Σ_1-definability at higher cardinals

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Introduction

Various fundamental results in descriptive theory show that simply definable sets of real numbers (e.g., Borel sets) possess a rich and canonical structure theory.

Seminal results show that canonical extensions of $\mathbf{ZFC}$ allow us to also establish these structural results for more complicated sets of reals.

Theorem (Shelah–Woodin)

The existence of a supercompact cardinal implies that there is no projective well-ordering of the reals.

Implications of this form are often used to measure the beneficial influence of candidates for new axioms for mathematics.

It is natural to ask whether these implications can be extended to classes of definable sets of higher cardinalities.

It turns out that the strong combinatorics of higher cardinals prevent us from directly generalizing central parts of the classical theory to them.

In particular, basically all theorems on the beneficial influence of strong axioms do not generalize to higher cardinals.

Remember that a set of reals is projective if and only if it is definable in the set $\mathrm{H}(\aleph_1)$ of all hereditary countable sets by a formula with parameters.

Theorem (S. Friedman – Holy)

If the existence of a supercompact cardinal is consistent with the axioms of $\mathbf{ZFC}$, then the existence of a well-ordering of $\mathcal{P}(\omega_1)$ that is definable in $\mathrm{H}(\aleph_2)$ is independent of this theory.

While a general transfer of the classical theory to higher cardinals is not possible, it turns out that deep and fruitful theories can be developed in certain restricted settings.

One such setting is provided by the work Woodin on large cardinal assumptions close to the *Kunen Inconsistency* that derives analogs of classical results at higher singular cardinals from very strong large cardinal assumptions (with completely different proofs).

Theorem (Woodin)

If there is a non-trivial elementary embedding

$$j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$$

with critical point below λ, then no well-ordering of $\mathcal{P}(\lambda)$ is definable in $\mathrm{H}(\lambda^+)$.

In this talk, I want to focus on a different setting that drastically restricts the complexity of the formulas and parameters used in definitions.

It then turns out that the beneficial influence of strong axioms is reflected in the structural properties of these simply definable sets.

Definition

A class X is *definable* by a formula $\varphi(v_0, \ldots, v_n)$ and parameters $z_0, \ldots, z_{n-1}$ if

$$X = \{y \mid \varphi(y, z_0, \ldots, z_{n-1})\}.$$

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.
- Given $n < \omega$, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \neg\varphi(x)$ for some Σ_n-formula φ.

In the following, we will consider subsets of $\mathcal{P}(\kappa)$ for uncountable cardinals κ that are defined by Σ_1-formulas with parameters in $\mathrm{H}(\kappa^+)$.

The main structural features of the class of sets definable in this way are:

- The Σ_1-Recursion Theorem.
- Σ_1-Upwards Absoluteness.

Unfortunately, it turns out that these classes can still be too large if we allow arbitrary subsets of κ as parameters:

Theorem (L.)

If the existence of a supercompact cardinal is consistent with the axioms of $\mathbf{ZFC}$, then the existence of a well-ordering of $\mathcal{P}(\omega_1)$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\aleph_2)$ is independent of this theory.

If we also restrict the parameters used in Σ_1-definitions, then we can recover the beneficial influence of canonical extensions of $\mathbf{ZFC}$ on definability:

Theorem (L.–Schindler–Schlicht)

The existence of a supercompact cardinal implies that no well-ordering of $\mathcal{P}(\omega_1)$ is definable by a Σ_1-formula that uses only the cardinal ω_1 and real numbers as parameters.

Theorem (L.–Schindler–Schlicht)

If there exists a supercompact cardinal, then the following statements are equivalent for every set A of real numbers:

- The set A is $\mathbf{\Sigma}^1_3$-definable.
- The set A is definable by a Σ_1-formula that only uses the cardinal ω_1 and real numbers as parameters.

Direct analogs of the above conclusion hold at higher cardinals with strong combinatorial properties:

Theorem (L.–Schindler–Schlicht)

If κ is a supercompact cardinal, then no well-ordering of $\mathcal{P}(\kappa)$ is definable by a Σ_1-formula that only uses the cardinal κ and elements of $\mathrm{H}(\kappa)$ as parameters.

Theorem (L.–Müller)

If κ is a limit of measurable cardinals, then no well-ordering of $\mathcal{P}(\kappa)$ is definable by a Σ_1-formula that only uses the cardinal κ and elements of $\mathrm{H}(\kappa)$ as parameters.

The techniques developed in the proofs of the above results can also be used to show that other types of pathological sets are not simply definable at uncountable cardinals with strong combinatorial properties.

In the following, I will present joint work with Omer Ben-Neria (Jerusalem) that studies Σ_1-definable closed unbounded sets and connects notions of stationarity given by these sets to important set-theoretic questions.

Σ_1-stationary sets

Recall that, given an infinite cardinal κ, a subset of κ is ...

- ... *closed unbounded* if it is unbounded in κ and contains all of its limit points in κ.
- ... *stationary* if it intersects every closed unbounded subset.

Central aspects of the combinatorics of uncountable regular cardinals κ are given by the fact that the collection of all non-stationary subsets of κ forms a normal ideal on κ.

In contrast, the above notions trivialize at cardinals of countable cofinality, i.e., given a cardinal κ of countable cofinality, a subset of κ is stationary if and only if it is cobounded.

Definition

Let κ be a uncountable cardinal, let $n < \omega$ and let A be a class.

- $S \subseteq \kappa$ is *$\Sigma_n(A)$-stationary in κ* if $C \cap S \neq \emptyset$ holds for every closed unbounded subset C of κ with the property that $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$.
- $S \subseteq \kappa$ is *$\mathbf{\Sigma}_n(A)$-stationary in κ* if it is $\Sigma_n(A \cup \mathrm{H}(\kappa))$-stationary in κ.
- $S \subseteq \kappa$ is *$\mathbf{\Sigma}_n$-stationary in κ* if it is $\mathbf{\Sigma}_n(\emptyset)$-stationary in κ.

We focus on the following two questions:

- How much can the collection of $\Sigma_1(A)$-stationary subsets of an uncountable cardinal κ differ from the collection of all stationary subsets of κ? What is the situation at cardinals of countable cofinality, where stationarity coincides with coboundedness?

- For which cardinals is it possible to develop a non-trivial structure theory for $\Sigma_1(A)$-stationary subsets?

Proposition

Assume that *Martin's Maximum* holds. Then a subset of ω_1 is $\mathbf{\Sigma}_1$-stationary in ω_1 if and only if it is stationary in ω_1.

Proof.

Woodin proved that *Martin's Maximum* implies *admissible club guessing*, i.e., for every closed unbounded subset C of ω_1, there is a real x with the property that

$$\{\alpha < \omega_1 \mid \mathrm{L}_\alpha[x] \models \mathrm{KP}\} \ \subseteq \ C$$

holds. $\square$

Jónsson cardinals

Definition

A cardinal κ is *Jónsson* if for every function $f : [\kappa]^{<\omega} \longrightarrow \kappa$, there is a proper subset H of κ of cardinality κ with $f[[H]^{<\omega}] \subseteq H$.

Questions

Does $\mathbf{ZFC}$ prove that ω_ω is not Jónsson?

Theorem (Ben-Neria – L.)

If ω_ω is Jónsson, then every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω.

Definition

Given uncountable cardinals $\mu < \kappa$, we say that the cardinal κ has the $\Sigma_1(\mu)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_1-formula with parameters in the set $\mathrm{H}(\mu) \cup \{\kappa\}$.

Lemma

Given uncountable cardinals $\mu < \kappa$, if the cardinal κ has the $\Sigma_1(\mu)$-undefinability property, then $\{\mu\}$ is $\Sigma_1(\mathrm{H}(\mu))$-stationary in κ.

Corollary

Let κ be a limit cardinal and $E \subseteq \kappa$ be a set of uncountable cardinals that is unbounded in κ. If κ has the $\Sigma_1(\mu)$-undefinability property for all $\mu \in E$, then E is $\boldsymbol{\Sigma}_1$-stationary in κ.

Definition

Given uncountable cardinals $\nu < \kappa$, the cardinal κ is ν-Rowbottom if and only if

$$\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \kappa, {<}\nu \rangle$$

holds for all $\lambda < \kappa$, i.e., given a countable first-order language $\mathcal{L}$ with a unary predicate symbol $\dot{R}$, every $\mathcal{L}$-structure A with domain κ and $|\dot{R}^A| = \lambda$ has an elementary substructure B of size κ with $|\dot{R}^B| < \nu$.

Lemma

- If κ is ν-Rowbottom for some $\nu < \kappa$, then κ is Jónsson.
- If κ is the least Jónsson cardinal, then κ is ν-Rowbottom for some $\nu < \kappa$.

Lemma

Let κ be a ν-Rowbottom cardinal with ν regular, let $y \in \mathrm{H}(\kappa^+)$ and let $z \in \mathrm{H}(\nu)$. Then there exists a transitive set M with $\kappa \in M$ and a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\kappa^+)$ satisfying $\mathrm{crit}(j) < \nu$, $y \in \mathrm{ran}(j)$, $j(\kappa) = \kappa$ and $j(z) = z$.

Lemma

If ω_ω is ω_n-Rowbottom for some $0 < n < \omega$, then ω_ω has the $\Sigma_1(\omega_n)$-undefinability property.

Theorem (Ben-Neria – L.)

If ω_ω is Jónsson, then every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\boldsymbol{\Sigma}_1$-stationary in ω_ω.

We can use the above methods to reduce the class of models of set theory in which ω_ω possesses strong partition properties.

Me specifically, we can show that ω_ω is not ω_2-Rowbottom in the standard models of strong forcing axioms, where the given axiom was forced over a model of the GCH by turning some large cardinal into ω_2.

Theorem (Ben-Neria – L.)

Assume that there are no special ω_2-Aronszajn trees and for all $2 < n < \omega$, there is a special ω_n-Aronszajn tree.

Then the set $\{\omega_2\}$ is definable by a Σ_1-formula with parameter ω_ω and the cardinal ω_ω is not ω_2-Rowbottom.

Consistency strength

Theorem (Ben-Neria – L.)

The following statements are equiconsistent over $\mathbf{ZFC}$:

- Every unbounded subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1(Ord)$-stationary in ω_ω.
- There is a singular cardinal κ of countable cofinality and a subset of κ that consists of cardinals and is $\mathbf{\Sigma}_1$-stationary in κ.
- There is a measurable cardinal.

In contrast, more measurable cardinals are required to obtain an analogous statement for singular cardinals of uncountable cofinality:

Theorem (Ben-Neria – L.)

The following statements are equiconsistent over $\mathbf{ZFC}$:

- There exists a singular cardinal κ of uncountable cofinality such that some non-stationary subset of κ is $\mathbf{\Sigma}_1$-stationary in κ.
- There exists a singular cardinal κ of uncountable cofinality such that some non-stationary subset of κ is $\mathbf{\Sigma}_1(Ord)$-stationary in κ.
- There exist uncountably many measurable cardinals.

Disjoint $\Sigma_1(A)$-stationary sets

At cardinals κ of uncountable cofinality, Solovay's theorem ensures that existence of bistationary (i.e. stationary and costationary) subsets of κ.

In contrast, all stationary subsets of singular cardinals of countable cofinality are cobounded and hence there are no bistationary subsets of these cardinals.

We now consider the question how bistationarity behaves in the definable context.

We can show that there exists a cardinal δ such that ...

- $\mathbf{ZFC}$ proves that for every set A of cardinality less than δ and every singular cardinal κ of countable cofinality, there are disjoint $\Sigma_1(A)$-stationary subsets of κ.

- The following statements are equiconsistent over $\mathbf{ZFC}$:
  - There is a singular cardinal κ of countable cofinality such that for every subset A of $\mathrm{H}(\kappa)$ of cardinality δ, there are disjoint $\Sigma_1(A)$-stationary subsets of κ.
  - There is a measurable cardinal.

The cardinal δ is ... the ***reaping number*** $\mathfrak{r}$.

Definition

The reaping number $\mathfrak{r}$ is the least cardinality of a subset A of $[\omega]^\omega$ with the property that for every $b \in [\omega]^\omega$, there is $a \in A$ such that either $a \setminus b$ or $a \cap b$ is finite.

Proposition

Let κ be a singular cardinal of countable cofinality and let A be a set of cardinality less than $\mathfrak{r}$. Then there exists a subset E of κ with the property that both E and $\kappa \setminus E$ are $\Sigma_1(A)$-stationary in κ.

Theorem (Ben-Neria – L.)

The following statements are equiconsistent over $\mathbf{ZFC}$:

- There is a measurable cardinal.
- There is a singular cardinal κ of countable cofinality such that for every subset A of $\mathrm{H}(\kappa)$ of cardinality $\mathfrak{r}$, there exists a subset E of κ such that both E and $\kappa \setminus E$ are $\Sigma_1(A)$-stationary.

Thank you for listening!