Weak compactness cardinals for strong logics

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We aim to further explore the deep connections between large cardinal axioms and compactness properties of strong logics, starting from the following classical result of Magidor:

Theorem (Magidor)

There exists an extendible cardinal if and only if second-order logic $\mathcal{L}^2$ has a strong compactness cardinal.

Given a cardinal κ, an $\mathcal{L}^2$-theory T is $<\kappa$*-consistent* if every subtheory of T of cardinality less than κ is consistent.

A cardinal κ is a *strong compactness cardinal for* $\mathcal{L}^2$ if every $<\kappa$-consistent $\mathcal{L}^2$-theory is consistent.

An *abstract logic* is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ consisting of

- a class function $\mathcal{L}$ that maps signatures σ to sets $\mathcal{L}(\sigma)$ of $\mathcal{L}$-sentences, and
- a satisfaction relation $\models_{\mathcal{L}}$ that determines which $\mathcal{L}$-sentences $\phi \in \mathcal{L}(\sigma)$ hold in σ-structures

that satisfy certain canonical rules about invariance under isomorphic copies, extensions of signatures, and boundedness of the sizes of signatures generating sentences.

Given an abstract logic $\mathcal{L}$ and a cardinal κ, an $\mathcal{L}$-theory T is *$<\kappa$-consistent* if every subtheory of cardinality less than κ is consistent.

A cardinal κ is a *strong compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-consistent $\mathcal{L}$-theory is consistent.

Theorem (Makowsky)

The following schemes are equivalent over $\mathbf{ZFC}$:

- Every abstract logic has a strong compactness cardinal.
- Vopěnka's Principle, i.e. for every proper class of graphs, there is an embedding between two distinct members of the class.

Weak compactness cardinals

A cardinal κ is a *weak compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-consistent $\mathcal{L}$-theory of cardinality κ is consistent.

Recent work of Boney, Dimopoulos, Gitman and Magidor connects this weaker property to the large cardinal notion of *subtleness*, introduced by Jensen and Kunen in their studies of strong diamond principles.

Definition (Jensen–Kunen)

A cardinal δ is *subtle* if for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Definition

"Ord *is subtle*" is the scheme of axioms stating that for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Ord}\rangle$ and every closed unbounded class C of ordinals, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Theorem (Boney–Dimopoulos–Gitman–Magidor)

The following schemes are equivalent over $\mathbf{ZFC}$ together with the existence of a definable global well-ordering:

- Ord is subtle.
- Every abstract logic has a stationary class of weak compactness cardinals.

This result leaves open several questions:

- Is it necessary to assume the existence of a global well-ordering?
- Can we characterize the existence of weak compactness cardinals for all abstract logics through large cardinal properties of Ord?
- Does the existence of weak compactness cardinals for all abstract logics imply the existence of an inaccessible cardinal?

Observation

The following statements are equivalent for every infinite cardinal δ:

- δ is subtle.
- For all closed unbounded subsets C of δ and all sequences $\langle \mathcal{E}_\gamma \mid \gamma < \delta \rangle$ with $\emptyset \neq \mathcal{E}_\gamma \subseteq \mathcal{P}(\gamma)$ for all $\gamma < \delta$, there are $\beta < \gamma$ in C and $E \in \mathcal{E}_\gamma$ with $E \cap \beta \in \mathcal{E}_\beta$.

Definition (Bagaria–L.)

We let "Ord *is essentially subtle*" denote the scheme of axioms stating that for every closed unbounded class C of ordinals and every class sequence $\langle \mathcal{E}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ with $\emptyset \neq \mathcal{E}_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$, there exist $\alpha < \beta$ in C and $E \in \mathcal{E}_\beta$ with $E \cap \alpha \in \mathcal{E}_\alpha$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$:

- Ord is essentially subtle.
- Every abstract logic has a stationary class of weak compactness cardinals.

Lemma (Bagaria–L.)

The following statements are equivalent for all cardinals δ with $H_\delta = V_\delta$:

- δ is either subtle or a limit of subtle cardinals.
- For every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and all $\xi < \delta$, there are cardinals $\xi < \mu < \nu < \delta$ with $A_\mu = A_\nu \cap \mu$.
- For every sequence $\langle \mathcal{E}_\gamma \mid \gamma < \delta \rangle$ such that $\emptyset \neq \mathcal{E}_\gamma \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma < \delta$ and all $\xi < \delta$, there are cardinals $\xi < \mu < \nu < \delta$ and $E \in \mathcal{E}_\nu$ with $E \cap \mu \in \mathcal{E}_\mu$.

Definition

We let "Ord *is essentially closure subtle*" denote the scheme of axioms stating that every class sequence $\langle \mathcal{E}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ with $\emptyset \neq \mathcal{E}_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$ and all $\xi \in \mathrm{Ord}$, there are cardinals $\xi < \mu < \nu$ and $E \in \mathcal{E}_\nu$ with $E \cap \mu \in \mathcal{E}_\mu$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$:

- Ord is essentially closure subtle.
- Every abstract logic has a weak compactness cardinal.

We now explore the differences between the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

and the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”}.$$

Proposition

If Φ is a sentence in the language of set theory with the property that $\mathbf{ZFC} + \Phi$ is consistent, then

$$\mathbf{ZFC} + \Phi \nvdash \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}.$$

Theorem

The following statements are equivalent:

- There exists a sentence Φ in the language of set theory such that the theory $\mathbf{ZFC} + \Phi$ is consistent and

$$\mathbf{ZFC} + \Phi \vdash \text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”}.$$

- $\mathbf{ZFC} + \text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”} \nvdash \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$.

- The theory

$$\mathbf{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals}\text{”}$$

  is consistent.

The techniques developed in the proofs of the above results also allow us to show that the existence of weak compactness cardinals for all abstract logics does not imply the existence of strongly inaccessible cardinals in V.

Theorem

The following schemes are equiconsistent over $\mathbf{ZFC}$:

- There is a proper class of subtle cardinals.
- Ord is essentially closure subtle and there are no inaccessible cardinals.

Weakly $C^{(n)}$-shrewd cardinals

We now relate the existence of weak compactness cardinals to large cardinal properties.

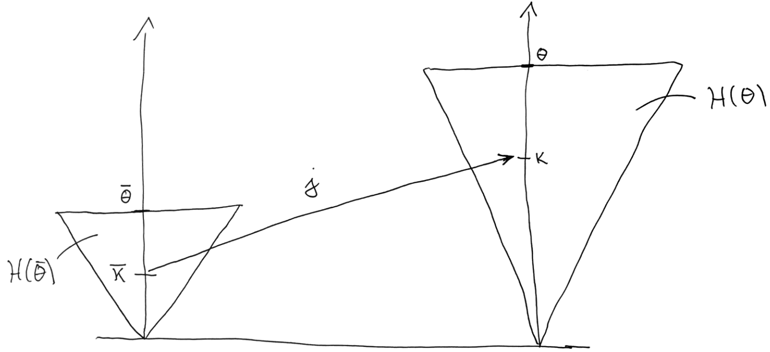
The starting point of these results is the following classical result:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ:

- κ is supercompact.
- For every cardinal $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist
  - cardinals $\bar{\kappa} < \bar{\theta} < \kappa$, and
  - an elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$

  such that $\mathrm{crit}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.



θ
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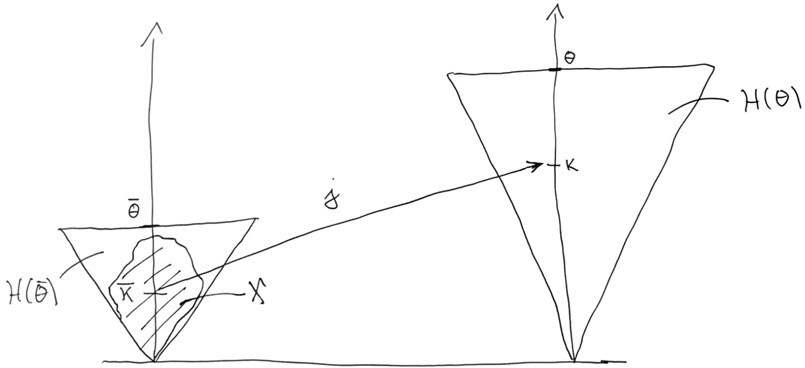
κ
j
θ̄
H(θ̄)
κ̄



Θ
$\mathcal{H}(\Theta)$
K
j
$\bar{\Theta}$
$\mathcal{H}(\bar{\Theta})$
$\bar{K}$
X

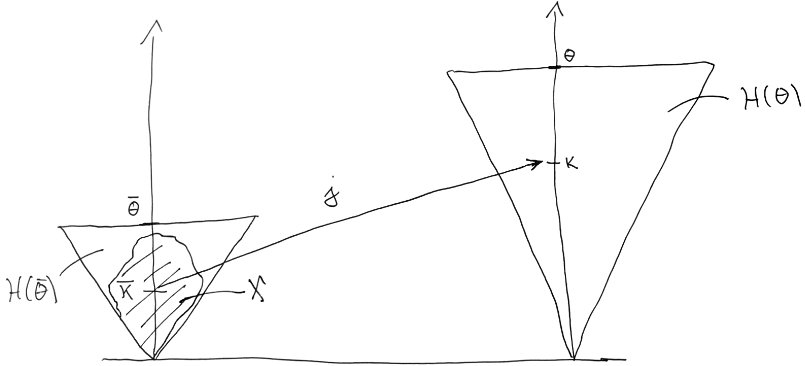
Theorem

The following statements are equivalent for every cardinal κ:

- For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist
  - cardinals $\bar{\kappa} < \bar{\theta} < \kappa$,
  - an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

  such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.
- κ is a strongly unfoldable cardinal.[1]
- κ is a shrewd cardinal.[2]

[1] Introduced by Villaveces.
[2] Introduced by Rathjen.


θ̄
H(θ̄)
K̄
X
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θ
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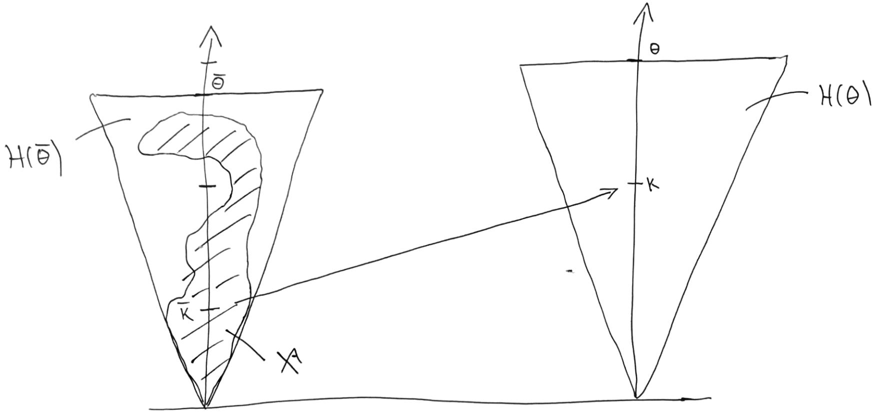

$\bar{\Theta}$
$H(\bar{\Theta})$
$\bar{K}$
X
Θ
$H(\Theta)$
K

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta$ with $V_\theta \prec_{\Sigma_n} V$ and every $z \in \mathrm{H}(\theta)$, there exists

- a cardinal $\bar{\theta}$ with $V_{\bar{\theta}} \prec_{\Sigma_n} V$,
- a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$:

- Ord is essentially closure subtle.
- For every natural number $n > 0$, there is a proper class of weakly $C^{(n)}$-shrewd cardinals.
- Every abstract logic has a weak compactness cardinal.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$:

- Ord is essentially subtle.
- For every natural number $n > 0$, there is a weakly $C^{(n)}$-shrewd cardinal κ with $V_\kappa \prec_{\Sigma_{n+1}} V$.
- Every abstract logic has a stationary class of weak compactness cardinals.

Thank you for listening!

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Definition

An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- κ is a weakly shrewd cardinal.
- For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist
  - cardinals $\bar{\kappa} < \bar{\theta}$,
  - an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

  with $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $z \in \mathrm{ran}(j)$.

Theorem

The following statements are equivalent for every cardinal δ:

- The cardinal δ is subtle.
- For every function $F : \delta \longrightarrow \mathrm{H}(\delta)$, there exists a cardinal $\kappa < \delta$ with the following properties:
  - $F[\kappa] \subseteq \mathrm{H}(\kappa)$.
  - For every $\gamma < \delta$ and every transitive set M of cardinality κ with $\kappa \cup \{\kappa, F \restriction \kappa\} \subseteq M$, there exists
    - a transitive set N with $\gamma \in N$, and
    - a non-trivial elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$ and $j(F \restriction \kappa) \restriction \gamma = F \restriction \gamma$.