) \in \mathbb{H}^n \times \mathbb{R}_+$$

such that $\rho(x) \leq \bar{R} + \bar{k}t$, $t \in \mathbb{R}_+$ which is equivalent to say that

$$u(x,t) \geq \bar{v}_0(x,t) \geq \bar{\eta} \ \text{ for all } \ (x,t) \in \mathbb{H}^n \times \mathbb{R}_+$$

such that $\rho(x) \leq \bar{R} + \bar{k}(t-h)$, $(t > h)$.
We now observe that, by (3.34)

$$\bar{k} = \bar{c} + (n-1)(1 + \coth \bar{R}) > \bar{c} + (n-1)(1 - 1 - \frac{\bar{c}-c}{n-1}) = \bar{c} - \bar{c} + c = 0$$

thus

$$\bar{k} > c.$$

It is then easy to verify that

$$ct < \bar{R} + \bar{k}(t-h) \quad \text{for any} \quad t > \tau := \max\left\{\frac{\bar{k}h - \bar{R}}{\bar{k} - c}, 0\right\},$$

so that it is clear that

$$\inf_{\rho(x)<ct} u(x,t) \geq \inf_{\rho \leq \bar{R}+\bar{k}(t-h)} u(x,t) \geq \bar{\eta}$$

for any $t > \tau$.
Now the conclusion easily follows if we observe that $u \geq 1 \in \mathbb{H}^n$, by hypothesis.

□

3.4 Asymptotical symmetry

In this section we will give a result concerning the level sets of the solution of problem (P2).

Theorem 3.5. *Let assumption (H_0) and either (H_1)-(H_3) or (H_2) be satisfied. Let $u_0 \not\equiv 0$ have compact support, and u_0 be suitably large if (H_2) holds. Moreover, assume $c^\star > n-1$. Then for any $a \in (0,1)$ and $t \in \mathbb{R}_+$ sufficiently large the following holds:*

i) the level set

$$\Gamma_a(u;t) := \{x \in \mathbb{H}^n | u(x,t) = a\} \ \ (t \in \mathbb{R}_+)$$

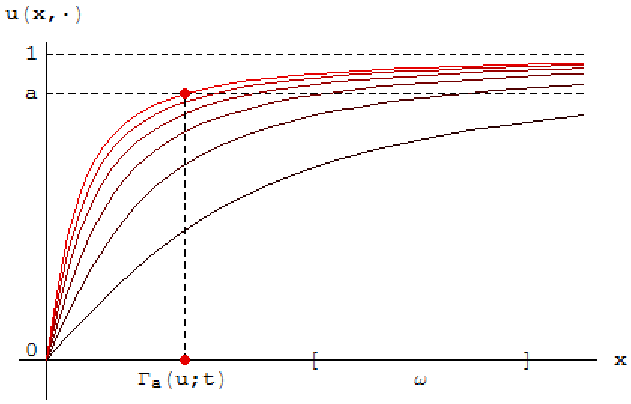
is a smooth $(n-1)$ dimensional submanifold of $\mathbb{H}^n$;

ii) every geodesic orthogonal to $\Gamma_a(u;t)$ intersects the convex hull of the support of u_0.

Proof. Let ω be the convex hull of the support of u_0 and $\pi \subseteq \mathbb{H}^n$ be an hyperplane such that $\pi \cap \omega = \emptyset$. Let $\mathbb{H}^n_\omega \subseteq \mathbb{H}^n$ be the half space containing ω.
Claim: the solution u of (P1) satisfies

$$\langle \nabla u, \nu \rangle_{\mathbb{H}} > 0 \text{ in } \pi \times \mathbb{R}_+ \tag{3.39}$$

If this claim is true then we can prove the two statements.
i) Since u_0 is compactly supported, by hypothesis, and $\lim_{t\to\infty} u(x,t) = 1$ uniformly in every compact subset of $\mathbb{H}^n$ then $\Gamma_a(u;t) \cap \omega = \emptyset$ for any $a \in (0,1)$ and $t \in \mathbb{R}_+$ sufficiently large.



Now set $x_0 \in \Gamma_a(u;t)$ then we can find an hyperplane $P \subseteq \mathbb{H}^n$ such that $x_0 \in P$ and $P \cap \omega = \emptyset$. If we think $\pi = P$ then we obtain $\nabla_{\mathbb{H}} u(x_0,t) \neq 0$. Hence, since x_0 is arbitrary, then $\Gamma_a(u;t)$ is smooth.
ii) We argue by contradiction. Suppose that $x_0 \in \Gamma_a(u;t)$ and that there exist an infinite geodesic γ orthogonal to $\Gamma_a(u;t)$ at x_0, which does not intersect ω. We can choose an hyperplane $Q \in \mathbb{H}^n$ such that $\gamma \subseteq Q$ and $Q \cap \omega = \emptyset$. Thinking $\pi = Q$ we deduce (by (3.39)) that $\langle \nabla_{\mathbb{H}} u(x_0,t), \tau \rangle_{\mathbb{H}} \neq 0$ where τ is a tangent vector to $\Gamma_a(u;t)$ at x_0. This contradicts the definition of $\Gamma_a(u;t)$, which is a level set of u.
We end up proving our Claim.
Let us define

$$\tilde{u}(x,t) := u(R_\pi(x),t) \text{ with } x \in \mathbb{H}^n$$

where R_π is the reflection through the hyperspace π (see definition (1.8)).
The functions u and $\tilde{u}$ satisfy the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + f(u) & \text{in} \quad \mathbb{H}^n_\omega \times \mathbb{R}_+, \\ u = u_0 \geq 0 = \tilde{u} & \text{in} \quad \mathbb{H}^n_\omega \times \{0\}, \\ u = \tilde{u} & \text{in} \quad \pi \times \mathbb{R}_+. \end{cases}$$

By the maximum principle we can state that $u > \tilde{u}$ in $\mathbb{H}^n_\omega \times \mathbb{R}_+$ and by the Hopf Lemma we can conclude that

$$\frac{\partial u}{\partial \nu} > \frac{\partial \tilde{u}}{\partial \nu} = -\frac{\partial u}{\partial \nu}$$

where ν is the vector field orthogonal to π pointing toward $\mathbb{H}^n_\omega$. Therefore we conclude that
$\langle \nabla u, \nu \rangle_{\mathbb{H}} > 0$ in $\pi \times \mathbb{R}_+$.

$\square$

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