Very large cardinals

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1. Tutorial

Large cardinals

Large cardinal axioms postulate the existence of cardinals possessing closure properties strong enough to imply the consistency of ZFC, and whose existence can therefore not be proven in ZFC by Gödel's Incompleteness Theorem.

These principles play an important role in the search for the right axioms for mathematics for two reasons:

First, they are themselves strong candidates for new axioms, because strong arguments can be made to intrinsically justify them and they can be shown to answer important questions left open by ZFC in the desired way.

Second, these notions can be used to measure and compare the strength of mathematical theories through equiconsistency results between such theories and extensions of ZFC by large cardinal axioms.

The simplest example of large cardinal axioms originates from Hausdorff's work on cardinal arithmetic:

Definition

An uncountable cardinal is *weakly inaccessible* if it is a regular limit cardinal.

If κ is weakly inaccessible, then the corresponding initial segment L_κ of Gödel's constructible universe L is a model of ZFC.

By Gödel's Incompleteness Theorem, this shows that, if ZFC is consistent, then this theory does not prove the existence of weakly inaccessible cardinals.

Another fundamental example of large cardinal axioms is given by Ulam's work on the *Measure Problem*:

Definition

An infinite cardinal is *measurable* if there exists a normal ultrafilter over κ.

Using an ultrapower construction, this large cardinal property can also be characterized by the existence of certain elementary embeddings:

Theorem (Keisler, Keisler–Tarski, Scott)

An infinite cardinal κ is measurable if and only if there exists an inner model M and a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with

$$\kappa \ = \ \mathrm{crit}(j) \ = \ \min\{\alpha \in \mathrm{Ord} \mid j(\alpha) > \alpha\}.$$

If we use a normal ultrafilter on a measurable cardinal κ in an ultrapower construction to obtain an elementary embedding $j : \mathrm{V} \longrightarrow M$ with critical point κ, then the resulting ultrapower M is closed under κ-sequences in V.

A canonical way to obtain stronger axioms is to postulate the existence of elementary embeddings into inner models with stronger closure properties.

Definition

Given uncountable cardinals $\kappa < \lambda$, the cardinal κ is *huge with target* λ if there is a $<\kappa$-complete, normal ultrafiler U over $\mathcal{P}(\lambda)$ with

$$\{x \in \mathcal{P}(\lambda) \mid \mathrm{otp}(x) = \kappa\} \ \in \ U.$$

Theorem

An uncountable cardinal κ is huge with target λ if and only if there exists an inner model M with ${}^{\lambda}M \subseteq M$ and a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\kappa = \mathrm{crit}(j)$ and $j(\kappa) = \lambda$.

The above property can be strengthened further in a canonical way:

Definition

Given a natural number $n > 0$ and uncountable cardinals

$$\kappa_0 \ < \ \kappa_1 \ < \ \ldots \ < \ \kappa_n,$$

the cardinal κ_0 is *n-huge with targets* $\kappa_1, \ldots, \kappa_n$ if there is a $<\kappa$-complete, normal ultrafiler U over $\mathcal{P}(\kappa_n)$ with

$$\{x \in \mathcal{P}(\kappa_n) \mid \mathrm{otp}(x \cap \kappa_{\ell+1}) = \kappa_\ell\} \ \in \ U$$

for all $\ell < n$.

Theorem

An uncountable cardinal κ_0 is n-huge with targets $\kappa_1, \ldots, \kappa_n$ if and only if there exists an inner model M with ${}^{\kappa_n}M \subseteq M$ and a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\kappa_0 = \mathrm{crit}(j)$ and $j(\kappa_\ell) = \kappa_{\ell+1}$ for all $\ell < n$.

It is now natural to consider transfinite versions of the above axioms.

Definition

Given a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$, the *critical sequence of* j is the sequence $\langle \kappa_\ell \mid \ell < \omega \rangle$ defined by $\kappa_0 = \mathrm{crit}(j)$ and $\kappa_{\ell+1} = j(\kappa_\ell)$ for all $\ell < \omega$.

Note that, in the situation of the definition, the supremum of the critical sequence of j is the least non-trivial fixed point of j, i.e., $\sup_{\ell < \omega} \kappa_\ell$ is the least ordinal $\lambda > \kappa_0$ with $j(\lambda) = \lambda$.

A surprising theorem of Kunen shows that the closure properties of the target inner models of elementary embeddings of V are severely restricted by the supremum of the corresponding critical sequence.

The Kunen Inconsistency

The Kunen's theorem can also be applied to embeddings that are not necessarily definable classes. To phrase the result in this wider context, we extend the language of set theory by a binary function symbol j and a unary relation symbol $\mathfrak{M}$, and consider the extension ZFC_* of ZFC that extends the Replacement and Separation Scheme to formulas in the enriched language.

Theorem (The Kunen Inconsistency, Version I., ZFC_*)

If $\mathfrak{M}$ is an inner model, $\mathrm{j} : \mathrm{V} \longrightarrow \mathfrak{M}$ is an elementary embedding and λ is the supremum of the critical sequence of j, then $j[\lambda] \notin \mathfrak{M}$.

Kunen's proof of the above result relies on the following concept:

Definition

Given an infinite cardinal λ, a function $f : [\lambda]^\omega \longrightarrow \lambda$ is *ω-Jónsson* if

$$f\,[[A]^\omega] \;=\; \lambda$$

holds for every $A \in [\lambda]^\lambda$.

Theorem (Erdős–Hajnal)

For every infinite cardinal λ, there exists an ω-Jónsson function $f : [\lambda]^\omega \longrightarrow \lambda$.

Proof.

Assume that there exists no ω-Jónsson function for λ.

Define a binary relation E on $[\lambda]^\omega$ by setting

$$x \; E \; y \iff \sup(x) = \sup(y) \; \wedge \; \exists \alpha < \sup(x) \; [x \setminus \alpha \; = \; y \setminus \alpha]$$

for all $x, y \in [\lambda]^\omega$. Then E is an equivalence relation on $[\lambda]^\omega$.

Using the Axiom of Choice, we can find a *selector for* E, i.e., a function $s : [\lambda]^\omega \longrightarrow [\lambda]^\omega$ such that

$$x \; E \; s(x)$$

and

$$x \; E \; y \implies s(x) \; = \; s(y)$$

hold for all $x, y \in [\lambda]^\omega$.

Given $x \in [\lambda]^\omega$, let $g(x)$ denote the least $\alpha \in s(x)$ with

$$x \setminus (\alpha + 1) \; = \; s(x) \setminus (\alpha + 1).$$

Our assumption now implies that for every $A \in [\lambda]^\lambda$, there is $B \in [A]^\lambda$ with the property that $A \nsubseteq g\,[[B]^\omega]$. This allows us to inductively define a descending sequence $\langle A_n \in [\lambda]^\lambda \mid n < \omega \rangle$ such that $A_0 = \lambda$ and there exists a strictly increasing sequence $\langle \alpha_n < \lambda \mid n < \omega \rangle$ with $\alpha_n < \min(A_{n+1})$ and

$$\alpha_n \in A_n \setminus g\,[[A_{n+1}]^\omega]$$

for all $n < \omega$. Define

$$x \;=\; \{\alpha_n \mid n < \omega\} \in [\lambda]^\omega.$$

Then there exists $m < \omega$ with

$$s(x) \setminus \alpha_m \;=\; \{\alpha_n \mid m \leq n < \omega\}$$

and

$$g(\{\alpha_n \mid m < n < \omega\}) \;=\; \alpha_m \;\in\; g\,[[A_{m+1}]^\omega]\,,$$

a contradiction. □

Proof of Kunen's Theorem.

Let $\mathfrak{M}$ be an inner model, let $\mathrm{j} : \mathrm{V} \longrightarrow \mathfrak{M}$ be a non-trivial elementary embeddings and let λ denote the supremum of the critical sequence of j.

Assume, towards a contradiction, that $\mathrm{j}[\lambda]$ is an element of M.

Fix an ω-Jónsson function $f : [\lambda]^\omega \longrightarrow \lambda$ for λ.

Since $\mathrm{j}[\lambda] \in ([\lambda]^\lambda)^{\mathfrak{M}}$ and elementarity implies that $\mathrm{j}(f)$ is an ω-Jónsson function for λ in $\mathfrak{M}$, we can find $y \in ([\mathrm{j}[\lambda]]^\omega)^{\mathfrak{M}}$ with $\mathrm{j}(f)(y) = \mathrm{crit}(\mathrm{j}) \notin \mathrm{j}[\lambda]$.

Set $x = \mathrm{j}^{-1}[y]$. Since $y \in [\mathrm{j}[\lambda]]^\omega$, we then have $x \in [\lambda]^\omega$ and

$$\mathrm{j}(x) \ = \ j[x] \ = \ y.$$

But this implies

$$\mathrm{crit}(\mathrm{j}) \ = \ \mathrm{j}(f)(y) \ = \ \mathrm{j}(f)(\mathrm{j}(x)) \ = \ \mathrm{j}(f(x)) \ \in \ \mathrm{j}[\lambda],$$

a contradiction. □

Since all ω-Jónsson functions for a singular cardinal λ of countable cofinality are elements of $\mathrm{V}_{\lambda+2}$, the above argument also yields the following ZFC-result:

Theorem (The Kunen Inconsistency, Version II.)

For every ordinal λ, there is no non-trivial elementary embedding from $\mathrm{V}_{\lambda+2}$ into itself.

Rank-into-rank embeddings

Motivated by Kunen's result, Kanamori, Reinhardt and Solovay initiated the study of large cardinal notion just below the Kunen Inconsistency:

Definition

- An I3-*embedding* is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An I2-*embedding* is a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with the property that $\mathrm{V}_\lambda \subseteq M$, where λ is the supremum of the critical sequence of j.
- An I1-*embedding* is a non-trivial elementary embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ for some ordinal λ.

If either $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is an I1-embedding or $j : \mathrm{V} \longrightarrow M$ is an I2-embedding whose critical sequence has supremum λ, then $j(\lambda) = \lambda$ and $j \upharpoonright \mathrm{V}_\lambda$ is an I3-embedding.

The definition of critical sequences also makes sense for I1- and I3-embeddings, and, if either $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is an I1-embedding or $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an I3-embedding, then the Kunen's Inconsistency implies that λ is the supremum of the critical sequence of j and hence $j(\gamma) > \gamma$ holds for all $\gamma \in [\mathrm{crit}(j), \lambda)$.

Moreover, if $\langle \kappa_\ell \mid \ell < \omega \rangle$ is the critical sequence of j in this situation, then we can inductively show that $\mathrm{V}_{\kappa_\ell} \prec \mathrm{V}_{\kappa_{\ell+1}}$ holds for all $\ell < \omega$. In particular, we have $\mathrm{V}_{\kappa_\ell} \prec \mathrm{V}_\lambda$ for all $\ell < \omega$ and, since κ_0 is strongly inaccessible, it follows that V_λ is a model of ZFC.

More far-reaching implications between the above three properties are less obvious. In order to isolate such implications, we use canonical liftings of I3-embeddings $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ to functions from $\mathrm{V}_{\lambda+1}$ to itself.

Definition

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an elementary embedding for some limit ordinal λ, then we define

$$j_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1};\ A \longmapsto \bigcup\{j(A \cap \mathrm{V}_\gamma) \mid \gamma < \lambda\}.$$

Theorem

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an elementary embedding for some limit ordinal λ, then j_+ is the unique Σ_0-elementary function from $\mathrm{V}_{\lambda+1}$ into itself that extends j.

We can now state and prove an alternative characterization of I2-embeddings:

Theorem

The following statements are equivalent for every I3-embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$:

- The map j can be extended to an I2-embedding $i : \mathrm{V} \longrightarrow M$.
- If R is a well-founded relation on V_λ, then $j_+(R)$ is a well-founded relation on V_λ.

Proof.

First, assume that $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ can be extended to an I2-embedding $i : \mathrm{V} \longrightarrow M$ and R is a well-founded relation on V_λ.

Then $i(R)$ is a well-founded relation on V_λ in M and, since well-foundedness is absolute between V and M, it follows that $i(R)$ is well-founded.

Given $A \in \mathrm{V}_{\lambda+1}$, elementarity implies that

$$i(A) \ = \ \bigcup \{ i(A \cap \mathrm{V}_\alpha) \mid \alpha < \lambda \}.$$

Since $i \restriction \mathrm{V}_\lambda = j$ and $i \restriction \mathrm{V}_{\lambda+1} : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is Σ_0-elementary, the uniqueness of j_+ implies that $j_+ = i \restriction \mathrm{V}_{\lambda+1}$.

In particular, we can conclude that $j_+(R)$ is well-founded.

Proof.

Now, assume that $j_+(R)$ is well-founded for every well-founded relation R on V_λ. Let $\langle \kappa_n \mid n < \omega \rangle$ denote the critical sequence of j.

Given $0 < n < \omega$, we let

$$U_n \;=\; \{A \subseteq \mathcal{P}(\kappa_n) \mid j[\kappa_n] \in j(A)\}$$

denote the n-huge filter over $\mathcal{P}(\kappa_n)$ induced by j.

Since each U_n is a $<\!\kappa$-complete ultrafilter, we can identify the corresponding ultrapower M_n with its transitive collapse and let

$$i_n : \mathrm{V} \longrightarrow M_n$$

denote the corresponding ultrapower embedding.

Given $0 < n < \omega$, standard arguments then show that:

- If $\gamma \leq \kappa_n$, then

$$[x \mapsto \mathrm{otp}(x \cap \gamma)]_{U_n} \ = \ \gamma.$$

  In particular, we have

$$i_n \restriction (\kappa_{n-1} + 1) \ = \ j \restriction (\kappa_{n-1} + 1)$$

  and $\mathrm{crit}(i_n) = \kappa_0$.

- $[\mathrm{id}_{\mathcal{P}(\kappa_n)}]_{U_n} = i_n[\kappa_n] \in M_n$.

- $\mathrm{V}_{\kappa_n} \in M_n$ and $i_n \restriction \mathrm{V}_{\kappa_{n-1}} = j \restriction \mathrm{V}_{\kappa_{n-1}}$.

Given $0 < m \leq n < \omega$, we define

$$i_{m,n} : M_m \longrightarrow M_n;\ [f]_{U_m} \longmapsto [x \mapsto f(x \cap \kappa_m)]_{U_n}.$$

Then each $i_{m,n}$ is an elementary embedding satisfying

$$i_{m,n} \circ i_m \ = \ i_n$$

and

$$i_{m,n} \restriction (\kappa_m + 1) \ = \ \mathrm{id}_{\kappa_m+1}.$$

By standard arguments, this implies that

$$i_{m,n} \restriction \mathrm{V}_{\kappa_m} \ = \ \mathrm{id}_{\mathrm{V}_{\kappa_m}}$$

holds for all $0 < m \leq n < \omega$.

Let

$$(\langle (M_n, \in) \mid 0 < n < \omega \rangle,\ \langle i_{m,n} : M_m \longrightarrow M_n \mid 0 < m \leq n < \omega \rangle)$$

denote the corresponding directed system and let

$$((M, E),\ \langle k_n : M_n \longrightarrow M \mid 0 < n < \omega \rangle)$$

denote its direct limit. Assume that (M, E) is ill-founded.

Then there exists a strictly increasing sequence $\langle n(\ell) \mid \ell < \omega \rangle$ and a sequence $\langle f_\ell : \mathcal{P}(\kappa_{n(\ell)}) \longrightarrow \mathrm{Ord} \mid \ell < \omega \rangle$ of functions such that

$$k_{n(\ell+1)}([f_{\ell+1}]_{U_{n(\ell+1)}}) \ \ E \quad k_{n(\ell)}([f_\ell]_{U_{n(\ell)}})$$

holds for all $\ell < \omega$.

Define

$$D \ = \ \bigcup\{\mathrm{ran}(f_\ell) \mid \ell < \omega\} \ \subseteq \ \mathrm{Ord}.$$

Then $|D| \leq \lambda$ and there is a well-founded relation R on λ such that there exists an order-embedding $e : (D, \in) \longrightarrow (\lambda, R)$.

Given $\ell < \omega$, we then have

$$[f_{\ell+1}]_{U_{n(\ell+1)}} \quad \in \quad i_{\kappa_{n(\ell),n(\ell+1)}}([f_\ell]_{U_{n(\ell)}})$$

and this implies that the set of all $y \in \mathcal{P}(\kappa_{n(\ell+1)})$ satisfying

$$(e \circ f_{\ell+1})(y) \quad R \quad (e \circ f_\ell)(y \cap \kappa_{n(\ell)})$$

is an element of $U_{\kappa_{n(\ell+1)}}$.

This allows us to conclude that

$$j_+(e \circ f_{\ell+1})(j[\kappa_{n(\ell+1)}]) \ \ j_+(R) \ \ j_+(e \circ f_\ell)(j[\kappa_{n(\ell)}])$$

holds for all $\ell < \omega$, contradicting the well-foundedness of $j_+(R)$.

We can now identify (M, E) with its transitive collapse. The universal property of direct limits then yields an elementary embedding

$$i : \mathrm{V} \longrightarrow M$$

satisfying

$$k_n \circ i_n \;=\; i$$

for all $0 < n < \omega$.

In combination with earlier computations, these equalities ensure that

$$i \restriction \mathrm{V}_\lambda \;=\; j.$$

This completes the proof of the theorem. $\square$

Corollary

Every I1-embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ can be extended to an I2-embedding.

Proof.

If R is a well-founded relation on V_λ, then $(j \restriction \mathrm{V}_\lambda)_+(R) = j(R)$ is a well-founded relation on V_λ.

By the above theorem, this shows that $j \restriction \mathrm{V}_\lambda$ can be extended to an I2-embedding $i : \mathrm{V} \longrightarrow M$.

The uniqueness of $(j \restriction \mathrm{V}_\lambda)_+$ now allows us to conclude that

$$j \;=\; (j \restriction \mathrm{V}_\lambda)_+ \;=\; i \restriction \mathrm{V}_{\lambda+1}$$

holds. □

2. Tutorial

Definition

- An I3-*embedding* is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An I2-*embedding* is a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with the property that $\mathrm{V}_\lambda \subseteq M$, where λ is the supremum of the critical sequence of j.
- An I1-*embedding* is a non-trivial elementary embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ for some ordinal λ.

Iterations of rank-into-rank embeddings

Recall the following concept from the last lecture:

Definition

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an elementary embedding for some limit ordinal λ, then we define

$$j_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1};\ A \longmapsto \bigcup\{j(A \cap \mathrm{V}_\gamma) \mid \gamma < \lambda\}.$$

Theorem

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an elementary embedding for some limit ordinal λ, then j_+ is the unique Σ_0-elementary function from $\mathrm{V}_{\lambda+1}$ into itself that extends j.

Note that, given a limit ordinal λ and elementary embeddings j and k from V_λ into itself, the map j is a subset of V_λ and hence we can apply k_+ to it.

Lemma

Let λ be a limit ordinal and let $k, j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ be elementary embeddings.

- The set $k_+(j)$ is an elementary embedding of V_λ into itself.
- If $j = \mathrm{id}_{\mathrm{V}_\lambda}$, then $k_+(j) = \mathrm{id}_{\mathrm{V}_\lambda}$.
- If j is an I3-embedding with critical sequence $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$, then $k_+(j)$ is an I3-embedding with critical sequence $k_+(\vec{\kappa}) = \langle k(\kappa_n) \mid n < \omega \rangle$.

Proof.

Since k_+ is a Σ_0-elementary embedding, we know that $k_+(j)$ is a function with domain V_λ.

Given a set-theoretic formula $\varphi(v_0, ..., v_{n-1})$, we have

$$\mathrm{V}_{\lambda+1} \models \forall x_0, \ldots, x_{n-1} \in \mathrm{V}_\lambda \\ [\varphi^{\mathrm{V}_\lambda}(x_0, \ldots, x_{n-1}) \longleftrightarrow \varphi^{\mathrm{V}_\lambda}(j(x_0), \ldots, j(x_{n-1}))]$$

and therefore the Σ_0-elementarity of k_+ implies that

$$\mathrm{V}_{\lambda+1} \models \forall x_0, \ldots, x_{n-1} \in \mathrm{V}_\lambda \\ [\varphi^{\mathrm{V}_\lambda}(x_0, \ldots, x_{n-1}) \longleftrightarrow \varphi^{\mathrm{V}_\lambda}(k_+(j)(x_0), \ldots, k_+(j)(x_{n-1}))].$$

The other statements can be proven in a similar way using the Σ_0-elementarity of k_+. $\square$

Given a limit ordinal λ, various statements can be proven about the algebraic structure of sets of elementary embeddings from V_λ into itself equipped with the operations of composition and application.

The following equality will be used in the subsequent constructions:

Proposition

If λ is a limit ordinal and $k, j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ are elementary embeddings, then

$$k_+ \circ j_+ \;=\; (k_+(j))_+ \circ k_+.$$

Fix an I3-embdding $j : \mathrm{V}_\lambda \longrightarrow \lambda$ with critical sequence $\langle \kappa_\ell \mid \ell < \omega \rangle$. The above lemma allows us to inductively define a sequence

$$\langle j_{n,n+1} : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda \mid n < \omega \rangle$$

of I3-embeddings by setting

- $j_{0,1} = j$, and
- $j_{n+1,n+2} \ = \ j_+(j_{n,n+1})$ for all $n < \omega$.

Given $n < \omega$, we then know that $\langle \kappa_{n+\ell} \mid \ell < \omega \rangle$ is the critical sequence of $j_{n,n+1}$.

Moreover, the above proposition shows that

$$j_{n+1,n+2} \ = \ j^+(j_{n,n+1}) \ = \ j^+_{n,n+1}(j_{n,n+1})$$

holds for all $n < \omega$.

We now let

$$\langle j_{m,n} : V_\lambda \longrightarrow V_\lambda \mid m \leq n < \omega \rangle$$

denote the directed system of elementary embeddings induced by the above maps.

Then each $j_{m,n}$ with $m < n$ is an I3-embedding with $\mathrm{crit}(j_{m,n}) = \kappa_m$ and

$$j_{n,n+k}(\kappa_{n+\ell}) \; = \; \kappa_{m+\ell+k}$$

holds for all $k, \ell, n < \omega$.

Let

$$\langle (M_j, E_j), \; \langle j_{n,\omega} : V_\lambda \longrightarrow M_j \mid n < \omega \rangle \rangle$$

denote the direct limit of this system.

In addition, let $wf(E_j)$ denote the well-founded part of (M_j, E_j) and identify it with its transitive collapse.

Easy computations now show that $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq wf(M_j)$ and $j_{0,\omega}(\kappa_0) = \lambda$.

Definition

A I3-embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is *iterable* if the corresponding direct limit (M_j, E_j) is well-founded.

The next result shows that the consistency strength of the existence of iterable I3-embeddings is strictly larger than the consistency strength of the existence of I3-embeddings:

Theorem

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an iterable I3-embedding and $\alpha < \lambda$, then there is an I3-embedding $i : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ for some $\alpha < \lambda' < \lambda$.

Proof.

Define T to be the set of all partial elementary embeddings $i : V_\lambda \xrightarrow{part} V_\lambda$ such that there exists a natural number $0 < \ell < \omega$ and a strictly increasing sequence $\langle \lambda_k \mid k \leq \ell \rangle$ of cardinals below λ with $\alpha < \lambda_\ell$, $\mathrm{dom}(i) = V_{\lambda_{\ell-1}}$, $\mathrm{ran}(i) \subseteq V_{\lambda_\ell}$, $i \upharpoonright \lambda_0 = \mathrm{id}_{\lambda_0}$, $i(\lambda_k) = \lambda_{k+1}$ and $V_{\lambda_k} \prec V_\lambda$ for all $k < \ell$.

By ordering T under inclusion, we can turn this set into a tree of height at most ω. Since the restrictions of j to sufficiently large elements of the critical sequence are elements of T, we know that T has height ω and there is a cofinal branch through T.

Since the tree T is definable in M_j, it is an element of M_j and the absoluteness of ill-foundedness yields a cofinal branch b through T in M_j.

By the definition of T, there is $\alpha < \lambda' \leq \lambda$ such that $\bigcup b : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ is an I3-embedding in M_j.

Since $\lambda = j_{0,\lambda}(\mathrm{crit}(j))$ is regular in M_j, it follows that $\lambda' < \lambda$. □

Theorem

If $i : \mathrm{V} \longrightarrow M$ is an I2-embedding and λ is the supremum of the critical sequence of j, then $i \restriction \mathrm{V}_\lambda$ is iterable.

Proof.

Set $j = i \restriction \mathrm{V}_\lambda$ and let $\langle j_{m,n} : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda \mid m \leq n < \omega \rangle$ denote the corresponding directed system of I3-embeddings.

Remember that, in the last session, we showed that from the parameter j, we can define an I2-embedding $i_{0,1} : \mathrm{V} \longrightarrow N_1$ with $i_{0,1} \restriction \mathrm{V}_\lambda = j$. Then $i_{0,1}(j) = j_+(j) = (j_{0,1})_+(j_{0,1}) = j_{1,2}$.

We can now repeat this construction in N_1 using the parameter $j_{1,2}$ and define an I2-embedding $i_2 : N_1 \longrightarrow N_2$ in N_1 with the property that $i_{1,2} \restriction \mathrm{V}_\lambda = j_{1,2}$ and $i_{1,2}(j_{1,2}) = (j_{1,2})_+(j_{1,2}) = j_{2,3}$.

By iteration this process, we obtain a directed system

$$(\langle N_n \mid n < \omega\rangle, \langle i_{m,n} : N_m \longrightarrow N_n \mid m \leq n < \omega\rangle)$$

of inner models and elementary embeddings with $N_0 = \mathrm{V}$ and if $n < \omega$, then $i_{n,n+1} : N_n \longrightarrow N_{n+1}$ is the I2-embedding constructed from $j_{n,n+1} = i_{0,n}(j)$ in N_n. In particular, we have

$$j_{m,n} \ = \ i_{m,n} \restriction \mathrm{V}_\lambda$$

for all $m \leq n < \omega$. In V, this system is definable from the parameter j.

Given $k < \omega$, if we carry out the same construction in N_k using the parameter $j_{k,k+1}$, then we obtain the directed system

$$\vec{N}_k \ = \ (\langle N_n \mid k \leq n < \omega\rangle, \langle i_{m,n} : N_m \longrightarrow N_n \mid k \leq m \leq n < \omega\rangle).$$

Moreover, for all $k, m, n < \omega$, we have $i_{0,k}[N_m] \subseteq N_{m+k}$ and

$$i_{0,k} \circ i_{m,n} \ = \ i_{m+k,n+k} \circ (i_{0,k} \restriction N_m).$$

Let

$$((N,E),\langle i_{n,\omega} : N_n \longrightarrow N \mid n < \omega\rangle)$$

denote the direct limit of $\vec{N}_0$.

Given $k < \omega$, we then know that

$$((N,E),\langle i_{k+n,\omega} : N_{k+n} \longrightarrow N \mid n < \omega\rangle)$$

is the direct limit of $\vec{N}$ in N_k and $i_{0,k}[N] \subseteq N$ holds.

Moreover, for all $n < \omega$, we have

$$i_{0,k} \circ i_{n,\omega} \ = \ i_{k+n,\omega} \circ (i_{0,k} \restriction N_n).$$

Assume that (N, E) is ill-founded and let α be the minimal ordinal ξ such that $i_{0,\omega}(\xi)$ is an element of the ill-founded part of N.

Given $k < \omega$, elementarity implies that $i_{0,k}(\alpha)$ is the minimal ordinal ξ such that $i_{k,\omega}(\xi)$ is an element of the ill-founded part of N.

The ill-founded part of N contains an ordinal smaller than $i_{0,\omega}(\alpha)$ and we can find $k < \omega$ and an ordinal β such that $i_{k,\omega}(\beta)$ is an element of the ill-founded part of N and $i_{k,\omega}(\beta) \mathrel{E} i_{0,\omega}(\alpha)$ holds.

Since this means that $\beta < i_{0,k}(\alpha)$, we obtained a contradiction.

This shows that (N, E) is well-founded. By construction, the direct limit (M_j, E_j) of

$$\langle j_{m,n} : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda \mid m \leq n < \omega \rangle$$

is an initial segment of (N, E) and we therefore know that (M_j, E_j) is well-founded. $\square$

Prikry forcing

Recall that, if U is a normal ultrafilter over a measurable cardinal κ, then *Prikry forcing with* U is the partial order $\mathbb{P}_U$ defined by the following clauses:

- Conditions in $\mathbb{P}_U$ are pairs $p = (s_p, A_p)$ with the property that $A_p \in U$ and $s_p : n_p \longrightarrow \min(A_p)$ is a strictly increasing function with $n_p < \omega$.
- Given conditions p and q in $\mathbb{P}_U$, we have $p \leq_{\mathbb{P}_U} q$ if and only if $A_p \subseteq A_q$, $n_q \leq n_p$, $s_p \upharpoonright n_q = s_q$ and $s_p(\ell) \in A_q$ for all $n_q \leq \ell < n_p$.

Theorem

Let U be a normal ultrafilter over a measurable cardinal κ and let G be $\mathbb{P}_U$-generic over V.

- V and $\mathrm{V}[G]$ have the same cardinals and contain the same bounded subsets of κ.
- The set

$$x_G = \bigcup \{\mathrm{ran}(s_p) \mid p \in G\}$$

is a cofinal subset of κ of order-type ω in $\mathrm{V}[G]$.

Lemma

If U is a normal ultrafilter over a measurable cardinal κ and G is $\mathbb{P}_U$-generic over V, then

$$U = \{A \in \mathcal{P}(\kappa)^{\mathrm{V}} \mid \exists \alpha < \kappa \quad x_G \setminus \alpha \subseteq A\}.$$

In particular, we have $\mathrm{V}[G] = \mathrm{V}[x_G]$.

Theorem (Mathias criterion)

If M is an inner model of ZFC and U is a normal ultrafilter over a measurable cardinal κ in M, then the following statements are equivalent for every cofinal subset x of κ of order-type ω:

- There exists G $\mathbb{P}_U^M$-generic over M with $x = x_G$.
- For all $A \in U$, there is $\alpha < \kappa$ with $x \setminus \alpha \subseteq A$.

Lemma

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an iterable I3-embedding with critical sequence $\vec{\kappa} = \langle \kappa_\ell \mid \ell < \omega \rangle$, $j_{0,\omega} : \mathrm{V}_\lambda \longrightarrow M_j$ is the corresponding ω-th iteration map and

$$U = \{A \in \mathcal{P}(\kappa_0) \mid \kappa_0 \in j(A)\}$$

is the induced normal ultrafilter over κ_0, then $M_j[\vec{\kappa}]$ is a $\mathbb{P}^{M_j}_{j_{0,\omega}(U)}$-generic extension of M_j.

Proof.

Fix $A \in j_{0,\omega}(U)$. Then there is $k < \omega$ and $A_k \subseteq \kappa_k$ with $A = j_{k,\omega}(A_k)$.

Given $k \leq \ell < \omega$, elementarity implies that

$$j_{k,\ell}(A_k) \in j_{0,\ell}(U) = \{B \in \mathcal{P}(\kappa_\ell) \mid \kappa_\ell \in j_{\ell,\ell+1}(B)\}$$

and hence $\kappa_\ell \in A$ for all $k \leq \ell < \omega$.

By the Mathias criterion, the cofinal subset $\{\kappa_\ell \mid \ell < \omega\}$ of λ induces a $\mathbb{P}^{M_j}_{j_{0,\omega}(U)}$-generic filter over M_j. □

3. Tutorial

In this talk, we will use the theory developed in the previous two sessions to prove that the consistency strength of a large cardinal principle isolated in recent joint work with Juan Aguilera (TU Wien) and Joan Bagaria (Barcelona) is surprisingly low.

As a motivation for the formulation of this notion, we consider a large cardinal property appearing in the work of Gabriel Goldberg and Farmer Schlutzenberg that, by the Kunen Inconsistency, contradicts the Axiom of Choice:

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

It now turns out that we can weaken this property in a canonical way in order to obtain an axiom that does not contradict the Kunen Inconsistency and still possesses key features of rank-Berkeleyness.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding

$$j : X \longrightarrow \mathrm{V}_\beta$$

with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

It can be easily show that the exactingness of a cardinal is witnessed by a single embedding whose domain is an elementary substructure of a sufficiently elementary initial segment of the set-theoretic universe.

Proposition

The following statements are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- There is an ordinal $\beta > \lambda$ with $\mathrm{V}_\beta \prec_{\Sigma_2} \mathrm{V}$, an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and an elementary embedding

  $$j : X \longrightarrow \mathrm{V}_\beta$$

  with $\mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

We continue by showing that the above notion can also be obtained by strengthening well-studied large cardinal notions defined through model-theoretic reflection properties.

Definition

A cardinal λ is Jónsson if every structure in a countable first-order language with domain λ has a proper elementary substructure of cardinality λ.

Theorem (Aguilera–Bagaria–L.)

The following are equivalent for each cardinal λ with $|V_\lambda| = \lambda$:

- λ is an exacting cardinal.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ isomorphic to a structure in $\mathcal{C}$.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ is isomorphic to a proper elementary substructure of a structure of cardinality λ in $\mathcal{C}$.

In another direction, exactingness can also be represented as a natural strengthening of the existence of I3-embeddings:

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- For every non-empty, ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exist $x, y \in A$ and a non-trivial elementary embedding

$$j : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y).$$

Proof.

First, assume that λ is an exacting cardinal and the second property fails.

Let A be the minimal non-empty subset of $\mathrm{V}_{\lambda+1}$ in the canonical well-ordering of OD with the property that for all $x, y \in A$, there is no elementary embedding $i : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y)$.

Then A is definable from the parameter λ.

Pick $\beta > \lambda$ such that V_β is sufficiently elementary. Then there exists an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding

$$j : X \longrightarrow \mathrm{V}_\beta$$

with $\mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

We then know that $A \in X$, $j(A) = A$ and there is $x \in A \cap X$.

Then $j(x) \in A$ and $j \restriction \mathrm{V}_\lambda : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, j(x))$ is a non-trivial elementary embedding, a contradiction.

Now, assume that λ has the property that for every non-empty, ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exist $x, y \in A$ and a non-trivial elementary embedding $j : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y)$. Then $|\mathrm{V}_\lambda| = \lambda$.

Pick $\beta > \lambda$ with $\mathrm{V}_\beta \prec_{\Sigma_2} \mathrm{V}$ and let A denote the set of all subsets of V_λ that code elementary submodels X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ in some fixed canonical way.

Then A is a non-empty, ordinal definable subset of $\mathrm{V}_{\lambda+1}$ and we can find $x, y \in A$ with the property that there exists a non-trivial elementary embedding $i : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y)$.

If we picked our coding of substructures in the right way, then this embedding yields elementary submodels X and Y of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X \cap Y$ and an elementary embedding

$$j : X \longrightarrow Y \prec \mathrm{V}_\beta$$

with $\mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$. □

The above result can be used to show that exacting cardinals have the surprising property that they imply the existence of sets that are not ordinal definable.

Theorem (Aguilera–Bagaria–L.)

If λ is an exacting cardinal, then λ is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Proof.

Assume that λ is singular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$. Then λ is singular in $\mathrm{HOD}_{\{z\}}$ for some $z \in \mathrm{V}_\lambda$. Moreover, since λ is a limit of inaccessible cardinals, a theorem of Vopenka shows that $\mathrm{HOD}_{\{z\}}$ is a forcing extension of HOD using a partial order of size strictly less than λ and therefore we know that λ is also singular in HOD.

Let c be the least cofinal subset of λ with order-type $\mathrm{cof}(\lambda)^{\mathrm{HOD}}$ and $\min(c) > \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ in the canonical well-ordering of HOD.

Then $\{c\}$ is a non-empty ordinal definable subset of $\mathrm{V}_{\lambda+1}$ and there exists a non-trivial elementary embedding $j : (\mathrm{V}_\lambda, \in, c) \longrightarrow (\mathrm{V}_\lambda, \in, c)$.

We then have $\mathrm{crit}(j) > \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ and $j \restriction c = \mathrm{id}_c$, a contradiction. $\square$

Lower bounds for the consistency strength of exactingness

We want to show that there are many I3-embeddings below an exacting cardinals.

Fur this purpose, we adapt the corresponding tree argument for iterable I3-embeddings by replacing the ω-th iterate M_j with the inner model $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Here, the additional complication arises that the Axiom of Choice might fail in $\mathrm{HOD}_{\mathrm{V}_\lambda}$ and therefore the ill-foundedness of trees is not equivalent to the existence of cofinal branches in this model.

This is taken care of with the following lemma:

Lemma

Let λ be a strong limit cardinal, let M be an inner model of ZF with $\mathrm{V}_\lambda \subseteq M$ and let $T \in M$ be a tree of height ω whose underlying set is a subset of V_λ. If T has an infinite branch in V and λ is regular in M, then T has an infinite branch in M.

Since we know that an exacting cardinal λ is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$, we can repeat an earlier argument using the tree of partial attempts to build an I3-embedding with this inner model and obtain the following result:

Theorem(Aguilera–Bagaria–Goldberg–L.)

If λ is an exacting cardinal and $\alpha < \lambda$, then there is an I3-embedding $i : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ for some $\alpha < \lambda' < \lambda$.

Upper bounds for the consistency strength of exactingness

The first consistency result for exacting cardinals contained in joint work with Aguilera and Bagaria derived this consistency from assumptions much stronger than the existence of an I1-embedding.

The following result shows that the consistency strength is surprisingly low:

Theorem (Aguilera–Bagaria–Goldberg–L.)

Let $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ be an iterable I3-embedding with critical point κ and let

$$U \ = \ \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

If G is $\mathbb{P}_U$-generic over V, then κ is an exacting cardinal in $\mathrm{V}[G]$.

The starting point of the proof of the above result is the observation that, in the directed system constructed from an iterable I3-embeddings, images of subsets of the critical point of the original embedding in the limit model are invariant under the original embedding:

Lemma

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an iterable I3-embedding, $j_{0,\omega} : \mathrm{V}_\lambda \longrightarrow M_j$ is the corresponding embedding into the direct limit and $A \subseteq \mathrm{V}_{\mathrm{crit}(j)}$, then

$$j_+(j_{0,\omega}(A)) \ = \ j_{0,\omega}(A).$$

Proof of the Theorem.

Let $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ be an iterable I3-embedding with critical point κ, let $j_{0,\omega} : \mathrm{V}_\lambda \longrightarrow M_j$ denote corresponding embedding into the direct limit of the induced directed system and set $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$.

Then M_j is a transitive set with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq M_\omega$, $\mathrm{crit}(j_{0,\omega}) = \kappa$ and $j_{0,\omega}(\kappa) = \lambda$.

Fix $\rho > \lambda$ such that V_ρ is sufficiently elementary in V and pick an elementary submodel X of V_ρ of cardinality κ with $\mathrm{V}_\kappa \cup \{U\} \subseteq X$.

Let $\pi : X \longrightarrow N$ denote the transitive collapse. Pick a well-founded relation R on κ such that $(N, \in)$ is the transitive collapse of (κ, R) and this collapse sends 0 to κ, 1 to $\pi(U)$ and $\omega \cdot \alpha$ to α for all $0 < \alpha < \kappa$.

Set $N_* = j_{0,\omega}(N)$, $R_* = j_{0,\omega}(R)$ and $U_* = j_{0,\omega}(\pi(U))$.

Then $\mathrm{V}_\lambda \subseteq N_*$ and the critical sequence $\vec{\kappa}$ of j is $\mathbb{P}^{N_*}_{U_*}$-generic over N_*.

We then have $j_+(R_*) = R_*$ and $(N_*, \in)$ is the transitive collapse of (λ, R_*) and this collapse τ sends 0 to λ, 1 to U_* and $\omega \cdot \alpha$ to α for $0 < \alpha < \lambda$.

We now know that

$$i \;=\; \tau \circ (j \restriction \lambda) \circ \tau^{-1} : N_* \longrightarrow N_*$$

is an elementary embedding with $i \restriction (\lambda+1) = j \restriction (\lambda+1)$ and $j(U_*) = U_*$.

In this situation, the embedding i can be canonically lifted to

$$i_* : N_*[\vec{\kappa}] \longrightarrow N_*[\vec{\kappa}].$$

Now, in $N_*[\vec{\kappa}]$, fix a non-empty subset A of $\mathrm{V}_{\lambda+1}$ that is definable by a formula with parameter λ. Pick $x \in A$ and set $y = i_*(x)$. Then $y \in A$ and i_* induces a non-trivial elementary embedding of $(\mathrm{V}_\lambda, \in, x)$ into $(\mathrm{V}_\lambda, \in, y)$.

Since λ has countable cofinality in $N_*[\vec{\kappa}]$ and this model satisfies a sufficiently large fragment of ZFC, a well-foundedness argument shows that such an embedding already exists in $N_*[\vec{\kappa}]$.

The characterization of exacting cardinals through I3-embeddings now shows that λ is an exacting cardinal in $N_*[\vec{\kappa}]$ and hence elementarity ensures that Prikry forcing with U over V turns κ into an exacting cardinal. $\square$

Thank you for listening!