# Games in Set Theory and Logic

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# Games in Set Theory and Logic

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*To My Father and My Mother,*

*who have been proud of me and ashamed of me since my birth.*

# Contents

# Acknowledgments

When I receive a thesis from a colleague of mine, I always start with acknowledgments and usually end with them :) This is not only because their work is not interesting to me but also because acknowledgments are the best part in their theses to see their personalities. Keeping this in mind, I decided to write not only about the list of people I am grateful but also about myself so that the readers can get more information about me. So this is longer than usual acknowledgments in Ph.D. theses and rather informal and filled with personal thoughts. I hope you can enjoy reading it.

To those who are also interested in mathematical parts of this thesis: The rest is formal and does not contain any personal thing. So please do not worry about being fed up with reading about my personality and enjoy the rest!

I first thank my father who introduced mathematics to me. When I was a kid, I was good at calculations (not anymore) and was eager to encounter difficult problems. When I entered some special private school for the exams of private junior high schools, they gave me lots of math problems I could not solve and I would often ask my father how to solve them (even when he was sleeping, I often woke him up and asked). Although he was not the best teacher in my life, he is the first person who taught me how to think in mathematics and since then, mathematics is not just a calculation to me and that led me to decide to become a mathematician when I was 10.

Secondly, I am grateful to Kurt Gödel for proving Incompleteness Theorems (the first and the second) which excited me a lot when I was a high school student. Although there was no person around me (including me) who knew about the precise statements of the theorems, they got me into foundations of mathematics and made me to study mathematical logic in my bachelor. Of course his work is much more than Incompleteness Theorems and it has been affecting my academic life even now, but this is the part of his work which changed my life the most.

I would like to thank Paul Cohen for inventing forcing and proving the independence of the Continuum Hypothesis from ZFC. After learning Gödel's Incompleteness Theorems, I got into axiomatic set theory thanks to the nice lectures of a professor in mathematical department and started to read a book in axiomatic set theory. In the last part of the book, it mentions the independence of CH from ZFC and I still remember how much I was shocked at it. While the Gödel sentence is a rather artificial statement to be independent from a suitable axiom system, CH seemed very natural to me and I wondered how on earth one could prove such statement to be independent, which led me to become a set theorist.

I am grateful to W. Hugh Woodin for developing such a beautiful connection between determinacy and large cardinals. When I was a senior undergraduate student, I was reading a book in set theory and found his theorem that the existence of a supercompact cardinal implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$, which was stunning to me and made me wonder how come such large objects could have strong effect on the world of real numbers. This had been a big mystery to me until I got into inner model theory and learned further results on determinacy and large cardinals and this is why I decided to major in descriptive set theory.

So after finishing my bachelor, I went to Nagoya, a city in Japan between Tokyo and Kyoto, to study set theory. I am indebted to Yo Matsubara, Yasuo Yoshinobu, and Sakaé Fuchino for their constant and patient support during my stay in Nagoya. Without their encouragement, I would not be able to imagine going abroad for my study. Besides them, I am also grateful to Tadatoshi Miyamoto, Hiroshi Sakai, and Toshimichi Usuba for arranging a warm atmosphere and for being always eager to teach me set theory and listen to me. Especially I have learned a lot from Sakai and Usuba by an enormous amount of discussions with them. I often recall the days when we talked about set theory until midnight (sometimes with alcohol), which is a precious memory to me.

I would like to thank Joan Bagaria for giving a mini course on forcing absoluteness in Kobe in 2005. Through his lecture, I got interested in forcing absoluteness and wrote a master's thesis on this topic. (The work in Chapter 2 is also the continuation of the work in my master's thesis.) He has been always quick to respond to my e-mails and always helpful to me, which also made me consider studying in Barcelona.

During the final stage of my master, I was more or less determined to study abroad and the first place I had in mind was California, which is the best place for studying the connection between determinacy and large cardinals. But since I had never been abroad at that time, my English was pretty poor and I would not be able to be a teaching assistant, which is a usual job to earn on their living for graduate students in the States. So I decided to try to find a position in Europe and I met Benedikt Löwe, who is my supervisor, the person whom I am most indebted to during my Ph.D. When I found an advertisement of Ph.D. positions in GLoRiClass project, I contacted him and after a few e-mails, he strongly encouraged me to apply for the position. Since then, he has been always helpful for me not only mathematically but also non-mathematically. Without his encouragement, I would not be able to study in Amsterdam and overcome many problems which happened to me during my Ph.D. He has been always patient to me no matter how lazy I am about my work and sometimes he gave me certain amount of pressure on my mathematical & non-mathematical work, which was very calm and not irritating at all. He always let me do whatever I wanted to do and watched me from a certain distant position, which I truly appreciate because it is one of the most difficult things to do for supervisors worrying about their students. Without his encouragement and patience, I would not be able to finish this thesis. Also, whenever I need help from him, he is always quick to respond and does his best for me, which is a surprising amount of work for him. I am very happy to have such a great academic father.

At the beginning of my Ph.D., I sometimes considered leaving Amsterdam as a Ph.D. student because it seemed hard to study inner model theory in Amsterdam, which was one of the main reasons why I decided to study abroad. That led Benedikt to introduce Ralf Schindler to me, who is my second supervisor, and I am really happy to have met him. His point of view so-called "Everything must come from mice" has changed my set theory lifestyle a lot as you see the cover of this thesis. When he talked to me about inner model theory, I often felt that mice are really living their lives in set theory since he treats them as if they were 'creatures', which was a nice experience and I have learned a lot from him especially about intuitions and pictures he has in mind on inner model theory. I am also grateful for his warmful hospitality in Münster for one and a half years and for taking care about me for such a long time. I thank him and people in Münster for introducing Hefeweizen and Skat to me, both of which I enjoy a lot even after my stay in Münster.

I am grateful to Jouko Väänänen for treating me as a good set theorist, for informing me a lot of events in mathematical logic, and for encouraging me to organize several events in set theory in Amsterdam. Without his encouragement, I would not have applied for the fall semester program in Mittag-Leffler and would not be able to enjoy such a wonderful semester with world-wide researchers in set theory and model theory. Also, without his suggestion, I would not have organized a series of seminars on $\Sigma^2_1$ forcing absoluteness and would not have given talks, which led me to understand the connection between saturations of ideals on $\omega_1$ and forcing absoluteness and the strong impact of Martin's Maximum on modern set theory.

I would like to thank Johan van Benthem and Yde Venema for sharing their knowledge with me and being open to me to discuss mathematical logic. Both of them are very energetic about their work, which attracted me to work on some problems in fixed point logics that you can see in Chapter 6 of this thesis. Thanks to them, I have not felt so isolated as a set theorist at ILLC.

Jörg Brendle was always helpful to me when I asked questions in set theory and he was very patient and careful to respond to my e-mails. Some of his responses

are included in the work of Chapter 2 of this thesis.

I thank W. Hugh Woodin once again for discussing Blackwell determinacy, determinacy, and large cardinals. His great insights on the above topics were very inspiring and I often gained more motivation and ideas from him.

When I came to Amsterdam, I needed to adjust myself to an European culture and it took me some time and energy to manage it. During the beginning of my life in Amsterdam, many people have helped me for surviving and enjoying life here. Among them, Ioanna Dimitriou, Stefan Bold, Jakub Szymanik, Andreas Witzel, Karol Oslowski, Ichiro Hasuo, Clemens Kupke, Fenrong Liu, Nick Bezhanishvili, Olivier Roy, Merlijn Sevenster, Aline Honingh, Leigh Smith, Ulle Endriss, Ingrid van Loon, Tanja Kassenaar, Jessica Pogorzelski, Marjan Veldhuisen, and Rene Goedman have been very helpful and I am grateful to all of them. Especially Jakub and Andi have been nice to me while we had difficult time during the first year of GLoRiClass project and I am glad that I had both of them as the same fellows.

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After one year in Amsterdam, I went to Münster for half a year to study inner model theory. Benjamin Claverie, Philipp Doebler, Gunter Fuchs, Philipp Schlicht, Daniel Busche, Thilo Volker Weinert, Christoph Duchhardt, and Giulia Uckelmann have been very nice friends and I had a nice time with them. I have been especially close to Ben, Philipp Doebler, and Philipp Schlicht and I am grateful to them. I also thank people in the boat house (Andi, Olivia, and Tikitu) for hosting me from time to time when I visited Amsterdam.

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whole year, which was a unique experience to me. Tomasz Samek, the house owner, is a very frank & cheerful person. He travels a lot and always brings nice alcohol as a souvenir. Jan Peltzer and Emre Burma have been always helpful and we often had fun with drinking alcohol or doing barbecue together. It was nice to talk with Tchelet Ram about Jewish cultures and art. Besides living in a German culture, it was a pleasure to chat with Ben about Japanese cultures and French cultures. It was also nice to meet new people at Universität Münster such as Philipp Lücke, Dominik Adolf, Lars Scheele, and Antongiulio Fornasiero.

After the second stay in Münster, I stayed at Institut Mittag-Leffler for 3.5 months to attend the semester program in model theory and set theory, where many famous researchers gathered and worked together. It was really an experience and nice to talk with such great people and work together. Besides the academic benefit, since we lived in the same institute, we had many opportunities to have parties and enjoy sightseeing. I really had a nice time with Jana Flašková, Vadim Kulikov, Philipp Schlicht, Marcin Sabok, Teppo Kankaanpää, Agatha C. Walczak-Typke, Benno van den Berg, Andrés Villaveces, Natasha Dobrinen, Meeri Kesälä, and Kaisa Kangas. I would like to thank Jouko and Juliette Kennedy for organizing such a great semester and great meetings with Saharon Shelah, W. Hugh Woodin, Stevo Todrcevic, and Menachem Magidor.

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I am so absent-minded and lazy that it has become one of the main ways to get close to people that I bring some troubles, they try to help me, and we get close each other. Certainly, at least some secretaries in academia should be mentioned here: Ingrid, Karin, Peter, Tanja, Marco Vervoort, Thés Smeets, Marie-Louise Koskull, Martina Pfeifer, and Alexander Koponen. The more they feel close to (or annoyed with) me, the more I owe them my gratitude.

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Amsterdam, April 2010,
Daisuke Ikegami

# Chapter 1

# Introduction

*Games* have been used in many areas of mathematics, especially mathematical logic as well as theoretical computer science. It was the Polish school of mathematicians who connected infinite games with analysis (e.g., Lebesgue measurability) and topology (e.g., the Baire property) and obtained many results. In this thesis, we give several results on games in set theory and logic or obtained by application of games.

## 1.1 Outline

In this thesis, we discuss the following topics. All the definitions and the notions given in this outline can be found in the later sections of this chapter.

In Chapter 2, entitled 'Games and Regularity Properties', we characterize almost all the known regularity properties for sets of reals via the Baire property for some topological spaces and use Banach-Mazur games to prove the general equivalence theorems between regularity properties, forcing absoluteness, and the transcendence properties over some canonical inner models. With the help of these equivalence results, we answer some open questions from set theory of the reals. Almost all the results in this chapter are contained in my paper [35].

In Chapter 3, entitled 'Games themselves', we compare the Axiom of Real Determinacy ($\mathsf{AD}_{\mathbb{R}}$) and the Axiom of Real Blackwell Determinacy ($\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$). We show that the consistency strength of $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ is strictly greater than that of the Axiom of Determinacy (AD) in § 3.1 and that $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ implies almost all the known regularity properties for every set of reals in § 3.2. In § 3.3, we discuss the possibility of the equivalence between $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ under ZF+DC. In § 3.4, we discuss the possibility of the equiconsistency between $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$. The results in § 3.1 are joint work with David de Kloet and Benedikt Löwe [36]. The results in § 3.2, § 3.3, and § 3.4 are joint work with Hugh Woodin.

In Chapter 4, entitled 'Games and Large Cardinals', we work on the connection between the determinacy of Gale-Stewart games and large cardinals. We investigate the upper bound of the consistency strength of the existence of alternating chains with length $\omega$, which are essential objects to prove projective determinacy from Woodin cardinals. This is joint work with Ralf Schindler.

In Chapter 5, entitled 'Wadge reducibility for the real line', we study the Wadge reducibility for the real line. Unlike the Wadge order for the Baire space, the Wadge order for the real line cannot be characterized by infinite games. We show that the Wadge Lemma for the real line fails and the Wadge order for the real line is ill-founded and we investigate more properties of the Wadge order for the real line. All the results in this chapter are joint work with Philipp Schlicht and Hisao Tanaka.

In Chapter 6, entitled 'Fixed-Point Logic and Product Closure', we define a product construction of an event model and a Kripke model and discuss the product closure of modal fixed point logics. We show that PDL, the modal $\mu$-calculus, and the continuous fragment of the modal $\mu$-calculus are product closed. Most of the results are joint work with Johan van Benthem [12].

In the remaining sections of this chapter, we give the mathematical background and results used in this thesis.

## 1.2 Choice principles

We use the following two types of choice principles in this thesis.

The first one is the family of the Choice Principles $\mathrm{AC}_X(Y)$. Let $X, Y$ be nonempty sets. The *Choice Principle* $\mathrm{AC}_X(Y)$ states that for any family $\{A_x \mid x \in X\}$ of nonempty subsets of $Y$, there is a function $f\colon X \to Y$ such that $f(x) \in A_x$ for every $x \in X$. The *Axiom of Choice* AC states that $\mathrm{AC}_X(Y)$ holds for all nonempty sets $X$ and $Y$. The following is easy to see:

**Remark 1.2.1.** Let $X, Y_1, Y_2$ be nonempty sets and suppose there is a surjection from $Y_2$ to $Y_1$. Then $\mathrm{AC}_X(Y_2)$ implies $\mathrm{AC}_X(Y_1)$.

Furthermore, we consider the Dependent Choice Principles $\mathrm{DC}_X$. Let $X$ be a nonempty set. The *Dependent Choice Principle* $\mathrm{DC}_X$ states that for any relation $R$ on $X$ (i.e., $R \subseteq X \times X$), if $(\forall x \in X)\ (\exists y \in X)\ (x, y) \in R$, then there is a function $f\colon \omega \to X$ such that $\big(f(n), f(n+1)\big) \in R$ for every $n \in \omega$. The *Axiom of Dependent Choice* DC states that $\mathrm{DC}_X$ holds for every nonempty set $X$.

Throughout this thesis, we work in $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$, where ZF is the axiom system of Zermelo-Fraenkel set theory. When we need more choice principles, we explicitly mention them (especially at the beginning of each chapter).

## 1.3 Trees

*Trees* are basic objects in mathematical logic, especially descriptive set theory and recursion theory. We fix some notation and introduce definitions about trees.

If $f$ is a function from $X$ to $Y$ and $A$ is a subset of $X$, then $f \upharpoonright A$ denotes the *restriction of $f$ to $A$*, i.e., $f \upharpoonright A = \{(a, f(a)) \mid a \in A\}$. For a relation $R$ between $X$ and $Y$ (i.e., $R \subseteq X \times Y$), $\mathrm{dom}(R) = \{x \in X \mid (\exists y)\ (x, y) \in R\}$ and $\mathrm{ran}(R) = \{y \in Y \mid (\exists x)\ (x, y) \in R\}$.

Given a nonempty set $X$, ${}^{<\omega}X$ denotes the set of all finite sequences of elements in $X$ and a nonempty subset $T$ of ${}^{<\omega}X$ is a *tree* on $X$ if it is closed under initial segments, i.e., if $s$ is in $T$ and $t$ is a subsequence of $s$ (i.e., $t = s \upharpoonright n$ for some $n$), then $t$ is in $T$. For a finite sequence $t$ of elements in $X$, $\mathrm{lh}(t)$ denotes the length of $t$.

By *nodes*, we mean elements of trees. For a tree $T$ on $X$, two nodes $s, t$ of $T$ are *incompatible* (denoted by $s \perp t$) if there is an $n$ in $\mathrm{dom}(s) \cap \mathrm{dom}(t)$ such that $s(n) \neq t(n)$. Note that $s, t$ are incompatible if and only if there is no $u$ in $T$ such that $s, t \subseteq u$. For a node $t$ of $T$ and an element $x$ of $X$, $t^\frown\langle x \rangle$ denotes the one-step extension of $t$ with $x$, i.e., $t^\frown\langle x \rangle = t \cup \{(\mathrm{lh}(t), x)\}$.

A tree $T$ on $X$ is called *perfect* if for any node $s$ in $T$, there are two nodes $t_1, t_2$ of $T$ such that $s \subseteq t_i$ for $i = 1, 2$ and $t_1 \perp t_2$. For a tree $T$ on $X$, $[T]$ denotes the set of all infinite paths through $[T]$, i.e., $[T] = \{x \in {}^{\omega}X \mid (\forall n \in \omega)\ x \upharpoonright n \in T\}$. For a tree $T$ on $X$ and a node $t$ in $T$, $t$ is called *splitting in $T$* if there are $x$ and $y$ in $X$ such that $x \neq y$ and both $t^\frown\langle x \rangle$ and $t^\frown\langle y \rangle$ are in $T$. For a tree $T$, the *stem of $T$* (denoted by $\mathrm{stem}(T)$) is the minimal splitting node in $T$ if it exists.

If $T$ is a tree on $X$ and $X$ is of the form $Y \times Z$, then we often identify a node $s$ of $T$ with the pair $(t_1, t_2)$ where $t_i = (s(0)_i, \dots s(n-1)_i)$ for $i = 1, 2$, $n = \mathrm{dom}(s)$, and $s(j) = (s(j)_1, s(j)_2)$ for $j < n$. The same identification will be applied in case $X$ is of the form $Y_1 \times \dots \times Y_m$ for a finite natural number $m \geq 1$.

## 1.4 General topology

*Topological spaces* are fundamental objects in mathematics. Throughout this thesis, we assume the basic theory of topological spaces which can be found in, e.g., [49]. We mainly use the following three types of topological spaces:

**The spaces ${}^{\omega}X$.** Let $X$ be a nonempty set. The set ${}^{\omega}X$ is the set of all $\omega$-sequences of elements in $X$ and we topologize it via the product topology where $X$ is always regarded as the discrete space. Hence for each finite sequence $s$ of elements in $X$, the set $[s] = \{x \in {}^{\omega}X \mid x \supseteq s\}$ (i.e., the set of all $\omega$-sequences of elements in $X$ extending $s$) is a basic open set in this topology and any open set is a union of basic open sets of this form.

Our main interest is when $X = 2$ (i.e., $\{0, 1\}$) or $\omega$. The space ${}^{\omega}2$ is called the *Cantor space* and the space ${}^{\omega}\omega$ is called the *Baire space*.

One of the special properties of this type of topological spaces is that closed sets have a *tree representation*: A subset $A$ of ${}^{\omega}X$ is closed if and only if there is a tree $T$ on $X$ such that $A = [T]$. Also, there is a one-to-one correspondence between perfect subsets of ${}^{\omega}X$ and perfect trees on $X$, where a subset $A$ of ${}^{\omega}X$ is *perfect* if it is closed and it has no isolated points: A subset $A$ of ${}^{\omega}X$ is perfect if and only if there is a perfect tree $T$ on $X$ such that $A = [T]$.

A subset $A$ of the Baire space or the Cantor space has the *perfect set property* if either it is countable or it contains a perfect set. It is easy to see that for any perfect set $C$, there is a bijection between $C$ and the Cantor space. Hence sets $A$ with the perfect set property satisfy Cantor's Continuum Hypothesis (CH), i.e., either $A$ is countable or there is a bijection between $A$ and the Cantor space. For this reason, it is interesting to see what kind of sets have the perfect set property. We discuss this in § 1.11.

**The spaces** $\mathrm{St}(\mathbb{P})$**.** *Stone spaces* are fundamental topological spaces not only in mathematical logic but also in general mathematics. We give basic definitions and the basic properties of Stone spaces of partial orders in our context.

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. A map $i\colon \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if it satisfies the following:

- $i$ preserves the order, i.e., if $p_1 \leq p_2$ in $\mathbb{P}$, then $i(p_1) \leq i(p_2)$ in $\mathbb{Q}$,
- $i$ preserves the incompatibility, i.e., given two elements $p_1, p_2$ of $\mathbb{P}$, if there is no $p$ in $\mathbb{P}$ with $p \leq p_1$ and $p \leq p_2$, then there is no $q$ in $\mathbb{Q}$ with $q \leq i(p_1)$ and $q \leq i(p_2)$, and
- the image of $i$ is dense, i.e., for any $q$ in $\mathbb{Q}$ there is a $p$ in $\mathbb{P}$ such that $i(p) \leq q$.

Dense embeddings are important in forcings in the sense that if there is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, then forcing with $\mathbb{P}$ and forcing with $\mathbb{Q}$ are essentially the same. (See § 1.9 about forcing.)

It is well known that if $\mathbb{P}$ is a partial order, then there is a complete Boolean algebra $\mathbb{B}$ and a dense embedding $i$ from $\mathbb{P}$ to $\mathbb{B}$. Moreover, the pair $(\mathbb{B}, i)$ is unique up to isomorphism in the sense that if there are two such pairs $(\mathbb{B}_1, i_1)$ and $(\mathbb{B}_2, i_2)$, then there is an isomorphism $i$ between $\mathbb{B}_1$ and $\mathbb{B}_2$ as complete Boolean algebras such that $i \circ i_1 = i_2$. We call such a pair $(\mathbb{B}, i)$ a *completion of* $\mathbb{P}$ and write $(B_{\mathbb{P}}, i_{\mathbb{P}})$ for $(\mathbb{B}, i)$.

Let $\mathbb{P}$ be a partial order. A nonempty subset $u$ of $\mathbb{P}$ is a *filter* on $\mathbb{P}$ if it is upward closed (i.e., if $p \in u$ and $p \leq q$, then $q$ is also in $u$) and any two elements of $u$ have an extension in $u$ (i.e., if $p$ and $q$ are in $u$, then there is an $r$ in $u$ such that $r \leq p$ and $r \leq q$). A filter $u$ on $\mathbb{P}$ is an *ultrafilter* if $u \neq \mathbb{P}$ and $u$ is maximal with respect to inclusions (i.e., if $v$ is a filter containing $u$, then $v = u$ or $v = \mathbb{P}$).

We now define Stone spaces of partial orders. Given a partial order $\mathbb{P}$, the set $\mathrm{St}(\mathbb{P})$ is the collection of all ultrafilters on $B_{\mathbb{P}}$. For each $b \in B_{\mathbb{P}}$, we define the set $O_b = \{u \in \mathrm{St}(\mathbb{P}) \mid u \ni b\}$ and the *Stone space* of $\mathbb{P}$ (also denoted by $\mathrm{St}(\mathbb{P})$) is the topology on the set $\mathrm{St}(\mathbb{P})$ generated by the set $\{O_b \mid b \in B_{\mathbb{P}}\}$.

For example, if $\mathbb{P}$ is the pair $({}^{<\omega}\omega, \supseteq)$, i.e., the set of all finite sequences of natural numbers ordered by reverse inclusion, then the Stone space of $\mathbb{P}$ is homeomorphic to the Cantor space ${}^{\omega}2$.

There are two advantages for taking ultrafilters on $B_{\mathbb{P}}$ rather than on $\mathbb{P}$ itself as a definition of the Stone space of $\mathbb{P}$: The first one is that it has several nice properties as topological spaces (e.g., it is a compact Hausdorff zero-dimensional space). The second is that it does not depend on the representation of $\mathbb{P}$, i.e., if there is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, then $\mathrm{St}(\mathbb{P})$ and $\mathrm{St}(\mathbb{Q})$ are homeomorphic.

**The real line $\mathbb{R}$.** We use $\mathbb{R}$ to denote the set of all real numbers except in Chapter 2, where we use it for Mathias forcing (we use $\mathbb{R}$ for Mathias forcing because it is closely related to the Ramsey property). As usual, the topology of the real line is generated by open intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ for $a, b \in \mathbb{R}$.

## 1.5 Borel sets, projective sets, and definability in the second-order arithmetics

Let $X$ be a topological space. Starting from open sets (or closed sets), we form the two hierarchies of sets of subsets of $X$. One is called the *Borel hierarchy* and the other is called the *projective hierarchy*:

**Definition 1.5.1.** Let $X$ be a topological space. The *Borel hierarchy* of $X$ $\big(\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi \mid 1 \leq \xi < \omega_1\big)$ is defined as follows:

**Case 1**: $\xi = 1$.

By $\mathbf{\Sigma}^0_1$, we mean the set of all open subsets of $X$ and $\mathbf{\Pi}^0_1$ denotes the set of all closed subsets of $X$. The set of all clopen subsets of $X$ is denoted by $\mathbf{\Delta}^0_1$.

**Case 2**: $\xi > 1$.

By $\mathbf{\Sigma}^0_\xi$, we mean the set of all countable unions of sets in $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta$, and $\mathbf{\Pi}^0_\xi$ denotes the set of all countable intersections of sets in $\bigcup_{\eta<\xi} \mathbf{\Sigma}^0_\eta$. The intersection of $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ is denoted by $\mathbf{\Delta}^0_\xi$.

Elements of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ are called *$\mathbf{\Sigma}^0_\xi$ sets, $\mathbf{\Pi}^0_\xi$ sets and $\mathbf{\Delta}^0_\xi$ sets* respectively. We set $\mathbf{B} = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi$ and elements of $\mathbf{B}$ are called *Borel sets*.

It is immediate that $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$ for each $1 \leq \xi < \omega_1$. By induction on $\xi$, it is easy to show that $\mathbf{\Pi}^0_\xi = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\xi\}$ for each $1 \leq \xi < \omega_1$. With the help of $\mathrm{AC}_\omega(\mathbb{R})$, it is easy to show that $\omega_1$ is a regular cardinal and hence that the set of all the Borel sets $\mathbf{B}$ is closed under complements and countable unions and it contains the empty set. Such a family of subsets of $X$ is called a *$\sigma$-algebra on $X$*. Note that the set of all the Borel subsets of $X$ is the smallest $\sigma$-algebra on $X$ containing all the open sets.

**Theorem 1.5.2** (Lebesgue)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then the following strict inclusions hold for each $1 \leq \xi < \omega_1$:

$$\begin{tikzcd} & \boldsymbol{\Sigma}^0_\xi \arrow[dr, hook, "\neq"] & \\ \boldsymbol{\Delta}^0_\xi \arrow[ur, hook, "\neq"] \arrow[dr, hook, "\neq"'] & & \boldsymbol{\Delta}^0_{\xi+1} \\ & \boldsymbol{\Pi}^0_\xi \arrow[ur, hook, "\neq"'] & \end{tikzcd}$$

*Proof.* See, e.g., [45, Theorem 22.4]. □

**Definition 1.5.3.** Let $X$ be a topological space. The *projective hierarchy* of $X$ $\big(\boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n, \boldsymbol{\Delta}^1_n \mid 1 \leq n < \omega\big)$ is defined as follows:

**Case 1**: $n = 1$.

By $\boldsymbol{\Sigma}^1_1$, we mean the set of all subsets $A$ of $X$ such that there is a closed subset $C$ of $X \times {}^\omega\omega$ such that $A$ is the first projection of $C$, i.e., $A = \mathrm{dom}(C)$, where $X \times {}^\omega\omega$ is topologized as the product space of $X$ and ${}^\omega\omega$. The set of all subsets $A$ of $X$ whose complements are in $\boldsymbol{\Sigma}^1_1$ is denoted $\boldsymbol{\Pi}^1_1$. The intersection between $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ is denoted $\boldsymbol{\Delta}^1_1$.

**Case 2**: $n > 1$.

By $\boldsymbol{\Sigma}^1_n$, we mean the set of all subsets $A$ of $X$ such that there is a subset $C$ of $X \times {}^\omega\omega$ in $\boldsymbol{\Pi}^1_{n-1}$ such that $A$ is the first projection of $C$. The set of all subsets $A$ of $X$ whose complements are in $\boldsymbol{\Sigma}^1_n$ is denoted $\boldsymbol{\Pi}^1_n$. The intersection between $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$ is denoted $\boldsymbol{\Delta}^1_n$.

Elements of $\boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n$, and $\boldsymbol{\Delta}^1_n$ are called $\boldsymbol{\Sigma}^1_n$ *sets,* $\boldsymbol{\Pi}^1_n$ *sets and* $\boldsymbol{\Delta}^1_n$ *sets* respectively. Sets in $\boldsymbol{\Sigma}^1_n$ for some $n$ are called *projective sets.*

Elements of $\boldsymbol{\Sigma}^1_1$ are also called *analytic sets*, and *co-analytic sets* are the same as $\boldsymbol{\Pi}^1_1$ sets. It is immediate that $\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$ for each $n$ and that $\boldsymbol{\Pi}^1_n = \{X \setminus A \mid A \in \boldsymbol{\Sigma}^1_n\}$ for each $n$.

**Theorem 1.5.4** (Suslin)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then $\mathbf{B} = \boldsymbol{\Delta}^1_1$.

*Proof.* See, e.g., [45, Theorem 14.11]. □

**Theorem 1.5.5** (Lusin)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then the following strict inclusions hold for each $1 \leq n < \omega$:

$$\begin{tikzcd} & \boldsymbol{\Sigma}^1_n \arrow[dr, hook, "\neq"] & \\ \boldsymbol{\Delta}^1_n \arrow[ur, hook, "\neq"] \arrow[dr, hook, "\neq"'] & & \boldsymbol{\Delta}^1_{n+1} \\ & \boldsymbol{\Pi}^1_n \arrow[ur, hook, "\neq"'] & \end{tikzcd}$$

In particular, every Borel set is a $\mathbf{\Sigma}^1_1$ set and there is a $\mathbf{\Sigma}^1_1$ set which is not a Borel set.[1]

*Proof.* See, e.g., [45, Theorem 37.7]. □

Definable sets in the second-order arithmetic are related to $\mathbf{\Sigma}^0_n$ sets, $\mathbf{\Pi}^0_n$ sets, $\mathbf{\Sigma}^1_n$ sets, and $\mathbf{\Pi}^1_n$ sets in the Baire space. By the *second-order structure*, we mean the two-sorted structure $\mathcal{A}^2 = (\omega, {}^\omega\omega, \mathrm{app}, +, \cdot, =, 0, 1)$, where app is the function from ${}^\omega\omega \times \omega$ to $\omega$ such that $\mathrm{app}(x, n) = x(n)$ and $+, \cdot, =$ are summation, multiplication, and equality on the natural numbers. By $\Sigma^0_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\exists^0 x_1)\ (\forall^0 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $\exists^0, \forall^0$ are the existential quantifier and the universal quantifier for natural numbers respectively, $Q_n$ is $\forall^0$ if $n$ is even and $\exists^0$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for natural numbers, and $\phi$ is a quantifier-free formula. By $\Pi^0_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\forall^0 x_1)\ (\exists^0 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $Q_n$ is $\exists^0$ if $n$ is even and $\forall^0$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for natural numbers, and $\phi$ is a quantifier-free formula. By *arithmetical formulas*, we mean $\Sigma^0_n$-formulas or $\Pi^0_n$-formulas for some natural number $n$. By $\Sigma^1_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\exists^1 x_1)\ (\forall^1 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $\exists^1, \forall^1$ are the universal quantifier and the existential quantifier for elements in the Baire space respectively, $Q_n$ is $\forall^1$ if $n$ is even and $\exists^1$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for elements in the Baire space, and $\phi$ is an arithmetical formula. By $\Pi^1_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\forall^1 x_1)\ (\exists^1 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $Q_n$ is $\exists^1$ if $n$ is even and $\forall^1$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for elements in the Baire space, and $\phi$ is an arithmetical formula. Let $n$ be a natural number with $n \geq 1$, $A$ be a subset of the Baire space and $a$ be an element of the Baire space. We say $A$ is a $\Sigma^0_n(a)$ *set* if there is a $\Sigma^0_n$-formula $\phi$ such that $A = \{x \mid \mathcal{A}^2 \vDash \phi(x, a)\}$. One can define $\Pi^0_n(a)$ *sets*, $\Sigma^1_n(a)$ *sets*, and $\Pi^1_n(a)$ *sets* in the same way. We also use $\Sigma^0_n(a), \Pi^0_n(a), \Sigma^1_n(a)$, and $\Pi^1_n(a)$ to denote the set of all $\Sigma^0_n(a)$ sets, $\Pi^0_n(a)$ sets, $\Sigma^1_n(a)$ sets, and $\Pi^1_n(a)$ sets respectively.

[1] The last statement is due to Suslin [82].

**Theorem 1.5.6.** Let $n$ be a natural number with $n \geq 1$. Then

$$\begin{aligned} \boldsymbol{\Sigma}^0_n &= \bigcup_{a \in {}^\omega\omega} \Sigma^0_n(a), & \boldsymbol{\Pi}^0_n &= \bigcup_{a \in {}^\omega\omega} \Pi^0_n(a), \\ \boldsymbol{\Sigma}^1_n &= \bigcup_{a \in {}^\omega\omega} \Sigma^1_n(a), & \boldsymbol{\Pi}^1_n &= \bigcup_{a \in {}^\omega\omega} \Pi^1_n(a). \end{aligned}$$

*Proof.* See, e.g., [66, 8B.5 & 8B.15]. □

## 1.6 Gale-Stewart games

In this section, we introduce *Gale-Stewart games*, which are infinite games with perfect information.

In 1913, Ernst Zermelo [93] investigated finite games with perfect information as a formalization of the game of chess and proved the determinacy of these games. In 1953, Gale and Stewart [27] developed the general theory of infinite games, so-called *Gale-Stewart games*, which are two-player zero-sum infinite games with perfect information. The theory of Gale-Stewart games has been investigated by many logicians and now it is one of the main topics in set theory and it has connections with other topics in set theory as well as model theory and computer science.

Let us start with the definition of Gale-Stewart games.

**Definition 1.6.1** (Gale-Stewart games)**.** Let $X$ be a nonempty set and $A$ be a subset of ${}^\omega X$. The *Gale-Stewart game* $G_X(A)$ is played by two players, player I and player II. They play elements of $X$ $\omega$-many times in turn, i.e., player I starts with choosing an element $x_0$ of $X$, then player II responds with $x_1 \in X$, then player I moves with $x_2 \in X$ and player II chooses $x_3$ and so on. After $\omega$ moves, they have produced an $\omega$-sequence $x = \langle x_n \mid n \in \omega \rangle \in {}^\omega X$. Player I wins if $x$ is in $A$ and player II wins if $x$ is not in $A$.

This game is an infinite zero-sum game with perfect information because one of the players always wins and when one player wins, the other loses, and because both players know what they have previously played and they can decide the next move considering their previous moves.

We are interested in whether one of the players has a winning strategy in the game $G_X(A)$, i.e., whether one of the players has a way to play this game such that no matter her opponent moves, she will always win this game. Let us formulate the notion of winning strategies.

**Definition 1.6.2.** A *strategy for player I* is a function $\sigma \colon X^{\mathrm{Even}} \to X$, where $X^{\mathrm{Even}}$ is the set of finite sequences of elements in $X$ with even length. A *strategy for player II* is a function $\tau \colon X^{\mathrm{Odd}} \to X$, where $X^{\mathrm{Odd}}$ is the set of finite sequences of elements in $X$ with odd length. Given a strategy $\sigma$ for player I and a strategy

$\tau$ for player II, one can produce the run $\sigma * \tau$ of the game $G_X(A)$ according to $\sigma$ and $\tau$ by letting player I follow $\sigma$ and player II follow $\tau$, more precisely, the run $\sigma * \tau$ of the game $G_X(A)$ is a unique $\omega$-sequence of elements in $X$ with the following property: For any natural number $n$,

$$(\sigma * \tau)(n) = \nu_{\sigma,\tau}\big((\sigma * \tau)\restriction n\big),$$

where for a finite sequence $s$ of elements in $X$, $\nu_{\sigma,\tau}(s) = \sigma(s)$ if the length of $s$ is even and $\nu_{\sigma,\tau}(s) = \tau(s)$ if the length of $s$ is odd. A strategy $\sigma$ for player I is *winning in the game* $G_X(A)$ if for any strategy $\tau$ for player II, $\sigma * \tau$ is in $A$. A strategy $\tau$ for player II is *winning in the game* $G_X(A)$ if for any strategy $\sigma$ for player I, $\sigma * \tau$ is not in $A$. A subset $A$ of ${}^\omega X$ is *determined* if one of the players has a winning strategy in the game $G_X(A)$.

Hence we are interested in what kind of sets $A$ are determined. Let us list some results concerning this question. Recall from § 1.4 that the topology of ${}^\omega X$ is given by the product topology where each coordinate (i.e., $X$) is seen as the discrete space.

**Theorem 1.6.3** (Gale and Stewart)**.** (AC) Let $X$ be a nonempty set.

1. Any closed subset of ${}^\omega X$ and any open subset of ${}^\omega X$ are determined. If $X$ is well-ordered, one does not need AC.

2. There is a subset of ${}^\omega\omega$ which is not determined.

*Proof.* See, e.g., [37, Lemma 33.1, Lemma 33.17]. □

**Theorem 1.6.4** (Martin)**.** (AC) Let $X$ be a nonempty set. Then every Borel subset of ${}^\omega X$ is determined.

*Proof.* See, e.g., [45, Theorem 20.5]. □

**Theorem 1.6.5** (Davis; Gödel and Addison)**.** ZFC cannot prove that every $\boldsymbol{\Sigma}^1_1$ subset of the Baire space is determined.

*Proof.* The statement follows from the combination of the following two results: The first is that if every $\boldsymbol{\Sigma}^1_1$ subset of the Baire space is determined, then every $\boldsymbol{\Pi}^1_1$ subset of the Baire space has the perfect set property and the second one is that ZFC cannot prove that every $\boldsymbol{\Pi}^1_1$ subset of the Baire space has the perfect set property. The first result is due to Davis [23] and the second result was announced by Gödel [28] and the details of the proof given by Addison [1]. For the proofs, see, e.g., [66, p. 224 & 225] and [37, Corollary 25.37]. □

Gale-Stewart games are general enough that they can be used to simulate several kinds of infinite games in mathematics (e.g., Banach-Mazur games; for the definition of Banach-Mazur games, see § 1.8). In particular, the determinacy of Gale-Stewart games implies that of several other kinds of games. From this, one can prove several properties of sets of reals assuming the determinacy of Gale-Stewart games such as Lebesgue measurability, the Baire property (for the definition, see § 1.8), and the perfect set property (for the definition, see § 1.4).

Mycielski and Steinhaus [68] introduced the *Axiom of Determinacy* (AD), which states that every subset of the Baire space is determined, and investigated the consequences of this axiom. They proved that AD implies that every set of reals is Lebesgue measurable and that every subset of the Baire space has the Baire property and the perfect set property where each of these statements contradicts the Axiom of Choice. Beside such properties for sets of reals, AD supplies a beautiful structural theory. Moreover, models of AD have been investigated for a long time and they are essential for the research on inner models with large cardinals (for inner models, see § 1.11). In this way, the study of AD has been one of the central topics in set theory despite the fact that AD contradicts AC.

One can define $\mathrm{AD}_X$ for a nonempty set $X$ as follows: Every subset of ${}^\omega X$ is determined. Let us list some known observations on $\mathrm{AD}_X$:

**Proposition 1.6.6.**

1. Let $X, Y$ be nonempty sets and assume that there is an injection from $X$ to $Y$. Then $\mathrm{AD}_Y$ implies $\mathrm{AD}_X$. In particular, $\mathrm{AD}_\mathbb{R}$ implies $\mathrm{AD}_\omega = \mathrm{AD}$.

2. The axioms $\mathrm{AD}_{\omega_1}$ and $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ are inconsistent.

*Proof.* The first statement is a folklore and it is easy. For the second statement, the inconsistency of $\mathrm{AD}_{\omega_1}$ is due to Mycielski [67] and that of $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ follows from the inconsistency of $\mathrm{AD}_{\omega_1}$, the fact that there is an injection from $\omega_1$ into $\mathcal{P}(\mathbb{R})$, and the first item of this proposition. (One can send a countable ordinal $\alpha$ to the set of all reals $x$ such that $(\omega, x)$ is isomorphic to $(\alpha, \in)$ and this is an injection from $\omega_1$ into $\mathcal{P}(\mathbb{R})$.) $\square$

We investigate AD and $\mathrm{AD}_\mathbb{R}$ further in Chapter 3.

## 1.7 Pointclasses, parametrization, and Recursion Theorem

As with Borel sets, one often looks at the properties of a class of sets of reals rather those of a set of reals. Such classes are called *pointclasses*. In this section, we introduce basic properties for pointclasses. When we are talking about "reals", we mean elements of the Cantor space ${}^\omega 2$ and we use $\mathbb{R}$ to denote the Cantor space.

A *pointclass* is the union of sets of subsets of $\omega^m \times \mathbb{R}^n$ for natural numbers $m \geq 0, n \geq 1$. If $\boldsymbol{\Gamma}$ is a pointclass, $\boldsymbol{\Gamma}$ is called a *boldface pointclass* if it is closed under continuous preimages, i.e., for natural numbers $m_1, m_2 \geq 0$ and $n_1, n_2 \geq 1$, a continuous function $f \colon \omega^{m_1} \times \mathbb{R}^{n_1} \to \omega^{m_2} \times \mathbb{R}^{n_2}$, and a subset $A \in \boldsymbol{\Gamma}$ of $\omega^{m_2} \times \mathbb{R}^{n_2}$, $f^{-1}(A)$ is also in $\boldsymbol{\Gamma}$. Closure under recursive preimages is similarly defined with recursive functions.

A pointclass $\Gamma$ is *$\omega$-parametrized* if for all natural numbers $m \geq 0$ and $n \geq 1$ there is a subset $G^{m,n}$ of $\omega^{m+1} \times \mathbb{R}^n$ in $\Gamma$ such that for any subset $A$ of $\omega^m \times \mathbb{R}^n$ in $\Gamma$, there is a natural number $e$ such that $A = G^{m,n}_e = \{(x,y) \mid (e,x,y) \in G^{m,n}\}$. The following lemma is useful: Let $\Gamma$ be a pointclass and $x$ be a real. Then the pointclass $\Gamma(x)$ is the set of all sets $A$ such that there is a set $C \in \Gamma$ such that $A = C_x$ where $C_x = \{y \in \mathbb{R} \mid (y,x) \in C\}$. Set $\boldsymbol{\Gamma} = \bigcup_{x \in \mathbb{R}} \Gamma(x)$.

**Lemma 1.7.1.** Suppose $\Gamma$ is an $\omega$-parametrized pointclass which is closed under recursive preimages. Then for each natural number $n \geq 1$, there is a set $G^n \subseteq \mathbb{R} \times \mathbb{R}^n$ in $\Gamma$ such that the following hold:

1. For each $n \geq 1$, $G^n$ is universal for subsets of $\mathbb{R}^n$ in $\boldsymbol{\Gamma}$, i.e., for any subset $A \in \boldsymbol{\Gamma}$, there is a real $x$ such that $A = G^n_x$,
2. For $A \subseteq \mathbb{R}^n$ in $\Gamma$, there is a recursive real $x$ such that $A = G^n_x$, and
3. For all natural numbers $n, m \geq 1$, there is a recursive function $S^{n,m} \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that for any real $a$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$, $G^{m+n}(a,x,y) \iff G^m(S^{n,m}(a,x),y)$.

*Proof.* See [66, 3H.1]. □

We fix some notions for projections. For natural numbers $m \geq 0$ and $n \geq 1$ and a subset $A$ of $\omega \times \omega^m \times \mathbb{R}^n$, let $\exists^\omega A = \{(x,y) \in \omega^m \times \mathbb{R}^n \mid (\exists e \in \omega)\ (e,x,y) \in A\}$ and $\forall^\omega A = \{(x,y) \in \omega^m \times \mathbb{R}^n \mid (\forall e \in \omega)\ (e,x,y) \in A\}$. The sets $\exists^{\mathbb{R}} A$ and $\forall^{\mathbb{R}} A$ are defined in the similar way. A pointclass $\Gamma$ is *closed under* $\exists^\omega$ if for any $A$ in $\Gamma$, $\exists^\omega A$ is in $\Gamma$. Closure under $\forall^\omega$, $\exists^{\mathbb{R}}$, and $\forall^{\mathbb{R}}$ is defined in the similar way.

**Definition 1.7.2.** A pointclass $\Gamma$ is a *Spector pointclass* if it satisfies the following:

1. It contains all the $\Sigma^0_1$ sets and it is closed under recursive substitutions, finite intersections and unions, $\exists^\omega$, and $\forall^\omega$,
2. It is $\omega$-parametrized,
3. It has the substitution property, and
4. It has the prewellordering property.

For the definition the substitution property and the basic theory of $\Gamma$-recursive functions, see [66, 3D & 3G]. For the definition of prewellordering property, see [66, 4B]. Typical examples of Spector pointclasses are $\Pi^1_1$ and $\Sigma^1_2$. Assuming the determinacy of all the projective sets, one can prove that $\Pi^1_{2n+1}$ and $\Sigma^1_{2n+2}$ are also Spector pointclasses for each natural number $n$.

We use the following general form of Kleene's Recursion Theorem for Spector pointclasses in Chapter 3:

**Theorem 1.7.3** (Recursion Theorem)**.** (Kleene) Let $\Gamma$ be a Spector pointclass and suppose $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is $\boldsymbol{\Gamma}$-recursive on its domain. Then there exists a fixed real $a^*$ such that for all reals $x$, if $f(a^*, x)$ is defined, then $f(a^*, x) = \{a^*\}(x)$, where $\{a^*\}$ is the $\boldsymbol{\Gamma}$-recursive function on its domain coded by $a^*$.

*Proof.* See [66, 7A.2]. □

## 1.8 The Baire property and Banach-Mazur games

In this section, we introduce the *Baire property* and *Banach-Mazur games* and discuss the connection between them. In the Scottish Café "Kawiarnia Szzkocka" in Lwów, Polish mathematicians in the Lwów School of Mathematics would often meet and spend their afternoons discussing mathematical problems in 1920s and 1930s. Their discussions produced the famous book so-called "the Scottish book of problems". In this book (see [63]), Mazur described infinite games nowadays called *Banach-Mazur games* and conjectured their connection to the Baire property. The conjecture was confirmed by Banach in 1935 and the statement was generalized to arbitrary topological space by Oxtoby [69] in 1957.

We start with the definition of the Baire property:

**Definition 1.8.1.** Let $X$ be a topological space and $A$ be a subset of $X$.

1. We say $A$ is *nowhere dense* if the interior of the closure of $A$ is empty.
2. We say $A$ is *meager* if it is a countable union of nowhere dense sets.
3. We say $A$ is *comeager* if the complement of $A$ is meager.
4. We say $A$ has the *Baire property* if there is an open subset $U$ of $X$ such that the symmetric difference between $A$ and $U$ (i.e., $\big((A\setminus U)\cup(U\setminus A)\big)$, denoted by $A\triangle U$) is meager.

Nowhere dense sets and meager sets are small in the sense of topology, e.g., on the Baire space, the Cantor space and the real line, any singleton is nowhere dense and any countable set is meager. Sets with the Baire property can be approximated by open sets modulo such small sets. But if some nonempty open set was meager, this property would not make sense. To avoid that problem, we introduce a property for topological spaces: A topological space $X$ is called a *Baire space* if any nonempty open subset of $X$ is not meager.[2] All the topological spaces that appear in this thesis will be Baire spaces.

If $X$ is a topological space, many subsets of $X$ have the Baire property in $X$: Trivially every open set has the Baire property, also every closed set has the Baire property (if we take $U$ to be the interior of the given closed set $A$, then symmetric difference between $A$ and $U$ is $A\setminus U$ and it is nowhere dense by the

[2]Note that being *a* Baire space is different from being *the* Baire space ${}^{\omega}\omega$. Being a Baire space is a property for topological spaces while the Baire space is one particular topological space.

definition of interior, hence meager). From this, we can conclude that the set of subsets of $X$ with the Baire property is closed under complements. Moreover, since the set of meager sets is closed under countable unions, the set of subsets with Baire property is also closed under countable unions and hence every Borel subset of $X$ has the Baire property.

It is natural to ask whether the converse is true for the Baire space, i.e., if a subset of the Baire space has the Baire property, then is it Borel? The answer is 'No'. In 1923, Lusin and Sierpinski [57] proved that every $\mathbf{\Sigma}^1_1$ set of reals has the Baire property and there is a $\mathbf{\Sigma}^1_1$ set of reals which is not Borel by Theorem 1.5.5. So one could ask, "How far can we go?" Actually, in the constructible universe L, there is a $\mathbf{\Delta}^1_2$ set of reals without the Baire property.[3] On the other hand, starting with a model of ZFC, one can construct a model of ZFC extending the given model such that every $\mathbf{\Delta}^1_2$ set has the Baire property. Hence the statement that every $\mathbf{\Delta}^1_2$ set of reals has the Baire property is *independent from* ZFC. Then one could naturally ask the following: When is it true and when is it not? We discuss this question in Chapter 2. Next, we introduce Banach-Mazur games, which characterize meagerness of topological spaces:

**Definition 1.8.2** (Banach-Mazur games)**.** Let $X$ be a topological space and $A$ be a subset of $X$. The *Banach-Mazur game* of $A$, denoted by $G^{**}(A)$ (or $G^{**}(A, X)$), is defined as follows: Players I and II choose alternatively nonempty open sets $V_n$ ($n \in \omega$) with $V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots$ in $\omega$ moves,

$$\begin{array}{llllll} \text{I} & V_0 & & V_2 & & \ldots \\ \text{II} & & V_1 & & V_3 & \ldots \end{array}$$

Player II wins this run of the game if $\bigcap_{n\in\omega} V_n \cap A = \emptyset$.

The notions of strategies and winning strategies are defined in the same way as for Gale-Stewart games in § 1.6.

**Theorem 1.8.3** (Banach and Mazur, Oxtoby)**.** Let $X$ be a topological space and $A$ be a subset of $X$. Then $A$ is meager if and only if player II has a winning strategy in the game $G^{**}(A)$.

*Proof.* See, e.g., [45, Theorem 8.33]. □

One can characterize when a subset $A$ of $X$ has the Baire property in $X$ in terms of Banach-Mazur games: Let $U_A$ be the union of all open sets $U$ in $X$ such that $U \setminus A$ is meager in $X$. Then $A$ has the Baire property if and only if the

[3] Although Gödel [28] announced the similar result for Lebesgue measurability in 1938 and seemed to know about this result at that time, it seems to have been first made explicit in [67, p. 216] (cf. [44, p. 169]).

set $A \setminus U_A$ is meager, hence if and only if player II has a winning strategy in the Banach-Mazur game $G^{**}(A \setminus U_A)$.

It is natural to ask whether one could characterize when player I has a winning strategy in Banach-Mazur games in terms of topology. The answer is: If $X$ is a completely metrizable topological space, then player I has a winning strategy in $G^{**}(A)$ if and only if there is a nonempty open subset $U$ of $X$ such that $U \setminus A$ is meager in $U$, where $U$ is equipped with the relative topology of $X$ in this case. (But this characterization is not true if $X$ is a general topological space. For the proof, see, e.g., [45, Theorem 8.33].) It follows from this result that player I cannot have a winning strategy in the Banach-Mazur game $G^{**}(A \setminus U_A)$. Hence we can conclude that a subset $A$ of $X$ has the Baire property if and only if the Banach-Mazur game $G^{**}(A \setminus U_A)$ is *determined*, i.e., either player I or II has a winning strategy in this game. Now we have reduced the problem of the Baire property of a given set to the problem of determinacy of Banach-Mazur games. This is how the Polish school of mathematics found out the following: Assume every Banach-Mazur game in the Baire space is determined, then every set of reals has the Baire property.

We also use a variant of Banach-Mazur games so-called the *unfolded Banach-Mazur games*:

**Definition 1.8.4** (The unfolded Banach-Mazur games)**.** Let $X$ be a topological space and $F$ be a subset of $X \times {}^{\omega}\omega$. Define the *unfolded Banach-Mazur game* $G_u^{**}(F)$ (or $G_u^{**}(F, X)$) as follows:

$$\begin{array}{llllll} \mathrm{I} & V_0, y_0 & & V_2, y_1 & & \dots \\ \\ \mathrm{II} & & V_1 & & V_3 & \dots \end{array}$$

Players I and II choose $V_0, V_1, \dots$ as in the Banach-Mazur game, but additionally I plays a natural number $y_n$ in her $n$th move. Let $y = \langle y_n \mid n \in \omega \rangle$. Player II wins if $\left(\bigcap_{n\in\omega} V_n \times \{y\}\right) \cap F = \emptyset$.

We have the same kind of characterization theorem as Banach-Mazur games:

**Theorem 1.8.5** (Folklore)**.** Let $X$ be a topological space and $F$ be a subset of $X \times {}^{\omega}\omega$. Let $A = \exists^{\mathbb{R}} F$.

1. If $A$ is meager in $X$, then player II has a winning strategy in the game $G_u^{**}(F)$.

2. Suppose that $F$ is of the form $(f \times \mathrm{id})^{-1}(C)$, where $f\colon X \to {}^{\omega}\omega$ is a continuous function, $f \times \mathrm{id}\colon X \times {}^{\omega}\omega \to {}^{\omega}\omega \times {}^{\omega}\omega$ is defined by $(f \times \mathrm{id})\,(x, y) = (f(x), y)$, and $C$ is a subset of ${}^{\omega}\omega \times {}^{\omega}\omega$. Then if player II has a winning strategy in the game $G_u^{**}(F)$, then $A$ is meager in $X$.

*Proof.* We show the first item. By Theorem 1.8.3, if $A$ is meager, then player II has a winning strategy $\tau$ in the game $G^{**}(A, X)$. But $\tau$ can be viewed as a winning strategy for player II in the game $G_u^{**}(F)$ by ignoring I's moves of $y_n$s.

Next we show the second item. The point is that given a winning strategy $\tau$ for player II in the game $G_u^{**}(F)$, she can modify $\tau$ so that in her $n$th move, she can decide the $n$th digit of $f(x)$ by the continuity of $f$. The rest of the argument is the same as in [45, Theorem 21.5]. $\square$

Using Theorem 1.8.5, one can characterize when player I has a winning strategy in the game $G_u^{**}(F)$ as well: Player I has a winning strategy in the game $G_u^{**}(F)$ if and only if there is a nonempty open set $U$ in $X$ such that $U \setminus A$ is meager in $U$. As before, it follows from this fact that a subset $A$ of $X$ has the Baire property if and only if the game $G_u^{**}(F')$ is determined, where $F'$ is a subset of $X \times {}^{\omega}\omega$ with $\exists^{\mathbb{R}} F' = A \setminus U_A$ and $U_A$ is the same as in the paragraphs after Theorem 1.8.3.

The advantage of the unfolded Banach-Mazur games over Banach-Mazur games is that one can reduce the complexity of the payoff sets (from $A$ to $F$ in the above definition). If $A$ is a $\mathbf{\Sigma}^1_1$ set in the Baire space, then $A \setminus U_A$ is also $\mathbf{\Sigma}^1_1$, hence there is a closed subset $F$ of ${}^{\omega}\omega \times {}^{\omega}\omega$ such that $\exists^{\mathbb{R}} F = A \setminus U_A$. Since there is no difference between playing basic open sets and playing open sets for Banach-Mazur games and the unfolded ones and basic open sets in the Baire space are easily coded by natural numbers, one can simulate the unfolded Banach-Mazur games by Gale-Stewart games in a simple way. By the first item of Theorem 1.6.3, all the closed Banach-Mazur games and the unfolded ones are determined. Hence we can conclude that every $\mathbf{\Sigma}^1_1$ set of reals has the Baire property.[4]

## 1.9 Forcing

While Zermelo-Fraenkel set theory with the axiom of choice (ZFC), which is a set-theoretic axiomatization for the foundation of mathematics, is a very good basis for most of mathematical practice, some mathematical questions remain undetermined by ZFC and one such typical question is whether the Continuum Hypothesis (CH) is true or not. In 1963, Cohen introduced *forcing* to prove that CH does not follow from ZFC and since then, forcing has been one of the most important basic tools in set theory. Starting from a model of ZFC (called the "ground model"), Cohen produced an extension of the given model (called a "generic extension") which is a model of ZFC and the negation of CH. This technique is so general that one can define a generic extension for each partial order in the given ground model, and one can change the truth-value of many mathematical statements between ground models and their generic extensions which yield the consistency and the independence of those statements from ZFC.

[4] This is not the original proof of Lusin. It is due to Solovay (cf. [44, Exercise 27.14]).

In Chapter 2 and Chapter 3, we assume the basic theory of forcing which can be found in, e.g., [52, § 7, 8]. Let us fix the notation concerning forcing and list the partial orders we will use in this thesis.

The *Universe* is the class of all sets and it is denoted by $V$. Let $M$ be a model of ZF, $\mathbb{P}$ be a partial order belonging to $M$, and $G$ be a $\mathbb{P}$-generic filter over $M$. By $M[G]$, we mean the generic extension of $M$ via $G$. For a $\mathbb{P}$-name $\tau$ in $M$, $\tau^G$ denotes the interpretation of $\tau$ via $G$. For a set $x$, $\check{x}$ denotes the standard $\mathbb{P}$-name for $x$, i.e., $\check{x}^G = x$ for any filter $G$.

The following is the list of partial orders we will use:

**Cohen forcing.** The partial order is $({}^{<\omega}\omega, \supseteq)$ denoted by $\mathbb{C}$ where $\supseteq$ is reverse inclusion on finite sequences of natural numbers. Given a model $M$ of ZF and a $\mathbb{C}$-generic filter $G$ over $M$, set $x_G = \bigcup\{p \in \mathbb{C} \mid p \in G\}$. By the genericity of $G$, $x_G$ is a function from $\omega$ to itself (i.e., an element of the Baire space). Such objects are called *Cohen reals over $M$*. Also one can reconstruct $G$ from $x_G$ and $\mathbb{C}$ as follows: $G = \{p \in \mathbb{C} \mid p \subseteq x_G\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{C}$-generic filters over $M$ and Cohen reals over $M$. We often identify these two objects.

**Random forcing** Elements of the partial order are Borel sets in the Baire space (or in the real line) with positive Lebesgue measure ordered by inclusion and it is denoted by $\mathbb{B}$. Given a model $M$ of ZF+AC$_\omega(\mathbb{R})$ and a $\mathbb{B}$-generic filter $G$ over $M$, the set $\bigcap\{B^{M[G]} \mid B \in G\}$ is a singleton $\{x_G\}$, where $B^{M[G]}$ is the interpretation of $B$ in $M[G]$ via Borel codes for $B$ in $M$.[5] Such reals $x_G$ are called *random reals over $M$*. As with Cohen reals, one can recover $G$ from $x_G$ and $M$ as follows: $G = \{B \in \mathbb{B} \mid x_G \in B^{M[G]}\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{B}$-generic filters over $M$ and random reals over $M$. We often identify these two objects.

**Hechler forcing.** Elements of the partial order are pairs $(n, f)$ where $n$ is a natural number and $f$ is a function from $\omega$ to itself and it is denoted by $\mathbb{D}$. Given $(n, f)$ and $(m, g)$ in $\mathbb{D}$, $(n, f) \leq (m, g)$ if $n \geq m$, $f \restriction m = g \restriction m$ and $f(k) \geq g(k)$ for any $k \geq m$. Given a model $M$ of ZF and a $\mathbb{D}$-generic filter $G$ over $M$, $x_G = \bigcup\{f \restriction n \mid (n, f) \in G\}$ is a function from $\omega$ to itself by the genericity of $G$. Such reals $x_G$ are called *Hechler reals over $M$*. One can recover $G$ from $x_G$ and $M$ as follows: $G = \{(n, f) \in \mathbb{D} \mid x_G \supseteq f \restriction n \text{ and } (\forall k \geq n)\ f(k) \leq x_G(k)\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{D}$-generic filters over $M$ and Hechler reals over $M$. We often identify these two objects.

[5] For the definition and the basic properties of Borel codes, see § 1.13.

**Mathias forcing.** Elements of the partial order are pairs $(s, A)$ where $s$ is a finite set of natural numbers and $A$ is an infinite set of natural numbers such that $\max(s) < \min(A)$ and the forcing is denoted $\mathbb{R}$.[6] Given $(s, A)$ and $(t, B)$ in $\mathbb{R}$, $(s, A) \leq (t, B)$ if $s \cap (n + 1) = t$, $A \subseteq B$ and $s \setminus t \subseteq B$, where $n = \max t$. Given a model $M$ of ZF and a $\mathbb{R}$-generic filter over $M$, $x_G = \bigcup\{s \mid (\exists A)\ (s, A) \in G\}$ is an infinite set of natural numbers by the genericity of $G$. Such reals are called *Mathias reals over* $M$. One can reconstruct $G$ from $x_G$ and $M$ as follows: $G = \{(s, A) \in \mathbb{R} \mid s \subseteq x_G \text{ and } x_G \subseteq s \cup A\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{R}$-generic filters over $M$ and Mathias reals over $M$. We often identify these two objects.

**Sacks forcing.** Elements of the partial order are perfect trees on 2 ordered by inclusion and it is denoted by $\mathbb{S}$. Given a model $M$ of ZF and an $\mathbb{S}$-generic filter $G$ over $M$, $x_G = \bigcup\{\text{stem}(T) \mid S \in G\}$ is a function from $\omega$ to 2 by the genericity of $G$. Such reals are called *Sacks reals over* $M$. One can recover $G$ from $x_G$ and $M$ as follows: $G = \{S \in \mathbb{S} \mid x_G \in [S]\}$. Hence there is a canonical one-to-one connection between $\mathbb{S}$-generic filters over $M$ and Sacks reals over $M$. We often identify these two objects.

**Silver forcing.** Elements of the partial order are uniform perfect trees on 2 ordered by inclusion and it is denoted by $\mathbb{V}$, where a perfect tree $T$ on 2 is *uniform* if for any $s$ and $t$ in $T$ with the same length and $i = 0, 1$, $s^\frown\langle i\rangle \in T$ if and only if $t^\frown\langle i\rangle \in T$. Given a model $M$ of ZF and a $\mathbb{V}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as Sacks reals and such reals are called *Silver reals over* $M$. There is a canonical one-to-one correspondence between $\mathbb{V}$-generic filters over $M$ and Silver reals over $M$ as in Sacks forcing. We often identify these two objects.

**Miller forcing.** Elements of the partial order are superperfect trees on $\omega$ ordered by inclusion and it is denoted by $\mathbb{M}$, where a tree $T$ on $\omega$ is *superperfect* if for any node $t$ of $T$, there is an extension $u$ of $t$ in $T$ such that $\{n \in \omega \mid u^\frown\langle n\rangle \in T\}$ is infinite. Given a model $M$ of ZF and a $\mathbb{M}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as Sacks reals and such reals are called *Miller reals over* $M$. There is a canonical one-to-one correspondence between $\mathbb{M}$-generic filters over $M$ and Miller reals over $M$ as in Sacks forcing. We often identify these two objects.

**Laver forcing.** Elements of the partial order are trees $T$ on $\omega$ such that for each node $t \supseteq \text{stem}(T)$ of $T$, the set $\{n \in \omega \mid t^\frown\langle n\rangle \in T\}$ is infinite and they are ordered by inclusion. The partial order is denoted by $\mathbb{L}$. Given a model $M$ of ZF and a $\mathbb{L}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as

[6] We use this notation only in Chapter 2 where we do not use $\mathbb{R}$ either for the real line, the Baire space or the Cantor space. Hence there will be no confusion for this notation.

Sacks reals and such reals are called *Laver reals over $M$*. There is a canonical one-to-one correspondence between $\mathbb{L}$-generic filters over $M$ and Laver reals over $M$ as in Sacks forcing. We often identify these two objects.

**Eventually different forcing.** Elements of the partial order are pairs $(s, F)$ where $s$ is a finite sequence of natural numbers and $F$ is a finite set of functions from $\omega$ to itself and it is denoted by $\mathbb{E}$. Given $(s, F)$ and $(t, F')$ in $\mathbb{E}$, $(s, F) \leq (t, F')$ if $s \supseteq t$, $F' \subseteq F$ and $(\forall f \in F') \big(\forall n \in \mathrm{dom}(s \setminus t)\big)\ s(n) \neq f(n)$. Given a model $M$ of ZF and a $\mathbb{E}$-generic filter $G$ over $M$, $x_G = \bigcup \{s \mid (\exists F)\ (s, F) \in G\}$ is a function from $\omega$ to itself by the genericity of $G$. Such reals are called *$\mathbb{E}$-generic reals over $M$* and one can reconstruct $G$ from $x_G$ and $M$ as follows: $G = \{(s, F) \in \mathbb{E} \mid s \subseteq x_G \text{ and } (\forall f \in F) \big(\forall n \geq \mathrm{dom}(s)\big)\ x_G(n) \neq f(n)\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{E}$-generic filters over $M$ and $\mathbb{E}$-generic reals over $M$. We often identify these two objects.

Next, we introduce useful classes of forcings that we use in Chapter 2. Let $\mathbb{P}$ be a partial order. For $p$ and $q$ in $\mathbb{P}$, $p$ and $q$ are *compatible* (denoted by $p||q$) if there is an $r$ in $\mathbb{P}$ such that $r \leq p$ and $r \leq q$. They are called *incompatible* (denoted by $p \perp q$) if they are not compatible. A subset $A$ of $\mathbb{P}$ is an *antichain* if any two different elements of $A$ are incompatible. A subset $D$ of $\mathbb{P}$ is *dense* if for any $p$ in $\mathbb{P}$ there is a $d$ in $D$ such that $d \leq p$. Let $D$ be a subset of $\mathbb{P}$ and $p$ be an element of $\mathbb{P}$. The set $D$ is *predense below* $p$ if for any $q \leq p$ in $\mathbb{P}$ there is a $d$ in $D$ such that $q$ and $d$ are compatible.

For a regular cardinal $\theta$, $\mathcal{H}_\theta$ denotes the set of all sets $a$ such that $|\mathrm{TC}(a)| < \theta$, where $\mathrm{TC}(a)$ denotes the transitive closure of $a$, i.e., the smallest set $b$ containing $a$ and which is transitive, i.e., $(\forall x \in b)\ x \subseteq b$.

**The countable chain condition (ccc).** A partial order $\mathbb{P}$ has the *countable chain condition* (or $\mathbb{P}$ is ccc) if every antichain of $\mathbb{P}$ is countable. Since the invention of forcing, ccc forcings have been fundamental partial orders and they enjoy many nice properties, e.g., they preserve cardinalities, i.e., given a ccc partial order $\mathbb{P}$ and a $\mathbb{P}$-generic filter $G$ over $V$, for any ordinal $\alpha$, $\alpha$ is a cardinal in $V$ if and only if it is a cardinal in $V[G]$. In particular, $\omega_1^V = \omega_1^{V[G]}$. Typical examples of ccc forcings are Cohen forcing, random forcing, Hechler forcing, and eventually different forcing. Mathias forcing, Sacks forcing, Silver forcing, Miller forcing, and Laver forcing are not ccc.

**Proper forcings.** A partial order $\mathbb{P}$ is *proper* if for any sufficiently large regular cardinal $\theta$ (e.g., $\theta \geq 2^{|\mathbb{P}|}$) and any countable elementary substructure $X$ of $\mathcal{H}_\theta$ with $\mathbb{P} \in X$, and any $p$ in $\mathbb{P} \cap X$, there is a $q \leq p$ in $\mathbb{P}$ such that $q$ is $(X, \mathbb{P})$-generic, i.e., for any dense set $D$ of $\mathbb{P}$ in $X$, $D \cap X$ is predense below $q$. Proper forcings were introduced by Shelah and they are also fundamental in modern set theory. They are a generalization of ccc forcings (i.e., every ccc forcing is proper) and they enjoy several properties ccc forcings satisfy, e.g., for a proper forcing $\mathbb{P}$, a $\mathbb{P}$-generic filter $G$ over $V$, and any countable set of ordinals $A$ in $V[G]$, there is a countable set of ordinals $B$ in $V$ such that $A \subseteq B$. In particular, $\omega_1^V = \omega_1^{V[G]}$. All the examples of forcings listed above are proper.

## 1.10 Large cardinals

*Large cardinals* are cardinals with certain transcendence properties over cardinals smaller than them. Many such properties are the analogies of the ones $\omega$ has over finite numbers. For the basics and background for large cardinals, we refer the reader to [44]. Let us list the large cardinals (or the large cardinal properties) we will use in this thesis:

**Inaccessible cardinals.** Inaccessible cardinals are the least and the oldest large cardinals. An uncountable cardinal $\kappa$ is *inaccessible* if it is regular, i.e., for any ordinal $\alpha < \kappa$ and any function $f\colon \alpha \to \kappa$, $f$ is bounded, i.e., there is a $\beta < \kappa$ such that $\operatorname{ran}(f) \subseteq \beta$, and it is strong limit, i.e., for any $\alpha < \kappa$, $2^\alpha < \kappa$. If $\kappa$ is inaccessible, then $V_\kappa$ is a model of ZFC. Hence the existence of an inaccessible cardinal implies the consistency of ZFC and by Gödel's Incompleteness Theorem, the consistency of ZFC+"There is an inaccessible cardinal" is strictly stronger than that of ZFC.

**Sharps.** Let $X$ be a set. By $X^\#$, we mean the complete theory of $\mathrm{L}(X)$ in the language $(\in, \{c_n\}_{n\in\omega}, \{d_a\}_{a\in\mathrm{TC}(X)})$ with some special properties, where $c_n$ is the constant for the $n$-th indiscernible for $\mathrm{L}(X)$ and $d_a$ is the constant for $a \in \mathrm{TC}(X)$. For the details, see, e.g., [22]. The existence of $X^\#$ is equivalent to the existence of a closed unbounded proper class of indiscernibles for $\mathrm{L}(X)$ with some properties. Also it is equivalent to the existence of an elementary embedding $j$ from $\mathrm{L}(X)$ to itself whose critical point is above the rank of $X$. (Here the critical point of $j$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$.) We say *every real has a sharp* if for any real $x$, $x^\#$ exists. We say *every set has a sharp* if for any set $X$, $X^\#$ exists.

**Measurable cardinals.** Measurable cardinals are one of the most fundamental large cardinals. An uncountable cardinal $\kappa$ is a *measurable cardinal* if there is an elementary embedding from $V$ to a transitive proper class whose critical point is $\kappa$. There is a first-order characterization of measurable cardinals: An uncountable cardinal $\kappa$ is measurable if and only if there is a non-trivial $\kappa$-complete ultrafilter on $\kappa$; here a filter is *non-trivial* if it is not principal and it is *$\kappa$-complete* if it is closed under intersections with $<\kappa$ many sets. It is easy to see that if $\kappa$ is a measurable cardinal, then for any set $X \in V_\kappa$, $X^\#$ exists.

**Strong cardinals.** Most large cardinals stronger than measurable cardinals assert the existence of elementary embeddings from $V$ to a transitive class $M$ with certain properties. The more $M$ is close to $V$, the stronger the large cardinal property is. Strong cardinals are one of the natural strengthening of measurable cardinals in this sense. Let $\alpha$ be an ordinal. An uncountable cardinal $\kappa$ is *$\alpha$-strong* if there is an elementary embedding $j$ from $V$ to $M$ such that $M$ is transitive, the critical point of $j$ is $\kappa$, and $V_\alpha \subseteq M$. An uncountable cardinal $\kappa$ is *strong* if it is $\alpha$-strong for any ordinal $\alpha$. It is immediate that any $\alpha$-strong cardinal is measurable. If $\kappa$ is $(\kappa+2)$-strong, then there are unboundedly many measurable cardinals below $\kappa$.

**Woodin cardinals.** Woodin cardinals were introduced when Shelah and Woodin tried to decide the optimal upper bound for the consistency strength of the saturation of the nonstationary ideal on $\omega_1$ and they are tightly connected to the determinacy of projective sets in Gale-Stewart games. Let $\alpha < \delta$ be ordinals and $A$ be a subset of $V_\delta$. An uncountable cardinal $\kappa < \delta$ is *$\alpha$-$A$-strong* if there is an elementary embedding $j$ from $V$ to a transitive class $M$ such that $\kappa$ is the critical point of $j$, $V_\alpha \subseteq M$, and $A \cap V_\alpha = j(A) \cap V_\alpha$. An uncountable cardinal $\kappa$ is *$<\delta$-$A$-strong* if it is $\alpha$-$A$-strong for every $\alpha < \delta$. An inaccessible cardinal $\delta$ is *Woodin* if it is a limit of $<\delta$-$A$-strong cardinals for any subset $A$ of $V_\delta$. If $\delta$ is Woodin, then $V_\delta$ satisfies "There is a proper class of strong cardinals".

## 1.11 Inner models and inner model theory

*Inner models* are transitive proper class models of ZF. The study of *inner model theory* is about canonical inner models with large cardinals. The *Gödel's Constructible Universe* L is the most basic canonical inner model. It always exists in ZF and it is the least inner model of ZFC. Gödel introduced L to prove the consistency of AC, CH, and moreover the Generalized Continuum Hypothesis (GCH) with ZF. Beside this fact, L has many interesting properties, e.g., in L, there is a $\Delta^1_2$ set of reals without the Baire property and which is not Lebesgue measurable, and there is a $\Pi^1_1$ set of reals without the perfect set property. As at the end of § 1.8, every $\mathbf{\Sigma}^1_1$ set of reals has the Baire property. Also every $\mathbf{\Sigma}^1_1$ set of reals is Lebesgue measurable and has the perfect set property. Hence the above facts about L show that $\mathbf{\Sigma}^1_1$ sets of reals are the limit for proving the above regularity properties in ZFC.

One can relativize the construction of L to any set in the following two ways: For a set $A$, $\mathrm{L}[A]$ denotes the least inner model such that $A \cap \mathrm{L}[A] \in \mathrm{L}[A]$ and $\mathrm{L}(A)$ denotes the least inner model containing $A$ as an element. The model $\mathrm{L}[A]$ is always a model of ZFC and $A$ might not belong to $\mathrm{L}[A]$ in general (e.g., $\mathrm{L}[\mathbb{R}] = \mathrm{L}$ and $\mathbb{R}$ does not belong to L in general) while $\mathrm{L}(A)$ might not be a model of AC (e.g., if there are $\omega$-many Woodin cardinals and a measurable cardinal above all of them, then AC fails in $\mathrm{L}(\mathbb{R})$). For a set of ordinals $A$, $\mathrm{L}[A] = \mathrm{L}(A)$.

Let us list the basic properties of L we use later:

**Lemma 1.11.1** (Gödel)**.**

1. The relation $\{(x, a) \in {}^{\omega}\omega \times {}^{\omega}\omega \mid x \in \mathrm{L}[a]\}$ is a $\Sigma^1_2$ set of reals.
2. For any real $a$, $\mathrm{L}[a] \vDash$ "There is a $\Delta^1_2(a)$ wellordering of the reals".

*Proof.* See, e.g., [66, Theorem 8F.7, 8F.23, 8F.24]. □

*Core models* are canonical inner models with the following special properties: first they are fine structural (constructed with Jensen's $J_\alpha$ Hierarchy), second, they are forcing invariant (they are absolute between ground models and their forcing extensions), and lastly they are close to $V$, e.g., they have covering properties or weak covering. If $0^\#$ does not exist, L is the basic core model. Unlike many canonical inner models, one needs to assume some anti-large cardinal hypothesis to prove the existence of core models. The following is a general result for the existence of the core model:

**Theorem 1.11.2** (Dodd and Jensen [24]; Koepke [50]; Jensen [38]; Mitchell [64]; Jensen [39]; Steel [79]; Jensen and Steel [41, 40])**.** Suppose every real has a sharp. If there is no inner model of ZFC with a Woodin cardinal, then the core model K exists. More generally, if $\boldsymbol{\Delta}^1_2$-determinacy fails, then there is a real $a_0$ such that for any $a \geq_{\mathrm{T}} a_0$, the $a$-relativized version of the core model $\mathrm{K}_a$ exists, where $\leq_{\mathrm{T}}$ is the Turing order.[7] Moreover, the core models have the following properties:

1. the relation $\{(x, a) \in {}^{\omega}\omega \times {}^{\omega}\omega \mid x \in \mathrm{K}_a\}$ is a $\Sigma^1_3$ set of reals, and
2. for any real $a$, $\mathrm{K}_a \vDash$ "There is a $\Delta^1_3(a)$ wellordering of the reals".

*Proof.* When there is a real $a$ such that $a^\dagger$ does not exist, see [24]. In the other case, see [79]. Note that in [79], Steel assumed the existence of a measurable cardinal to construct K. But Jensen and Steel [41, 40] omitted this assumption. □

To build core models, one needs to study fragments of core models or more general objects, which are called *mice*. Standard examples of mice are L and the core model K. For a set $a$, there are $a$-relativized version of mice called *a-mice*. Basic examples are $\mathrm{L}[a]$ and $\mathrm{K}_a$. The following two theorems are essential to study mice:

**Theorem 1.11.3** (Comparison Lemma)**.** Let $M, N$ be mice and $\theta = \max\{|M|^+, |N|^+\}$. After $<\theta$ steps of coiterations, one of them is an initial segment of the other.

[7] Note that $\Delta^1_2$-determinacy (lightface) is equivalent to the existence of an inner model of ZFC with a Woodin cardinal. This is why we said "More generally,".

*Proof.* See, e.g., [92, Lemma 9.1.8]. □

**Theorem 1.11.4** (Dodd-Jensen Lemma)**.** Let $M$ be a mouse and $i\colon M \to M'$ be an iteration map according to the unique iteration strategy of $M$. Suppose there is a $\Sigma^*$ preserving map $\sigma\colon M \to M'$. Then

1. there is no drop in the iteration tree for $i$, and
2. for any ordinal $\xi$ in $M$, $i(\xi) \leq \sigma(\xi)$.

In particular, any two iteration maps without drops from a mouse to a mouse are the same.

*Proof.* See, e.g., [92, Lemma 9.2.10]. □

## 1.12 Absoluteness

We speak of *absoluteness* if a sentence or a class of sentences does not change truth values of mathematical statements between models of set theory and it is one of the basic and central notions in set theory. Given models of set theory $M$ and $N$ with $M \subseteq N$ and a formula $\phi$, $\phi$ is *absolute between $M$ and $N$* if for any finite sequence of elements $\vec{x}$ in $M$, $M \vDash \phi(\vec{x})$ if and only if $N \vDash \phi(\vec{x})$. For example, the formula "$x$ is $\omega$" is absolute between any two transitive models of ZF. The first nontrivial and important absolute notion is *wellfoundedness*. A relation $R$ on a set $A$ is *wellfounded* if for any nonempty subset $B$ of $A$, there is an $R$-minimal element of $B$, i.e., there is a $b \in B$ such that for any element $a$ of $B$, $(a, b) \notin R$.

**Lemma 1.12.1.** The formula "$R$ is a wellfounded relation on $A$" is absolute between any two transitive models of ZF.

*Proof.* See, e.g., [37, Lemma 13.11] and the two paragraphs preceding it. □

Given a $\Pi^1_1$ formula $\phi$, one can recursively compute a tree $T$ on $\omega \times \omega$ such that $\{x \mid \mathcal{A}^2 \vDash \phi(x)\} = \{x \mid [T_x] = \emptyset\}$, where $T_x = \{t \in {}^{<\omega}\omega \mid \big(x \restriction \mathrm{dom}(t), t\big) \in T\}$ in ZF+$\mathrm{AC}_\omega(\mathbb{R})$. But $[T_x] = \emptyset$ if and only if $(T_x, \supseteq)$ is wellfounded. Hence $\mathcal{A}^2 \vDash \phi(x)$ if and only if $(T_x, \supseteq)$ is wellfounded. Hence the problem of membership for a $\Pi^1_1$ set is reduced to the one for the wellfoundedness of certain trees. Combining with Lemma 1.12.1,

**Theorem 1.12.2** (Mostowski)**.** Every $\Pi^1_1$ formula is absolute between transitive models of ZF+$\mathrm{AC}_\omega(\mathbb{R})$. Hence every $\Sigma^1_1$ formula is also absolute between transitive models of ZF+$\mathrm{AC}_\omega(\mathbb{R})$.

*Proof.* See, e.g., [37, Theorem 25.4]. □

In general, a $\Pi^1_2$ formula is not absolute between transitive models of ZF. Shoenfield proved that any $\Pi^1_2$ formula is absolute between inner models of ZF+ $\mathrm{AC}_\omega(\mathbb{R})$:

**Theorem 1.12.3** (Shoenfield)**.** For any $\Pi^1_2$ formula $\phi$ and real $a$, there is a tree $T$ on $\omega\times\omega_1$ in $\mathrm{L}[a]$ such that for any real $x$, $\mathcal{A}^2 \vDash \phi(x,a)$ if and only $T_x$ is wellfounded. This tree is called a *Shoenfield tree* and one can construct a Shoenfield tree in any inner model of ZF+$\mathrm{AC}_\omega(\mathbb{R})$ and the construction depends only on $\phi$, $a$, and a fixed uncountable ordinal (in this case, $\omega_1^V$).

Hence Shoenfield trees are absolute and thus every $\Pi^1_2$ formula (and $\Sigma^1_2$ formula) is absolute between between inner models of ZF+$\mathrm{AC}_\omega(\mathbb{R})$, especially between L and $V$.

*Proof.* See, e.g., [66, 8F.8, 8F.9, 8F.10]. □

In general, a $\Pi^1_3$ formula is not absolute between L and $V$, e.g., the statement "Every real is in L" is equivalent to a $\Pi^1_3$ formula and one can add nonconstructible real (e.g., a Cohen real over L) via forcing starting from L. Using sharps for reals, Martin and Solovay constructed a tree called *Martin-Solovay tree* for a $\Pi^1_3$ formula which is like a Shoenfield tree for a $\Pi^1_2$ formula. We will give a sufficient condition for the absoluteness of Martin-Solovay trees. Assume every real has a sharp. For a real $a$, let $I_a$ be the closed unbounded class of indiscernibles derived from $a^\#$ and set $I = \bigcap_{a\in{}^\omega\omega} I_a$. The class $I$ is called the *class of uniform indiscernibles* and $u_2$ denotes the second element of $I$ and is called the *second uniform indiscernible*.

**Theorem 1.12.4** (Martin and Solovay)**.** Let $M$, $N$ be inner models of ZFC+ "Every real has a sharp". If $u_2^M = u_2^N$ with $M \subseteq N$, then Martin-Solovay trees are absolute between $M$ and $N$ and hence every $\Pi^1_3$ formula (and $\Sigma^1_3$ formula) is absolute between $M$ and $N$.

*Proof.* See, e.g., [33, Theorem 2.1]. □

Every $\Sigma^1_3$ formula is absolute between the core model K and $V$ when K exists:

**Theorem 1.12.5** (Dodd and Jensen; Steel)**.** Assume every real has a sharp. If $\boldsymbol{\Delta}^1_2$-determinacy fails, then there is a real $a_0$ such that for any $a \geq_{\mathrm{T}} a_0$, the $a$-relativized version of the core model $\mathrm{K}_a$ exists and every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_a$ and $V$.

*Proof.* In case there is a real $a$ such that $a^\dagger$ does not exist, this is due to Dodd and Jensen [24]. If every real has a dagger, then this is due to Steel [79, Theorem 7.9].[8] □

[8]In [79, Theorem 7.9], he assumed two measurable cardinals. But one can replace this assumption with daggers for reals. See [71, Theorem 0.1].

Before closing this section, we discuss the absoluteness of being a winning strategy for Gale-Stewart games with closed payoff sets:

**Theorem 1.12.6** (Folklore)**.** Let $X$ be a nonempty set and $M$ be a transitive model of ZF with $X \in M$. For any closed subset $A$ of ${}^{\omega}X$, given a strategy $\sigma$ for player I in $M$, $M \vDash$ "$\sigma$ is winning in $A$" if and only if $V \vDash$ "$\sigma$ is winning in $A$". The same holds for player II.

*Proof.* As described in [45, 20.B], if there is a winning strategy for player I in the game $G_X(A)$ for a closed set $A$, then there is a canonical winning quasistrategy $\Sigma_A$ for player I and a strategy $\sigma$ for I is winning for the game $G_X(A)$ if and only if $\sigma \subseteq \Sigma_A$. Since the construction of $\Sigma_A$ is absolute between transitive models of ZF, the statement "$\sigma$ is winning in $A$" is absolute between transitive models of ZF, as desired. □

## 1.13 Borel codes and $\infty$-Borel codes

If $X$ is the Baire space, the Cantor space, or the real line, it is easy to show that there is a surjection from the Cantor space to the set of all Borel subsets of $X$. (By induction on $1 \leq \xi < \omega_1$, one can construct surjections from the Cantor space to $\mathbf{\Sigma}^0_\xi$ subsets of $X$ and one can amalgamate them into one surjection.) *Borel codes* are effective realizations of such surjections introduced by Solovay. To introduce them, we first fix some notions and notations. Let $Y$ be a set. A tree $T$ on $Y$ is *wellfounded* if $(T, \supseteq)$ is wellfounded. A node $s$ of $T$ is *terminal* if there is no node $t$ in $T$ extending $s$. Let $\mathrm{Term}(T)$ denote the set of all terminal nodes of $T$. Let $s, t$ be nodes of $T$. The node $t$ is a *successor of $s$ in $T$* if $t$ extends $s$ and $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$. For a node $s$ of $T$, $\mathrm{Succ}_T(s)$ denotes the set of successors of $s$ in $T$.

We introduce Borel codes for Borel subsets of the Cantor space. One can introduce Borel codes for the Baire space and the real line in the same way. *Borel codes* are pairs $(T, f)$ where $T$ is a wellfounded tree on $\omega$ and $f$ is a function from $\mathrm{Term}(T)$ to ${}^{<\omega}2$. One can simply regard Borel codes as elements of the Cantor space by identifying trees on $\omega$ with a map from ${}^{<\omega}\omega$ to $\{0, 1\}$ and fixing a simple bijection between ${}^{<\omega}\omega$ and $\omega$. With this identification, we regard Borel codes as elements of the Cantor space. Given a Borel code $c = (T, f)$, the *decode* $B_c$ is defined as follows: For each node $t$ of $T$,

$$B_t = \begin{cases} [f(t)] & \text{if } t \in \mathrm{Term}(T) \\ {}^{\omega}2 \setminus B_s & \text{if } (\exists s \in T)\ \{s\} = \mathrm{Succ}_T(t) \\ \bigcup_{s \in \mathrm{Succ}_T(t)} B_s & \text{otherwise.} \end{cases}$$

We set $B_c = B_\emptyset$. This is well-defined because $T$ is wellfounded. One can easily check any Borel set is of the form $B_c$ for some Borel code $c$. The following are basic observations on Borel codes:

**Lemma 1.13.1** (Solovay)**.** The set of Borel codes and the relations $x \in B_c$, $x \notin B_c$ are $\Pi^1_1$ sets and hence they are absolute between transitive models of ZF+AC$_\omega(\mathbb{R})$.

*Proof.* See, e.g., [37, Lemma 25.44 & Lemma 25.55]. $\square$

Infinitary Borel codes ($\infty$-Borel codes) are a transfinite generalization of Borel codes: Let $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ be the language allowing arbitrary many conjunctions and disjunctions and no quantifiers with atomic sentences $\mathbf{a}_n$ for each $n \in \omega$. The $\infty$-*Borel codes* are the sentences in $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ belonging to any $\Gamma$ such that

- the atomic sentence $\mathbf{a}_n$ is in $\Gamma$ for each $n \in \omega$,
- if $\phi$ is in $\Gamma$, then so is $\neg\phi$, and
- if $\alpha$ is an ordinal and $\langle \phi_\beta \mid \beta < \alpha\rangle$ is a sequence of sentences each of which is in $\Gamma$, then $\bigvee_{\beta<\alpha} \phi_\beta$ is also in $\Gamma$.

To each $\infty$-Borel code $\phi$, we assign a set of reals $B_\phi$ in the same way as decoding Borel codes:

- if $\phi = \mathbf{a}_n$, then $B_\phi = \{x \in {}^\omega 2 \mid x(n) = 1\}$,
- if $\phi = \neg\psi$, then $B_\phi = {}^\omega 2 \setminus B_\psi$, and
- if $\phi = \bigvee_{\beta<\alpha} \psi_\beta$, then $B_\phi = \bigcup_{\beta<\alpha} B_{\psi_\beta}$.

A set of reals $A$ is called $\infty$-*Borel* if there is an $\infty$-Borel code $\phi$ such that $A = B_\phi$.

As Borel codes, one can regard $\infty$-Borel codes as wellfounded trees with atomic sentences $\mathbf{a}_n$ on terminal nodes and decode them by assigning sets of reals on each node recursively from terminal nodes. (If a node has only one successor, then it means "negation" and if a node has more than one successors, then it means "disjunction".) The only difference between Borel codes and $\infty$-Borel codes is that trees are on $\omega$ for Borel codes while trees are on ordinals for $\infty$-Borel codes. From this visualization, it is easy to see that the statement "$\phi$ is an $\infty$-Borel code" is absolute between any transitive models of ZF by Lemma 1.12.1.

Given an $\infty$-Borel code $\phi$ and a real $x$, the problem whether $x$ is in $B_\phi$ can be easily translated into the following kind of satisfaction game using the above visualization of $\infty$-Borel codes via wellfounded trees: Let us regard $\phi$ as a wellfounded tree $T_\phi$ on ordinals with terminal nodes labeled by atomic sentences. In the game $G_c(T_\phi)$, there are two players, Spoiler and Duplicator, and a counter designating which player should move next. We start with the top node (the empty sequence) with the counter designating Duplicator. If the node has only one successor, no player is supposed to decide anything and they move to the unique successor and exchange the name in the counter. (This is for the negation.) If the node has more than one successors, then the player designated by the counter chooses one of the successors and keeps the name of the counter. (This is for the disjunction.) If the node is a terminal node, then look at the atomic sentence labeled at the node, say $\mathbf{a}_n$. If the real $x$ satisfies that $x(n) = 1$, then the player designated by the counter wins, otherwise the other player wins. It is fairly easy to see that a real $x$ is in $B_\phi$ if and only if Duplicator has a winning strategy in the game $G_c(T_\phi)$. By the fact that the payoff set of this game is a clopen subset of ${}^\omega\gamma$ for some ordinal $\gamma$, being a winning strategy in this game is absolute in any transitive model of ZF by Theorem 1.12.6. Hence the statement "a real $x$ is in $B_\phi$" is absolute between transitive models of ZF.

The following characterization of $\infty$-Borel sets is very useful:

**Fact 1.13.2** (Folklore)**.** Let $A$ be a set of reals. Then the following are equivalent:

1. $A$ is $\infty$-Borel, and
2. There is a formula $\phi$ in the language of set theory and a set $S$ of ordinals such that for each real $x$,

$$x \in A \iff \mathrm{L}[S, x] \vDash \phi(x).$$

*Proof.* See [80]. □

Standard examples of $\infty$-Borel sets are Suslin sets. A set of reals $A$ is *Suslin* if there are an ordinal $\gamma$ and a tree $T$ on $2 \times \gamma$ such that $A = \mathrm{p}[T]$, where $\mathrm{p}[T]$ is the projection of $[T]$ to the first coordinate, i.e.,

$$\mathrm{p}[T] = \{x \in {}^\omega 2 \mid (\exists f \in {}^\omega\gamma)\ (x, f) \in [T]\}.$$

By the above fact, every Suslin set is $\infty$-Borel. Assuming the Axiom of Choice, it is easy to see that every set of reals is Suslin, in particular $\infty$-Borel. Hence the property $\infty$-Borelness is trivial in the ZFC context while it is nontrivial and powerful in a determinacy world, as we will see in Chapter 3.

## 1.14 Blackwell games

In this section, we introduce *Blackwell games*, which are infinite games with imperfect information and compare them with Gale-Stewart games.

In 1928, John von Neumann proved his famous *minimax theorem* which is about finite games with imperfect information. Infinite versions of von Neumann's games were introduced by David Blackwell [15] where he proved the analogue of von Neumann's theorem for $\mathrm{G}_\delta$ sets of reals (i.e., $\mathbf{\Pi}^0_2$ sets of reals). The games he introduced are called *Blackwell games* and they were called by him "games with slightly imperfect information" in his paper [16].

We start with the definition of Blackwell games.[9] Let $X$ be a nonempty set and assume $\mathrm{AC}_\omega({}^\omega\mathbb{R})$. Recall from § 1.4 that the topology of ${}^\omega X$ is given by the product topology where each coordinate (i.e., $X$) is seen as the discrete space. In Blackwell games, players choose probabilities on $X$ instead of elements of $X$ and with those probabilities, one can deduce a Borel probability on ${}^\omega X$, i.e., a measure assigning probability to each Borel subset of ${}^\omega X$. Player I wins if the probability of a given payoff set is 1 and player II wins if the probability of the payoff set is 0. Let us formulate this in detail.

**Definition 1.14.1.** A *mixed strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}_\omega(X)$, where $\mathrm{Prob}_\omega(X)$ is the set of functions $\mu\colon X \to [0,1]$ with $\sum_{x\in X}\mu(x) = 1$.[10] A *mixed strategy for player II* is a function $\tau\colon X^{\mathrm{Odd}} \to \mathrm{Prob}_\omega(X)$.

Given mixed strategies $\sigma$, $\tau$ for player I and II respectively, let $\nu(\sigma,\tau)\colon {}^{<\omega}X \to \mathrm{Prob}_\omega(X)$ be as follows: For each finite sequence $s$ of elements of $X$,

$$\nu(\sigma,\tau)(s) = \begin{cases} \sigma(s) & \text{if } s \in X^{\mathrm{Even}}, \\ \tau(s) & \text{if } s \in X^{\mathrm{Odd}}. \end{cases}$$

For each finite sequence $s$ of elements of $X$, define

$$\mu_{\sigma,\tau}([s]) = \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s\restriction i)\ \big(s(i)\big).$$

Recall that $[s]$ denotes the set of $x \in {}^\omega X$ such that $x \supseteq s$ and these sets are basic open sets in the space ${}^\omega X$. With the help of $\mathrm{AC}_\omega({}^\omega X)$, we can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability on ${}^\omega X$, i.e., the probability whose domain is the set of all Borel sets in the space ${}^\omega X$. Let us also use $\mu_{\sigma,\tau}$ for denoting this Borel probability.

Let $A$ be a subset of ${}^\omega X$. A mixed strategy $\sigma$ for player I is *optimal in A* if for any mixed strategy $\tau$ for player II, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 1$. A mixed strategy $\tau$ for player II is *optimal in A* if for any mixed strategy $\sigma$ for player I, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 0$. A set $A$ is *Blackwell-determined* if one of the players has an optimal strategy in $A$. The axiom $\mathrm{Bl\text{-}AD}_X$ states that every subset of ${}^\omega X$ is Blackwell-determined. We write Bl-AD for $\mathrm{Bl\text{-}AD}_\omega$.

Note that since there is a bijection between $\mathbb{R}$ and ${}^\omega\mathbb{R}$, by Remark 1.2.1, $\mathrm{AC}_\omega(\mathbb{R})$ implies $\mathrm{AC}_\omega({}^\omega\mathbb{R})$ and hence one can formulate Blackwell games in ${}^\omega\mathbb{R}$ and $\mathrm{Bl\text{-}AD}_\mathbb{R}$ within $\mathrm{ZF}+\mathrm{AC}_\omega(\mathbb{R})$. The following is an analogy with Proposition 1.6.6:

[9] Our definitions of Blackwell games and Blackwell determinacy are different from the original ones given by Blackwell [16] where Blackwell determinacy is formulated as an extension of von Neumann's minimax theorem, but our formulation is equivalent to the original one when it is about the Cantor space (i.e., when $X = 2$). For the original formulation of Blackwell games and Blackwell determinacy, see, e.g., [56, § 3 & § 5].

[10] We use $\mathrm{Prob}_\omega(X)$ to denote such functions because they are the same as Borel probabilities $\mu$ on $X$ with countable support, i.e., there is a countable subset $A$ of $X$ with $\mu(A) = 1$.

**Proposition 1.14.2.**

1. Let $X, Y$ be nonempty sets and suppose that there is an injection from $X$ to $Y$ and assume $\mathrm{AC}_\omega({}^\omega Y)$. Then $\text{Bl-AD}_Y$ implies $\text{Bl-AD}_X$. In particular, $\text{Bl-AD}_{\mathbb{R}}$ implies Bl-AD.

2. The axioms Bl-AD and $\text{Bl-AD}_2$ are equivalent.

*Proof.* The first item is easy to see. For the second item, see [55, Corollary 4.4]. □

As for Gale-Stewart games, one could ask what kind of subsets of ${}^\omega X$ are Blackwell-determined for a nonempty set $X$. After proving that every $\mathrm{G}_\delta$ subset of the Cantor space is Blackwell-determined, Blackwell asked whether every Borel subset of the Cantor space is determined. It was Donald Martin who found a general connection between the determinacy of Gale-Stewart games and Blackwell determinacy.[11]

**Theorem 1.14.3** (Martin)**.** Let $X$ be a set and assume $\mathrm{AC}_\omega({}^\omega X)$. If there is a winning strategy for player I (resp., II) in a subset $A$ of ${}^\omega X$, then there is an optimal strategy for player I (resp., II) in $A$. In particular, AD implies that Bl-AD and $\mathrm{AD}_{\mathbb{R}}$ implies that $\text{Bl-AD}_{\mathbb{R}}$.

*Proof.* Given a strategy $\sigma$ for player I (resp., II), one can naturally translate $\sigma$ into a mixed strategy $\hat{\sigma}$ for player I (resp., II) by setting $\hat{\sigma}(s)$ to be the Dirac measure concentrating on $\sigma(s)$. It is easy to see that if $\sigma$ is winning in $A$, then $\hat{\sigma}$ is optimal in $A$. □

By Theorem 1.6.4, every Borel subset of the Cantor space is Blackwell-determined in ZFC and this answers the question of Blackwell. After proving Theorem 1.14.3, Martin conjectured the following:

**Conjecture 1.14.4** (Martin)**.** Bl-AD implies AD.

This conjecture is still not known to be true. The best known result toward AD from Bl-AD is as follows: Recall the notion of Suslinness from § 1.13. A set of reals is *co-Suslin* if its complement is Suslin.

**Theorem 1.14.5** (Martin, Neeman, and Vervoort)**.** Assume Bl-AD. Then every Suslin and co-Suslin set of reals is determined.

*Proof.* See [59, Lemma 4.1].[12] □

[11] In [58], Martin proved the Blackwell determinacy in the original formulation as mentioned in Footnote 9, not in our formulation.

[12] In [59, Lemma 4.1], they assume the Blackwell determinacy for sets of reals in a weakly scaled pointclass. But the argument shows the statement in Theorem 1.14.5.

Together with the following result, one can establish the equiconsistency between AD and Bl-AD:

**Theorem 1.14.6** (Kechris and Woodin)**.** Assume that every Suslin and co-Suslin set of reals is determined. Then $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ holds.

*Proof.* See [46]. □

**Corollary 1.14.7** (Martin, Neeman, and Vervoort)**.** In $\mathrm{L}(\mathbb{R})$, AD and Bl-AD are equivalent. In particular, AD and Bl-AD are equiconsistent.

Also, Bl-AD has some consequence on regularity properties:

**Theorem 1.14.8** (Vervoort)**.** Assume Bl-AD. Then every set of reals is Lebesgue measurable.

*Proof.* See [86]. □

We discuss the connection between Blackwell determinacy and other regularity properties such as the Baire property in § 3.2.

It is not difficult to see that if finite games are Blackwell determined, then they are determined. As a corollary, one can obtain the following:

**Theorem 1.14.9** (Löwe)**.** Assume $\text{Bl-AD}_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a function.

*Proof.* See [56, Theorem 9.3]. □

Since there is a relation on the reals which cannot be uniformized by a function in $\mathrm{L}(\mathbb{R})$, $\text{Bl-AD}_{\mathbb{R}}$ does not hold in $\mathrm{L}(\mathbb{R})$. Since $\text{Bl-AD}_{\mathbb{R}}$ implies Bl-AD by the first item of Remark 1.14.2 and Bl-AD implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ by Corollary 1.14.7, AD does not imply $\text{Bl-AD}_{\mathbb{R}}$.

In Chapter 3, we discuss the connection between $\mathrm{AD}_{\mathbb{R}}$ and $\text{Bl-AD}_{\mathbb{R}}$.

## 1.15 Wadge reducibility and Wadge games

When we study descriptive set theory, we often would like to compare given two sets of reals via some measure of complexity, i.e., we would like to ask the question "Which set of reals is more complex than the other?". In 1972, Wadge [88] introduced *Wadge reducibility* for sets of reals in the Baire space, which is an analogue of many-one reducibility in recursion theory: A set of reals $A$ is *Wadge reducible* to a set of reals $B$ if there is a continuous function $f$ from the Baire space to itself such that $A = f^{-1}(B)$. After its introduction, set theorists in California developed a beautiful theory of Wadge reducibility under the Axiom of Determinacy (AD) plus the principle of Dependent Choice (DC). Nowadays this theory is one of the basic tools in the research of determinacy and is essential to the study of descriptive set theory. The key tool of the analysis of Wadge reducibility is a type of infinite games called *Wadge games*, which characterize continuous functions from the Baire space to itself.

For a subset $A$ of a topological space $X$, $A^{\mathrm{c}}$ denotes the complement of $A$ and $\overline{A}$ denotes the closure of $A$ in $X$.

We start with the definition of Wadge reducibility for a general topological space. Let $X$ be a topological space and $A, B$ be subsets of $X$. The set $A$ is *Wadge reducible to* $B$ (write $A \leq^X_{\mathrm{W}} B$) if there is a continuous function $f\colon X \to X$ such that $A = f^{-1}(B)$. Hence the problem of the membership of $A$ can be reduced to that of the membership of $B$ via a continuous function, and in this sense $B$ is more complicated than (or as complicated as) $A$. This notion reminds us of the many-one reducibility for subsets of $\omega$ in recursion theory given by replacing continuous functions with recursive functions. We define three other notions of Wadge reducibility. A subset $A$ of $X$ is *Wadge equivalent to* a subset $B$ of $X$ ($A \equiv^X_{\mathrm{W}} B$) if $A \leq^X_{\mathrm{W}} B$ and $B \leq^X_{\mathrm{W}} A$. A subset $A$ of $X$ is *strictly Wadge reducible to* a subset $B$ of $X$ ($A <^X_{\mathrm{W}} B$) if $A \leq^X_{\mathrm{W}} B$ and $B \not\leq^X_{\mathrm{W}} A$. A subset $A$ of $X$ is *Wadge comparable to* a subset $B$ of $X$ if $A \leq^X_{\mathrm{W}} B$ or $B \leq^X_{\mathrm{W}} A$ holds. It is easy to see that the Wadge order $\leq^X_{\mathrm{W}}$ is a preorder (i.e., reflexive and transitive) and that the Wadge equivalence $\equiv^X_{\mathrm{W}}$ is an equivalence relation on subsets of $X$. An equivalence class of this equivalence relation is called a *Wadge degree*.

When $X$ is the Baire space, the study of Wadge degrees is interesting to descriptive set theorists in the way that Turing degrees are interesting to recursion theorists. Since each boldface pointclass is closed under continuous preimages, it consists of an initial segment of all the subsets of reals via Wadge reducibility and hence the study of Wadge degrees gives us a finer analysis of boldface pointclasses such as Borel classes $\mathbf{\Sigma}^0_\xi$ and projective classes $\mathbf{\Sigma}^1_n$. Wadge introduced Wadge games to analyze Wadge reducibility for the Baire space. Given two set of reals $A, B$ in the Baire space, the *Wadge game* $G_{\mathrm{W}}(A, B)$ is played by two players I and II in the following way: I plays a natural number $x_0$, then II plays a natural number $y_0$ or she can pass, then I plays again a natural number $x_1$ and II plays a natural number or she can pass. After $\omega$ rounds of this process, they will produce sequences $x = \langle x_n \mid n \in \omega\rangle$ and $y = \langle y_n \mid n < i\rangle$ where $i \leq \omega$. Player II wins if $i = \omega$ (i.e., player II plays natural numbers infinitely often) and $x \in A \iff y \in B$. Otherwise player I wins. It is easy to see that $A \leq^{\omega^\omega}_{\mathrm{W}} B$ if and only if player II has a winning strategy in the Wadge game $G_{\mathrm{W}}(A, B)$. Since Wadge games can be easily simulated by Gale-Stewart games, under AD, we can conclude the following:

**Theorem 1.15.1** (Wadge's Lemma)**.** Assume AD and let $A, B$ be two sets of reals in the Baire space. Then either $A \leq^{\omega^\omega}_{\mathrm{W}} B$ or $B \leq^{\omega^\omega}_{\mathrm{W}} A^{\mathrm{c}}$ holds.

*Proof.* Suppose $A \not\leq^{\omega^\omega}_{\mathrm{W}} B$. Then by the above observation, player I has a winning strategy in the game $G_{\mathrm{W}}(A, B)$. But using this strategy, player II can win the game $G_{\mathrm{W}}(B, A^{\mathrm{c}})$ because the negation of $x \in A \iff y \in B$ is the same as

$y \in B \iff x \in A^{\mathrm{c}}$. Hence player II has a winning strategy in the game $G_{\mathrm{W}}(B, A^{\mathrm{c}})$ and $B \leq_{\mathrm{W}}^{\omega^\omega} A^{\mathrm{c}}$. □

By the above theorem, we can deduce that the Wadge order $\leq_{\mathrm{W}}^{\omega^\omega}$ is almost linear in the following sense: Let $X$ be a topological space and $A$ be a subset of $X$. We say $A$ is *selfdual* if $A \leq_{\mathrm{W}}^{X} A^{\mathrm{c}}$ (equivalently $A^{\mathrm{c}} \leq_{\mathrm{W}}^{X} A$) and *non-selfdual* if $A \not\leq_{\mathrm{W}}^{X} A^{\mathrm{c}}$ (equivalently $A^{\mathrm{c}} \not\leq_{\mathrm{W}}^{X} A$). Let $A$ be a selfdual set of reals and $B$ be a set of reals in the Baire space. Then either i) $B <_{\mathrm{W}}^{\omega^\omega} A$, ii) $B \equiv_{\mathrm{W}}^{\omega^\omega} A$, or iii) $A <_{\mathrm{W}}^{\omega^\omega} B$ holds. Let $A$ be a non-selfdual set of reals and $B$ be a set of reals in the Baire space. Then either i) $B <_{\mathrm{W}}^{\omega^\omega} A$ and $B <_{\mathrm{W}}^{\omega^\omega} A^{\mathrm{c}}$, ii) $B \equiv_{\mathrm{W}}^{\omega^\omega} A$, iii) $B \equiv_{\mathrm{W}}^{\omega^\omega} A^{\mathrm{c}}$, or iv) $A <_{\mathrm{W}}^{\omega^\omega} B$ and $A^{\mathrm{c}} <_{\mathrm{W}}^{\omega^\omega} B$ holds.

Donald Martin and Leonard Monk proved that the Wadge order $\leq_{\mathrm{W}}^{\omega^\omega}$ is wellfounded. Hence we can measure the complexity of sets of reals via ordinals by taking their rank in the Wadge order.

**Theorem 1.15.2** (Martin and Monk)**.** Assume AD+$\mathrm{DC}_{\mathbb{R}}$. Then the Wadge order $\leq_{\mathrm{W}}^{\omega^\omega}$ is wellfounded.

*Proof.* See, e.g., [83, Theorem 2.2]. □

The above two theorems are essential parts of the basic theory of the Wadge order for the Baire space. In Chapter 5, we show that both theorems fail for the Wadge order for the real line.

# Chapter 2

# Games and Regularity Properties

In this chapter, we focus on the connection between infinite games and *regularity properties* for sets of reals. Roughly speaking, a set of reals with a regularity property can be approximated by some simple sets (e.g., open sets or Borel sets) modulo some small sets.

We characterize almost all the known regularity properties for sets of reals via the Baire property for some topological spaces and use Banach-Mazur games to prove the general equivalence theorems between the regularity properties, forcing absoluteness, and the transcendence properties over some canonical inner models. With the help of these equivalence results, we answer some open questions from set theory of the reals.

In this chapter, we work in ZFC. We assume that readers are familiar with the elementary theories of forcing and descriptive set theory. (For basic definitions not given in this paper, see [37, 66].) When we are talking about "reals", we mean elements of the Baire space ${}^{\omega}\omega$ or of the Cantor space ${}^{\omega}2$. In this chapter, we use $\mathbb{R}$ for Mathias forcing and we will not use it for the real line or the set of all reals.

## 2.1 $\mathbb{P}$-Baireness and $\mathbb{P}$-measurability

In this section, we introduce two kinds of regularity properties for sets of reals for a wide class of forcing notions $\mathbb{P}$ and compare them. The first one is called $\mathbb{P}$-*Baireness*, which was implicitly mentioned in the paper by Feng, Magidor, and Woodin [25]. The idea of $\mathbb{P}$-Baireness is to reduce properties for sets of reals to the Baire property in the Stone space of $\mathbb{P}$ by taking the continuous preimages of sets of reals in the Stone space of $\mathbb{P}$. Sets of reals with the $\mathbb{P}$-Baireness behave nicely in forcing extensions by $\mathbb{P}$ because continuous functions from the Stone space of $\mathbb{P}$ to the reals correspond to $\mathbb{P}$-names for reals. The second one is called $\mathbb{P}$-*measurability*, which is a generalization of almost all the known regularity properties for sets of reals. Since almost all the known regularity properties come from tree-type forcings, we first introduce a wide class of tree-type forcings called *strongly arboreal forcings*. As is mentioned in the introduction of this chapter, a set of reals with a regularity property can be approximated by some simple sets modulo small sets. To each strongly arboreal forcing $\mathbb{P}$, we will associate a $\sigma$-ideal $I_{\mathbb{P}}$ which will be the set of small sets in this context and give the definition of $\mathbb{P}$-measurability. After introducing these two regularity properties, we will investigate the connection between them.

From now on, we work with only *separative* partial orders: A partial order $\mathbb{P}$ is *separative* if for any two elements $p, q$ of $\mathbb{P}$, if $p \nleq q$, then there is an $r \leq p$ with $r \perp q$. Every Boolean algebra is separative. The advantage of working with separative partial orders is that one can identify $\mathbb{P}$ and its image via $i_{\mathbb{P}}$ in $B_{\mathbb{P}}$ where $(B_{\mathbb{P}}, i_{\mathbb{P}})$ is a completion of $\mathbb{P}$, namely the embedding $i$ is isomorphic between $\mathbb{P}$ and its image. From now on, we always identify $\mathbb{P}$ and its image inside a completion of $\mathbb{P}$.

We start with $\mathbb{P}$-Baireness. We recall the definition of Stone spaces from § 1.4. For a partial order $\mathbb{P}$, the *Stone space* of $\mathbb{P}$ (denoted by $\mathrm{St}(\mathbb{P})$) is the set of all ultrafilters on $B_{\mathbb{P}}$ equipped with the topology generated by $\{O_b \mid b \in B_{\mathbb{P}}\}$, where $B_{\mathbb{P}}$ is a completion of $\mathbb{P}$ and $O_b = \{u \in \mathrm{St}(\mathbb{P}) \mid u \ni b\}$. For example, if $\mathbb{P}$ is Cohen forcing $\mathbb{C}$, then $\mathrm{St}(\mathbb{C})$ is homeomorphic to the Cantor space ${}^{\omega}2$. Dense sets in $\mathbb{P}$ are the same as open dense subsets in $\mathrm{St}(\mathbb{P})$: If $D$ is a dense subset of $\mathbb{P}$, then the set $\bigcup\{O_p \mid p \in D\}$ is open dense in $\mathrm{St}(\mathbb{P})$, where $i$ is a unique dense embedding from $\mathbb{P}$ to $B_{\mathbb{P}}$. Conversely, if $U$ is an open dense subset of $\mathrm{St}(\mathbb{P})$, then $\{p \in \mathbb{P} \mid O_p \subseteq U\}$ is a dense open subset of $\mathbb{P}$.

Next, we discuss meagerness and the Baire property in $\mathrm{St}(\mathbb{P})$. We should first observe that this space meets our requirement:

**Lemma 2.1.1.** Let $\mathbb{P}$ be a separative partial order. Then $\mathrm{St}(\mathbb{P})$ is a Baire space, i.e., any nonempty open set in $\mathrm{St}(\mathbb{P})$ is not meager.

*Proof.* We show that $O_b$ is not meager for each $b$ in $B_{\mathbb{P}}$. Since $\mathbb{P}$ is dense in $B_{\mathbb{P}}$, it suffices to show that $O_p$ is not meager for each $p$ in $\mathbb{P}$. Since any nowhere dense set is a subset of a closed nowhere dense set (the closure of a nowhere dense set is again nowhere dense by definition) and the complement of a closed nowhere dense set is an open dense set, it suffices to show that $O_p$ intersects with the countable intersection of any open dense sets in $\mathrm{St}(\mathbb{P})$ for each $p \in \mathbb{P}$.

Take any $p \in \mathbb{P}$ and let $\{U_n \mid n \in \omega\}$ be a countable set of open dense subsets of $\mathrm{St}(\mathbb{P})$. We would like to prove that the intersection $O_p$ with $\bigcap_{n\in\omega} U_n$ is nonempty. We construct a descending sequence $\langle p_n \in \mathbb{P} \mid n \in \omega\rangle$ such that $p_0 \leq p$ and $O_{p_n} \subseteq U_n$ for each $n \in \omega$. This is possible because each $U_n$ is open dense in $\mathrm{St}(\mathbb{P})$. Then consider any ultrafilter $u$ extending $\{p_n \mid n \in \omega\}$ (we use Zorn's Lemma here). Then $u$ belongs to $O_p$ and $U_n$ for each $n \in \omega$. Hence the intersection $O_p$ with $\bigcap_{n\in\omega} U_n$ is nonempty. □

Before defining $\mathbb{P}$-Baireness, let us see the connection between Baire measurable functions from $\mathrm{St}(\mathbb{P})$ to the reals and $\mathbb{P}$-names for reals. Let $X, Y$ be topological spaces. Then a function $f\colon X \to Y$ is *Baire measurable* if for any open set $U$ in $Y$, $f^{-1}(U)$ has the Baire property in $X$. Baire measurable functions are the same as continuous functions modulo meager sets: Let $X, Y$ be topological spaces and assume $Y$ is second countable, i.e., there is a countable base for the topology of $Y$. Then it is fairly easy to see that a function $f\colon X \to Y$ is Baire measurable if and only if there is a comeager set $D$ in $X$ such that $f \restriction D$ is continuous.

There is a natural correspondence between Baire measurable functions from $\mathrm{St}(\mathbb{P})$ to the reals and $\mathbb{P}$-names for reals:

**Lemma 2.1.2** (Feng, Magidor, and Woodin)**.** Let $\mathbb{P}$ be a separative partial order.

1. If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then

$$\tau_f = \bigl\{ ((m,n)\check{}, p) \mid O_p \setminus \{u \in \mathrm{St}(\mathbb{P}) \mid f(u)(m) = n\} \text{ is meager} \bigr\}$$

is a $\mathbb{P}$-name for a real.

2. Let $\tau$ be a $\mathbb{P}$-name for a real. Define $f_\tau$ as follows: For $u \in \mathrm{St}(\mathbb{P})$ and $m, n \in \omega$,

$$f_\tau(u)(m) = n \iff (\exists p \in u)\ p \Vdash \tau(\check{m}) = \check{n}.$$

Then the domain of $f_\tau$ is comeager in $\mathrm{St}(\mathbb{P})$ and $f_\tau$ is continuous on the domain. Hence it can be uniquely extended to a Baire measurable function from $\mathrm{St}(\mathbb{P})$ to the reals modulo meager sets.

3. If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then $f_{\tau_f}$ and $f$ agree on a comeager set in $\mathrm{St}(\mathbb{P})$. Also, if $\tau$ is a $\mathbb{P}$-name for a real, then $\Vdash \tau_{f_\tau} = \tau$.

*Proof.* The result is due to Feng, Magidor, and Woodin [25, Theorem 3.2]. For the sake of completeness, we will give a proof.

Let us first fix some notation. When $f$ is a function from $\mathrm{St}(\mathbb{P})$ to ${}^{\omega}\omega$ and $m, n$ are natural numbers, we write $A^f_{m,n} = \{u \in \mathrm{St}(\mathbb{P}) \mid f(u)(m) = n\}$.

Let us start with proving the first item. We show that $\tau_f$ is a $\mathbb{P}$-name for a real assuming $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is Baire measurable. Take any $\mathbb{P}$-generic filter $G$ over $V$. We prove that $\tau_f^G$ is a function from $\omega$ to $\omega$. By the definition of $\tau_f$, it is easy to show that $\tau_f^G$ is a subset of $\omega \times \omega$.

We first claim that it is a function. Suppose $(m, n_1), (m, n_2) \in \tau_f^G$ for natural numbers $m, n_1$, and $n_2$. We show that $n_1 = n_2$. By the assumption, there are $p_i \in G$ $(i = 1, 2)$ such that $((m, n_i)\check{}, p_i) \in \tau_f$ for $i = 1, 2$ . By the definition of $\tau_f$, $O_{p_i} \setminus A^f_{m,n_i}$ is meager in $\mathrm{St}(\mathbb{P})$ for $i = 1, 2$. Since $p_1, p_2 \in G$ and $G$ is a filter, there is a $p$ such that $p \le p_1, p_2$. Hence $O_p \setminus A^f_{m,n_i}$ is meager in $\mathrm{St}(\mathbb{P})$ for $i = 1, 2$. By Lemma 2.1.1, $O_p$ is not meager in $\mathrm{St}(\mathbb{P})$. Hence $O_p \cap A^f_{m,n_1} \cap A^f_{m,n_2}$ is not meager and especially non-empty. Take any element $u$ from $O_p \cap A^f_{m,n_1} \cap A^f_{m,n_2}$. By the definition of $A^f_{m,n_i}$ for $i = 1, 2$, $n_1 = f(u)(m) = n_2$, as desired.

We prove that $m \in \mathrm{dom}(\tau_f^G)$ for every natural number $m$. Fix an $m$. Since $f$ is Baire measurable, the set $D = \{p \in \mathbb{P} \mid (\exists n \in \omega)\ O_p \setminus A^f_{m,n} \text{ is meager}\}$ is dense.

By the genericity of $G$, there is a $p$ both in $G$ and $D$. Then $O_p \setminus A^f_{m,n}$ is meager for some $n$ and hence $((m,n)\check{}, p) \in \tau_f$ which means that $\tau_f^G(m) = n$, as desired.

We show the second item. Let $\tau$ be a $\mathbb{P}$-name for a real. We first show that the domain of $f_\tau$ is comeager in $\mathrm{St}(\mathbb{P})$. If we set $D_m = \{p \in \mathbb{P} \mid (\exists n)\ p \Vdash \tau(\check{m}) = \check{n}\}$ and $U_m = \bigcup\{O_p \mid p \in D_m\}$ for each $m \in \omega$, $\mathrm{dom}(f_\tau) = \bigcap_{m\in\omega} U_m$. Since $\tau$ is a $\mathbb{P}$-name for a real, $D_m$ is dense and hence $U_m$ is open dense in $\mathrm{St}(\mathbb{P})$ for each $m$. So $\mathrm{dom}(f_\tau)$ is comeager in $\mathrm{St}(\mathbb{P})$.

We next show that $f_\tau$ is a function. Let $u \in \mathrm{dom}(f_\tau)$ and assume $f_\tau(u)(m) = n_1$ and $f_\tau(u)(m) = n_2$ for natural numbers $m, n_1$, and $n_2$. We show that $n_1 = n_2$. By the definition of $f_\tau$, there are $p_i \in u$ such that $p_i \Vdash \tau(\check{m}) = \check{n}_i$ for $i = 1, 2$. Since $u$ is a filter, there is a $p$ such that $p \leq p_i$ for each $i = 1, 2$, which yields $p \Vdash \check{n}_1 = \tau(\check{m}) = \check{n}_2$. Hence $n_1 = n_2$.

We finally show that $f_\tau$ is Baire measurable. We prove that $A^{f_\tau}_{m,n}$ has the Baire property in $\mathrm{St}(\mathbb{P})$ for all natural numbers $m$ and $n$. Let $U = \bigcup\{O_p \mid p \Vdash \tau(\check{m}) = \check{n}\}$. We show that $U \cap \mathrm{dom}(f_\tau) = A^{f_\tau}_{m,n} \cap \mathrm{dom}(f_\tau)$. If $u$ is in $U \cap \mathrm{dom}(f_\tau)$, then there is a $p \in u$ such that $p \Vdash \tau(\check{m}) = \check{n}$. By the definition of $f_\tau$, $f_\tau(u)(m) = n$ and hence $u \in A^{f_\tau}_{m,n} \cap \mathrm{dom}(f_\tau)$. Conversely, if $u$ is in $A^{f_\tau}_{m,n} \cap \mathrm{dom}(f_\tau)$, then $f_\tau(u)(m) = n$ and there is a $p \in u$ such that $p \Vdash \tau(\check{m}) = \check{n}$. Hence $u \in U \cap \mathrm{dom}(f_\tau)$.

We prove the third item. We first show that $f_{\tau_f}$ and $f$ agree on a comeager set if $f$ is Baire measurable. First note that if $O_p \setminus A^f_{m,n}$ is meager, then $f_{\tau_f}$ and $f$ agree on $O_p \cap A^f_{m,n}$. For let $u$ be in $O_p \cap A^f_{m,n}$. Since $O_p \setminus A^f_{m,n}$ is meager, $((m,n)\check{}, p) \in \tau_f$, in particular, $p \Vdash \tau_f(\check{m}) = \check{n}$. By the definition of $f_{\tau_f}$, $f_{\tau_f}(u)(m) = n$, as desired. Since $f$ is Baire measurable, the set $D = \{p \in \mathbb{P} \mid (\exists n \in \omega)\ O_p \setminus A^f_{m,n} \text{ is meager}\}$ is dense and hence the set $A = \bigcup\bigcup_{n\in\omega}\{O_p \cap A^f_{m,n} \mid O_p \setminus A^f_{m,n} \text{ is meager}\}$ is comeager. But $f_{\tau_f}$ and $f$ agree on $A$, as desired.

We next show that $\tau^G_{f_\tau} = \tau^G$ for each $\mathbb{P}$-name $\tau$ for a real and a $\mathbb{P}$-generic filter $G$ over $V$. Suppose $\tau^G_{f_\tau}(m) = n$. We show that $\tau^G(m) = n$. Since $\tau^G_{f_\tau}(m) = n$, there is a $p \in G$ such that $((m,n)\check{}, p) \in \tau_{f_\tau}$. By the definition $\tau_{f_\tau}$, $O_p \setminus A^{f_\tau}_{m,n}$ is meager. Then by the definition of $f_\tau$, the set $\{u \in \mathrm{St}(\mathbb{P}) \mid (\exists p' \in u)\ p' \Vdash \tau(\check{m}) = \check{n}\}$ is comeager in $O_p$, which means that the set $\{p' \leq p \mid p' \Vdash \tau(\check{m}) = \check{n}\}$ is dense below $p$. Since $p \in G$, by the genericity of $G$, there is a $p' \in G$ such that $p' \Vdash \tau(\check{m}) = \check{n}$. Hence $\tau^G(m) = n$, as desired. $\square$

Now we define the property $\mathbb{P}$-Baireness. Let $\mathbb{P}$ be a separative partial order and $A$ be a set of reals. Then $A$ is *$\mathbb{P}$-Baire* if for any Baire measurable function $f \colon \mathrm{St}(\mathbb{P}) \to {}^\omega\omega$, $f^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$. It is easy to see that every Borel set of reals is $\mathbb{P}$-Baire for any $\mathbb{P}$ by the same argument as for the Baire property we gave in the paragraphs after Definition 1.8.1.

Next we introduce $\mathbb{P}$-measurability. We start with defining a class of tree-type forcings we will work on from now on. A partial order $\mathbb{P}$ is *arboreal* if its conditions are perfect trees on $\omega$ (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \{{}^{<\omega}\omega\}$. We need the following stronger notion:

**Definition 2.1.3.** A partial order $\mathbb{P}$ is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P})\ (\forall t \in T)\ T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

Note that every strongly arboreal forcing is separative (if $S \nleq T$, then there is an $s \in S \setminus T$ and hence $S_s \leq S$ and $S_s \perp T$).

With strongly arboreal forcings, we can code generic objects by reals in the standard way: Let $\mathbb{P}$ be strongly arboreal and $G$ be $\mathbb{P}$-generic over $V$. Let $x_G = \bigcup\{\mathrm{stem}(T) \mid T \in G\}$. Then $x_G$ is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through $T$. Hence $V[x_G] = V[G]$. We call such real $x_G$ a $\mathbb{P}$*-generic real over* $V$.

Almost all typical forcings related to regularity properties are strongly arboreal:

**Example 2.1.4.**

1. Cohen forcing $\mathbb{C}$: Let $T_0$ be ${}^{<\omega}\omega$. Consider the partial order $\big(\{(T_0)_s \mid s \in {}^{<\omega}\omega\}, \subseteq\big)$. Then this is strongly arboreal and equivalent to Cohen forcing.

2. Random forcing $\mathbb{B}$: Consider the set of all perfect trees $T$ on 2 such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.

3. Hechler forcing $\mathbb{D}$: For $(n, f) \in \mathbb{D}$, let

$$T_{(n,f)} = \Big\{t \in {}^{<\omega}\omega \mid \text{either } t \subseteq f{\restriction}n \text{ or } \Big(t \supseteq f{\restriction}n \text{ and } \big(\forall m \in \mathrm{dom}(t)\big)\ t(m) \geq f(m)\Big)\Big\}.$$

Then the partial order $(\{T_{(n,f)} \mid (n, f) \in \mathbb{D}\}, \subseteq)$ is strongly arboreal and equivalent to Hechler forcing.

4. Mathias forcing $\mathbb{R}$: For a condition $(s, A)$ in $\mathbb{R}$, let

$$T_{(s,A)} = \{t \in {}^{<\omega}\omega \mid t \text{ is strictly increasing and } s \subseteq \mathrm{ran}(t) \subseteq s \cup A\}.$$

Then $\{T_{(s,A)} \mid (s, A) \in \mathbb{R}\}$ is a strongly arboreal forcing equivalent to Mathias forcing.

5. Eventually different forcing $\mathbb{E}$: For a condition $(s, F)$ in $\mathbb{E}$, let

$$T_{(s,F)} = \{t \in {}^{<\omega}\omega \mid \text{either } t \subseteq s \text{ or } \Big(t \supseteq s \text{ and } (\forall f \in F)\ \big(\forall n \in \mathrm{dom}(t \setminus s)\big)\ t(n) \neq f(n)\Big)\}.$$

Then $\{T_{(s,F)} \mid (s, F) \in \mathbb{E}\}$ is a strongly arboreal forcing equivalent to eventually different forcing.

6. Sacks forcing $\mathbb{S}$, Silver forcing $\mathbb{V}$, Miller forcing $\mathbb{M}$, Laver forcing $\mathbb{L}$: These forcings can be naturally seen as strongly arboreal forcings.

We now introduce a $\sigma$-ideal $I_{\mathbb{P}}$ on the reals expressing "smallness" for each strongly arboreal forcing $\mathbb{P}$.

**Definition 2.1.5.** Let $\mathbb{P}$ be a strongly arboreal forcing. A set of reals $A$ is $\mathbb{P}$-*null* if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. Let $N_{\mathbb{P}}$ denote the set of all $\mathbb{P}$-null sets and $I_{\mathbb{P}}$ denote the $\sigma$-ideal generated by $\mathbb{P}$-null sets, i.e., the set of all countable unions of $\mathbb{P}$-null sets.

**Example 2.1.6.**

1. Cohen forcing $\mathbb{C}$: $\mathbb{C}$-null sets are the same as nowhere dense sets in the Baire space ${}^{\omega}\omega$ and $I_{\mathbb{C}}$ is the ideal of meager sets in the Baire space.
2. Random forcing $\mathbb{B}$: $\mathbb{B}$-null sets are the same as Lebesgue null sets in the Baire space and $I_{\mathbb{B}}$ is the Lebesgue null ideal.
3. Hechler forcing $\mathbb{D}$: $\mathbb{D}$-null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{[s, f] \mid (s, f) \in \mathbb{D}\}$ where

$$[s, f] = \{x \in {}^{\omega}\omega \mid s \subseteq x \text{ and } (\forall n \geq \mathrm{dom}(s))\ x(n) \geq f(n)\}.$$

Hence $I_{\mathbb{D}}$ is the meager ideal in the dominating topology.

4. Eventually different forcing $\mathbb{E}$: $\mathbb{E}$-null sets are the same as nowhere dense sets in the eventually different topology $\mathcal{E}$, i.e., the topology generated by $\{[s, F] \mid (s, F) \in \mathbb{E}\}$ where

$$[s, F] = \{x \in {}^{\omega}\omega \mid s \subseteq x \text{ and } (\forall f \in F)\ (\forall n \geq \mathrm{dom}(s))\ x(n) \neq f(n)\}.$$

Hence $I_{\mathbb{E}}$ is the meager ideal in the topology $\mathcal{E}$.

5. Mathias forcing $\mathbb{R}$: A set of reals $A$ is $\mathbb{R}$-null if and only if $\{\mathrm{ran}(x) \mid x \in A \cap A_0\}$ is Ramsey null or meager in the Ellentuck topology, where $A_0$ is the set of strictly increasing infinite sequences of natural numbers. Hence $I_{\mathbb{R}} = N_{\mathbb{R}}$.
6. Sacks forcing $\mathbb{S}$: In this case, $I_{\mathbb{S}} = N_{\mathbb{S}}$ by a standard fusion argument. The ideal $I_{\mathbb{S}}$ is called the Marczewski ideal and often denoted by $s_0$.

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_{\mathbb{P}} = N_{\mathbb{P}}$. In the case of ccc forcings, $I_{\mathbb{P}}$ is often different from $N_{\mathbb{P}}$ (e.g., Cohen forcing and Hechler forcing).

We now introduce $\mathbb{P}$-measurability:

**Definition 2.1.7.** Let $\mathbb{P}$ be strongly arboreal. A set of reals $A$ is $\mathbb{P}$-*measurable* if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

As is expected, $\mathbb{P}$-measurability coincides with a known regularity property for $\mathbb{P}$ when $\mathbb{P}$ is ccc:

**Proposition 2.1.8.** Let $\mathbb{P}$ be a strongly arboreal, ccc forcing and let $A$ be a set of reals. Then $A$ is $\mathbb{P}$-measurable if and only if there is a Borel set $B$ such that $A \triangle B \in I_{\mathbb{P}}$, where $A \triangle B$ is the symmetric difference between $A$ and $B$.

*Proof.* The direction from right to left follows from the fact that every Borel set of reals is $\mathbb{P}$-measurable which will be proved in Lemma 2.1.15.

For the other direction, suppose $A$ is $\mathbb{P}$-measurable and we will find a Borel set approximating $A$ modulo $I_{\mathbb{P}}$. Since $A$ is $\mathbb{P}$-measurable, the set $D = \{T \in \mathbb{P} \mid$ either $[T] \cap A \in I_{\mathbb{P}}$ or $[T] \setminus A \in I_{\mathbb{P}}\}$ is dense. We take a maximal antichain $\mathcal{A}$ in $D$ and define $B = \bigcup\{[T] \mid T \in \mathcal{A}$ and $[T] \setminus A \in I_{\mathbb{P}}\}$. Then since $\mathcal{A}$ is countable by the ccc-ness of $\mathbb{P}$, $B$ is Borel and $A \triangle B \in I_{\mathbb{P}}$ because $D$ is dense. □

This argument does not work for non-ccc forcings such as Sacks forcing. For example, assuming every $\mathbf{\Pi}^1_1$ set has the perfect set property (i.e., either the set is countable or contains a perfect subset), there is no $\mathbf{\Sigma}^1_1$ Bernstein set (i.e., a set where neither it nor its complement contains a perfect subset) but for a $\mathbf{\Sigma}^1_1$ set of reals $A$, $A$ is approximated by a Borel set modulo $I_{\mathbb{S}}$ if and only if $A$ is Borel. This is because $I_{\mathbb{S}}$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals by the assumption that every $\mathbf{\Pi}^1_1$ set has the perfect set property.

But $\mathbb{P}$-measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals $A$ is $\mathbb{R}$-measurable if and only if $\{\mathrm{ran}(x) \mid x \in A \cap A_0\}$ is completely Ramsey (or has the Baire property in the Ellentuck topology), where $A_0$ is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

**Proposition 2.1.9** (Brendle, Löwe)**.** Let $\mathbf{\Gamma}$ be a topologically reasonable pointclass on the Cantor space ${}^{\omega}2$, i.e., it is a set of subsets of the Cantor space closed under continuous preimages on the Cantor space and any intersection between a set in $\mathbf{\Gamma}$ and a closed set in the Cantor space. Then every set in $\mathbf{\Gamma}$ is $\mathbb{S}$-measurable if and only if there is no Bernstein set in $\mathbf{\Gamma}$.[1]

*Proof.* See [20, Lemma 2.1]. □

We now introduce a (possibly finer) ideal ${I_{\mathbb{P}}}^*$ which will be central to our theorems:

**Definition 2.1.10.** Let $\mathbb{P}$ be a strongly arboreal forcing. A set of reals $A$ is in ${I_{\mathbb{P}}}^*$ if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A$ is in $I_{\mathbb{P}}$.

**Question 2.1.11.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Can we prove $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$?

We give some easy observations concerning Question 2.1.11:

[1] In general, the property not being a Bernstein set does not imply $\mathbb{S}$-measurability while the converse is true. By using the axiom of choice, we can construct a set of reals which is not $\mathbb{S}$-measurable but is not a Bernstein set.

**Lemma 2.1.12.** Let $\mathbb{P}$ be a strongly arboreal forcing.

1. The ideal $I_{\mathbb{P}}$ is a subset of ${I_{\mathbb{P}}}^*$.

2. A set of reals $A$ is $\mathbb{P}$-measurable if and only if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in {I_{\mathbb{P}}}^*$ or $[T'] \setminus A \in {I_{\mathbb{P}}}^*$ holds. Hence we get the same notion of measurability even if we replace $I_{\mathbb{P}}$ by ${I_{\mathbb{P}}}^*$ in the definition of $\mathbb{P}$-measurability.

3. If $\mathbb{P}$ is ccc, then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$.

4. If $I_{\mathbb{P}} = N_{\mathbb{P}}$, then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$. Hence $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$ for any typical non-ccc tree-type forcing admitting a fusion argument.

5. (Brendle) Suppose $\mathbb{P}$ satisfies the following condition: For any maximal antichain $\mathcal{A}$ in $\mathbb{P}$, there is a maximal antichain $\mathcal{A}'$ such that for any two distinct elements $T, T'$ of $\mathcal{A}'$, $[T]$ and $[T']$ are disjoint and $\mathcal{A}'$ refines $\mathcal{A}$, i.e., for any $T'$ in $\mathcal{A}'$ there is a $T$ in $\mathcal{A}$ with $T' \subseteq T$. Then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$.

Sacks forcing is a typical example of the condition in 5. But we do not know of any strongly arboreal $\mathbb{P}$ satisfying the condition but which are neither ccc nor satisfying $I_{\mathbb{P}} = N_{\mathbb{P}}$.

*Proof.* We will prove only 5. The rest are straightforward. Suppose $\mathbb{P}$ satisfies the above condition and let $A$ be in ${I_{\mathbb{P}}}^*$. We prove $A$ is in $I_{\mathbb{P}}$. Since $A$ is in ${I_{\mathbb{P}}}^*$, the set of all $T$ in $\mathbb{P}$ such that $[T] \cap A \in I_{\mathbb{P}}$ is dense in $\mathbb{P}$. Hence we can take a maximal antichain $\mathcal{A}$ contained in this set. By the condition, we may assume for any two distinct elements $T_1$, $T_2$ of $\mathcal{A}$, $[T_1]$ and $[T_2]$ are pairwise disjoint. For each $T$ in $\mathcal{A}$, $[T] \cap A \in I_{\mathbb{P}}$. So we can pick $\{N_{n,T} \mid n \in \omega\}$ such that each $N_{n,T}$ is $\mathbb{P}$-null and $\bigcup_{n\in\omega} N_{n,T} = [T] \cap A$. Let $N_n = \bigcup_{T\in\mathcal{A}} N_{n,T}$ for each $n \in \omega$. Since $A \setminus \bigcup_{n\in\omega} N_n$ is $\mathbb{P}$-null, the proof is complete if we prove the following:

**Claim 2.1.13.** For each $n \in \omega$, $N_n$ is $\mathbb{P}$-null.

*Proof of Claim 2.1.13.* Take any $T'$ in $\mathbb{P}$. Since $\mathcal{A}$ is a maximal antichain, we can take a $T \in \mathcal{A}$ such that $T$ and $T'$ are compatible. Take a common extension $T''$ of $T$ and $T'$. Then $[T''] \cap N_n = [T''] \cap N_{n,T}$ because of the property of $\mathcal{A}$. But we know that $N_{n,T}$ is $\mathbb{P}$-null. Hence we can take a further extension of $T''$ disjoint from $N_n$. □ (Claim 2.1.13)

□

Before investigating the relation between $\mathbb{P}$-Baireness and $\mathbb{P}$-measurability, we first look at the $\mathbb{P}$-name for a generic real we defined in the paragraph after Definition 2.1.3 and its corresponding Baire measurable function from $\mathrm{St}(\mathbb{P})$ to the reals given in Lemma 2.1.2. Recall that $x_G$ is a generic real constructed from a generic object $G$ for any strongly arboreal forcing $\mathbb{P}$. Let $\dot{x_G}$ be a canonical $\mathbb{P}$-name for $x_G$.

**Example 2.1.14.** Let $\mathbb{P}$ be strongly arboreal. Then $f_{\dot{x_G}}(u)(m) = n$ if and only if there is a $T$ in $u$ such that $\mathrm{stem}(T)(m) = n$, where $f_{\dot{x_G}}$ is the corresponding

Baire measurable function from $\mathrm{St}(\mathbb{P})$ to the reals given in Lemma 2.1.2. Hence $f_{\dot{x_G}}(u) = \bigcup\{\mathrm{stem}(T) \mid T \in u\}$ for $u \in \mathrm{dom}(f_{\dot{x_G}})$, as is expected.

From now on, we use $\pi$ for denoting $f_{\dot{x_G}}$ throughout this chapter.

We give the relation between $\mathbb{P}$-Baireness and $\mathbb{P}$-measurability. Recall that $I_{\mathbb{P}}{}^*$ is a technical ideal introduced in Definition 2.1.10 which is the same as $I_{\mathbb{P}}$ for most cases.

**Lemma 2.1.15** ($\mathbb{P}$-Baireness vs. $\mathbb{P}$-measurability)**.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing and $A$ be a set of reals. Then

1. $A$ is in $I_{\mathbb{P}}{}^*$ if and only if $\pi^{-1}(A)$ is meager in $\mathrm{St}(\mathbb{P})$, and
2. $A$ is $\mathbb{P}$-measurable if and only if $\pi^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$.

In particular, if $A$ is $\mathbb{P}$-Baire, then $A$ is $\mathbb{P}$-measurable. Hence every Borel set is $\mathbb{P}$-measurable by the paragraph after Lemma 2.1.2.

Note that $\mathbb{P}$-measurability does not imply $\mathbb{P}$-Baireness in general.[2]

*Proof of Lemma 2.1.15.* Note that the domain of $\pi$ is comeager in $\mathrm{St}(\mathbb{P})$ and $\pi$ is continuous on it by Lemma 2.1.2.

The following are useful for the proof:

**Claim 2.1.16.** (a) For $T$ in $\mathbb{P}$ and $u \in \mathrm{dom}(\pi)$, if $T \in u$, then $\pi(u) \in [T]$.

(b) For $T$ in $\mathbb{P}$, the converse of (a) holds for comeager many $u$, i.e., for comeager many $u$ in $\mathrm{St}(\mathbb{P})$, $u$ is in the domain of $\pi$ and if $\pi(u) \in [T]$, then $T \in u$.

*Proof of Claim 2.1.16.* For (a), suppose $T \in u$. We prove $\pi(u)\restriction n \in T$ for each $n \in \omega$. Fix a natural number $n$. Then by Example 2.1.14, there is a $T'$ in $u$ such that $\mathrm{stem}(T') \supseteq \pi(u)\restriction n$. Since both $T$ and $T'$ are in $u$, they are compatible, especially $\mathrm{stem}(T') \in T$ (otherwise $[T] \cap [T'] = \emptyset$). Hence $\pi(u)\restriction n \in T$.

For (b), take any $T$ in $\mathbb{P}$. Then the set $D_T = \{T' \in \mathbb{P} \mid T' \subseteq T \text{ or } [T'] \cap [T] = \emptyset\}$ is dense in $\mathbb{P}$. (Take any $T'$. If $T' \not\subseteq T$, then there is a $t' \in T' \setminus T$. By strong arborealness of $\mathbb{P}$, $T'_{t'} \in \mathbb{P}$ and $[T'_{t'}] \cap [T] = \emptyset$.) Since $D_T$ is dense, the set $\{u \mid u \cap D_T \neq \emptyset\}$ is open dense in $\mathrm{St}(\mathbb{P})$. Hence it suffices to show that if $u$ is in $\mathrm{dom}(\pi)$, $u \cap D_T \neq \emptyset$ and $\pi(u) \in [T]$, then $T \in u$. Suppose $T \notin u$. Then since $u \cap D_T \neq \emptyset$, there is a $T' \in u$ such that $[T'] \cap [T] = \emptyset$. By (a), $\pi(u) \in [T']$, hence $\pi(u) \notin [T]$, a contradiction. □ (Claim 2.1.16)

We prove the first item of Lemma 2.1.15. We start with the direction from left to right.

We first show that $\pi^{-1}(A)$ is meager if $A$ is in $N_{\mathbb{P}}$. If $A$ is in $N_{\mathbb{P}}$, then the set $D = \{T \mid [T] \cap A = \emptyset\}$ is dense in $\mathbb{P}$. Hence the set of all $u \in \mathrm{dom}(\pi)$ with

[2] For example, if $A$ is a $\Sigma^1_2$ (lightface) set of reals universal for $\boldsymbol{\Sigma}^1_2$ (boldface) sets of reals and if every $\Sigma^1_2$ (lightface) set of reals has the Baire property but there is a $\boldsymbol{\Sigma}^1_2$ (boldface) set of reals without the Baire property, then $A$ is $\mathbb{C}$-measurable by Proposition 2.1.8, but $A$ is not $\mathbb{C}$-Baire because every $\boldsymbol{\Sigma}^1_2$ subset of the Cantor space is a continuous preimage of $A$ and every continuous preimage of $A$ has to have the Baire property in the Cantor space for the $\mathbb{C}$-Baireness of $A$.

$u \cap D \neq \emptyset$ is comeager. But if $u$ is in the comeager set, then there is a $T \in u \cap D$ and by Claim 2.1.16 (a), $\pi(u) \in [T]$ and $[T] \cap A = \emptyset$, in particular $\pi(u) \notin A$. Therefore $\pi^{-1}(A)$ is meager.

We have seen that $\pi^{-1}(A)$ is meager assuming $A$ is in $N_{\mathbb{P}}$. Since $I_{\mathbb{P}}$ is the $\sigma$-ideal generated by sets in $N_{\mathbb{P}}$, $\pi^{-1}(A)$ is meager for all $A$ in $I_{\mathbb{P}}$.

We show that $\pi^{-1}(A)$ is meager if $A$ is in $I_{\mathbb{P}}{}^*$. Since $A$ is in $I_{\mathbb{P}}{}^*$, the set $D' = \{T \mid [T] \cap A \in I_{\mathbb{P}}\}$ is dense in $\mathbb{P}$. We use the following well-known fact:

**Fact 2.1.17.** Let $X$ be a topological space and $A$ be a subset of $X$. Then $\left(\bigcup\{U \mid U \text{ is open and } U \cap A \text{ is meager }\}\right) \cap A$ is meager.

*Proof of Fact 2.1.17.* See, e.g., [45, Theorem 8.29]. □

Since $D'$ is dense, $\bigcup\{O_T \mid T \in D'\}$ is open dense. By the above fact, it suffices to prove that $O_T \cap \pi^{-1}(A)$ is meager for any $T$ in $D'$.

Take any $T$ in $D'$. By the definition of $D'$, we know that $[T] \cap A$ is in $I_{\mathbb{P}}$. Hence $\pi^{-1}([T] \cap A)$ is meager in $\mathrm{St}(\mathbb{P})$. But by Claim 2.1.16 (a), $O_T \cap \pi^{-1}(A) \cap \mathrm{dom}(\pi) \subseteq \pi^{-1}([T] \cap A)$. Since $\mathrm{dom}(\pi)$ is comeager in $\mathrm{St}(\mathbb{P})$, $O_T \cap \pi^{-1}(A)$ is almost included in the meager set $\pi^{-1}([T] \cap A)$. Therefore, $O_T \cap \pi^{-1}(A)$ is meager as desired.

Next, we see the direction from right to left for the equivalence of the first item of Lemma 2.1.15. Suppose $\pi^{-1}(A)$ is meager. Take any $T$ in $\mathbb{P}$ and we will find an extension $T'$ of $T$ such that $[T'] \cap A$ is in $I_{\mathbb{P}}$. Since $\pi^{-1}(A)$ is meager, then there is a sequence $\langle U_n \mid n \in \omega\rangle$ of open dense sets in $\mathrm{St}(\mathbb{P})$ such that $\bigcap_{n \in \omega} U_n \cap \pi^{-1}(A) = \emptyset$. For each $n \in \omega$, let $D_n = \{S \in \mathbb{P} \mid O_S \subseteq U_n\}$. Since $U_n$ is open dense in $\mathrm{St}(\mathbb{P})$, $D_n$ is dense open in $\mathbb{P}$. We choose a sequence $\langle \mathcal{A}_n \mid n \in \omega\rangle$ of maximal antichains such that $\mathcal{A}_n \subseteq D_n$, for each element $S$ of $\mathcal{A}_n$, the length of $\mathrm{stem}(S)$ is greater than $n$, and $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, i.e., every element of $\mathcal{A}_{n+1}$ is below some element in $\mathcal{A}_n$.

Now we use the properness of $\mathbb{P}$ to treat each $\mathcal{A}_n$ as “countable”. Let $\theta$ be a sufficiently large regular cardinal and $X$ be a countable elementary substructure of $\mathcal{H}_\theta$ such that $\mathbb{P}, T, \langle \mathcal{A}_n \mid n \in \omega\rangle$ are in $X$. By properness, there is an $(X, \mathbb{P})$-generic condition $T'$ below $T$. We show that $[T'] \cap A$ is in $I_{\mathbb{P}}$, which will complete the proof of the first item of Lemma 2.1.15.

Consider the set

$$B = \bigcap_{n \in \omega} \bigcup\{[S] \mid S \in \mathcal{A}_n \cap X\} \setminus \bigcup_{n \in \omega}\{[S] \cap [S'] \mid S, S' \in \mathcal{A}_n \cap X \text{ and } S \neq S'\}.$$

So $B$ is the set of all $x$s uniquely deciding which condition from $\mathcal{A}_n$ contains it for each $n$. By the property of $\langle \mathcal{A}_n \mid n \in \omega\rangle$, it will generate a filter coming from elements in $\mathcal{A}_n$s. The point is that any ultrafilter $u$ extending that filter satisfies $\pi(u) = x$, the given element, and that $u$ is in $U_n$ for each $n$. This will play a role for the argument.

Now we claim $[T'] \setminus B \in I_{\mathbb{P}}$ and $B \cap A = \emptyset$. We will be done if we prove them. The fact that $[T'] \setminus B \in I_{\mathbb{P}}$ follows from the fact that $\{S \mid S \in \mathcal{A}_n \cap X\}$ is predense below $[T']$ for each $n$ because $T'$ is $(X, \mathbb{P})$-generic and from that $[S] \cap [S'] \in I_{\mathbb{P}}$ for each $S, S' \in \mathcal{A}_n \cap X$ with $S \neq S'$ because $\mathcal{A}_n$ is an antichain, and from that $\mathcal{A}_n \cap X$ is countable for each $n$.

To prove $B \cap A = \emptyset$, take any element $x$ from $B$. As we mentioned above, for each $n \in \omega$, there is a unique element $S_n$ in $\mathcal{A}_n \cap X$ with $x \in [S_n]$. Since $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, $S_{n+1} \leq S_n$ for each $n$. Hence the set $\{S_n \mid n \in \omega\}$ generates a filter $F_x$. Take any ultrafilter $u$ extending $F_x$. We claim that $\pi(u) = x$ and $u \in U_n$ for each $n$. By the property of $\langle \mathcal{A}_n \mid n \in \omega \rangle$, the length of $\mathrm{stem}(S_n)$ is greater than $n$. Hence, by Example 2.1.14, $\pi(u)$ is already decided to be $x$ by $S_n$ $(n \in \omega)$. The fact that $u \in U_n$ for each $n$ follows from the fact that $S_n \in \mathcal{A}_n \subseteq D_n$ and the definition of $D_n$. Since we have assumed that $\bigcap_{n \in \omega} U_n \cap \pi^{-1}(A) = \emptyset$, $x$ does not belong to $A$. Hence we have seen $B \cap A = \emptyset$ as desired.

We have shown the first item of Lemma 2.1.15. Next, we show the equivalence in the second item of Lemma 2.1.15. For left to right, we assume $A$ is $\mathbb{P}$-measurable. Then the set

$$D = \{T \in \mathbb{P} \mid \text{ either } [T] \cap A \in I_{\mathbb{P}} \text{ or } [T] \setminus A \in I_{\mathbb{P}}\}$$

is dense and the set $U = \bigcup\{O_T \mid T \in D\}$ is open dense in $\mathrm{St}(\mathbb{P})$. Let $U_1 = \bigcup\{O_T \mid [T] \setminus A \in I_{\mathbb{P}}\}$ and $U_2 = \bigcup\{O_T \mid [T] \cap A \in I_{\mathbb{P}}\}$. Then $U = U_1 \cup U_2$.

We claim that $U_1 \cap U_2 = \emptyset$. First we note that $[T] \notin I_{\mathbb{P}}$ (even $[T] \notin {I_{\mathbb{P}}}^*$) for any $T \in \mathbb{P}$. If $[T]$ is in $I_{\mathbb{P}}$ for some $T$, then $\pi^{-1}([T])$ is meager in $\mathrm{St}(\mathbb{P})$ by the first item of Lemma 2.1.15. Since $O_T \subseteq \pi^{-1}([T])$ by Claim 2.1.16 (a), $O_T$ would be also meager in $\mathrm{St}(\mathbb{P})$, which would contradict Lemma 2.1.1. Hence $[T] \notin I_{\mathbb{P}}$ for any $T \in \mathbb{P}$. We show that $U_1 \cap U_2 = \emptyset$. Suppose there is a $u$ in $U_1 \cap U_2$. Then there are $T_1, T_2 \in u$ with $[T_1] \setminus A \in I_{\mathbb{P}}$ and $[T_2] \cap A \in I_{\mathbb{P}}$. Since $u$ is a filter, there is a $T_3$ in $u$ with $T_3 \leq T_1, T_2$. But then $[T_3] \setminus A$ and $[T_3] \cap A$ are both in $I_{\mathbb{P}}$, which means $[T_3]$ itself is in $I_{\mathbb{P}}$. Contradiction!

Hence, it suffices to show that $U_1 \setminus \pi^{-1}(A)$, $U_2 \cap \pi^{-1}(A)$ are meager because that will imply $U_1 \triangle \pi^{-1}(A)$ is meager. We will only see that $U_2 \cap \pi^{-1}(A)$ is meager. The case for $U_1 \setminus \pi^{-1}(A)$ being meager is similar. By Fact 2.1.17, it suffices to see that $O_T \cap \pi^{-1}(A)$ is meager when $[T] \cap A \in I_{\mathbb{P}}$. But if $[T] \cap A \in I_{\mathbb{P}}$, then $O_T \cap \pi^{-1}(A) \subseteq \pi^{-1}([T] \cap A)$ and $\pi^{-1}([T] \cap A)$ is meager by Claim 2.1.16 (a) and the first item of Lemma 2.1.15. Hence we are done.

Now we see the direction from right to left. Assume $\pi^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$. Then there are open sets $U_1$, $U_2$ such that $U_1 \triangle \pi^{-1}(A)$, $U_2 \triangle \pi^{-1}({}^{\omega}\omega \setminus A)$ are meager. By Lemma 2.1.1, $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2$ is open dense in $\mathrm{St}(\mathbb{P})$. Let $D_i = \{T \in \mathbb{P} \mid O_T \subseteq U_i\}$ for $i = 1, 2$. Then $D_1 \cup D_2$ is dense in $\mathbb{P}$. Hence by Lemma 2.1.12 (2), it suffices to prove that $[T] \setminus A \in {I_{\mathbb{P}}}^*$ for each $T$ in $D_1$ and that $[T] \cap A \in {I_{\mathbb{P}}}^*$ for each $T$ in $D_2$.

We only prove $[T] \setminus A \in {I_{\mathbb{P}}}^*$ for each $T$ in $D_1$. By the first item of Lemma 2.1.15, it is enough to see that $\pi^{-1}([T] \setminus A)$ is meager in $\mathrm{St}(\mathbb{P})$. But by Claim 2.1.16, $\pi^{-1}([T] \setminus A)$ is almost the same as $O_T \setminus \pi^{-1}(A)$. Since $T$ is in $D_1$, by the definition of $U_1$, $O_T \setminus \pi^{-1}(A)$ is meager. This completes the proof of the second item of Lemma 2.1.15. $\square$

Note that if $\mathbb{P}$ satisfies the condition in Lemma 2.1.12 (5), then we do not need the properness of $\mathbb{P}$ for the proofs of Lemma 2.1.15.

Before closing this section, let us mention the connection between our framework and Zapletal's setting. In [91], Zapletal starts from a $\sigma$-ideal $I$ on a Polish space $X$ (a separable, completely metrizable space) and considers the quotient of the set of all Borel sets in $X$ modulo $I$ and develops the general theory of this forcing so-called "idealized forcing" as a Boolean algebra. The following proposition shows that our forcings are all idealized forcings:

**Proposition 2.1.18.** Suppose $\mathbb{P}$ is a strongly arboreal, proper forcing. Then the map $i\colon \mathbb{P} \to \big(\mathbf{B}/I_{\mathbb{P}}{}^*\big) \setminus \{0\}$ defined by

$$i(T) = \text{ the equivalence class represented by } [T],$$

is a dense embedding, where $\mathbf{B}$ denotes the set of all Borel sets of the reals and $\mathbf{B}/I_{\mathbb{P}}{}^*$ is the quotient Boolean algebra via $I_{\mathbb{P}}{}^*$.

Hence, our situation is a special case of Zapletal's.[3]

*Proof of Proposition 2.1.18.* First we see that the map $i$ is well-defined, i.e., $[T]$ is not in $I_{\mathbb{P}}{}^*$ for each $T$ in $\mathbb{P}$. But this is just the same argument as the proof of $[T] \notin I_{\mathbb{P}}$ for each $T$ in $\mathbb{P}$ in Lemma 2.1.15.

It is clear that if $T_1 \leq T_2$, then $i(T_1) \leq i(T_2)$. To show the converse, assume $T_1 \nleq T_2$ and we prove that $i(T_1) \nleq i(T_2)$. Since $T_1 \nleq T_2$, there is a $t \in T_1$ which is not in $T_2$. By strong arborealness of $\mathbb{P}$, $(T_1)_t \in \mathbb{P}$ and $[(T_1)_t] \cap [T_2] = \emptyset$. Hence $i((T_1)_t) \nleq i(T_2)$. Since $(T_1)_t \leq T_1$, $i((T_1)_t) \leq i(T_1)$. Therefore, $i(T_1) \nleq i(T_2)$.

So it suffices to show that the range of $i$ is dense in $\big(\mathbf{B}/I_{\mathbb{P}}{}^*\big) \setminus \{0\}$. Let $B$ be a Borel set which is not in $I_{\mathbb{P}}{}^*$. We will find a $T$ in $\mathbb{P}$ with $[T] \setminus B \in I_{\mathbb{P}}{}^*$. By Lemma 2.1.15, $B$ is $\mathbb{P}$-measurable. Since $B$ is not in $I_{\mathbb{P}}{}^*$, there is a $T$ such that $[T] \setminus B \in I_{\mathbb{P}}$, hence $[T] \setminus B \in I_{\mathbb{P}}{}^*$ by Lemma 2.1.12, as desired. $\square$

## 2.2 Forcing absoluteness

Recall from § 1.12 that absoluteness is one of the central notions in set theory, and it is the unchangingness of the truth-values of statements between models of set theory. Forcing absoluteness is the absoluteness between ground models and their generic extensions, which plays an important role in many areas in set

[3] In [91, Corollary 2.1.5], Zapletal proved a more general result. In the corollary, the ideal $I$ he constructed is essentially the same as our $I_{\mathbb{P}}{}^*$in the following sense: If we use $b_n = |\dot{x}_{\text{gen}}(\check{n}) = 1|$ $(n \in \omega)$ instead of $b_t$ $(t \in {}^{<\omega}2)$ for the generators of $C$, then Zapletal's $I$ is exactly the same as our $I_{\mathbb{P}}{}^*$ on Borel sets.

theory. In this section, we focus on the forcing absoluteness of statements in second-order arithmetic. We start with its definition:

**Definition 2.2.1** ($\mathbf{\Sigma}^1_n$-$\mathbb{P}$-absoluteness)**.** Let $\mathbb{P}$ be a forcing notion and $n$ be a natural number with $n \geq 1$. Then $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-*absoluteness* is the following statement:

> For any $\Sigma^1_n$ formula $\varphi$, real $r$ in $V$, and $\mathbb{P}$-generic filter $G$ over $V$, $V \vDash \varphi(r)$ if and only if $V[G] \vDash \varphi(r)$.

By definition, it is immediate that $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-absoluteness is equivalent to $\mathbf{\Pi}^1_n$-$\mathbb{P}$-absoluteness for each $\mathbb{P}$ and each $n \geq 1$, where $\mathbf{\Pi}^1_n$-$\mathbb{P}$-absoluteness is defined similarly. By Theorem 1.12.3, $\mathbf{\Sigma}^1_2$-$\mathbb{P}$-absoluteness holds for any $\mathbb{P}$. How about $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness? In L, $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness *fails* if $\mathbb{P}$ adds a new real, i.e., there is a new real in a generic extension by $\mathbb{P}$. This is because the statement "There is a non-constructible real" is $\Sigma^1_3$ and it is true in a generic extension of L by $\mathbb{P}$ while it is false in L. On the other hand, typical forcing axioms imply $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness for many $\mathbb{P}$, e.g., $\mathrm{MA}_{\aleph_1}$ implies $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness for any ccc forcing $\mathbb{P}$.[4] Since one can force $\mathrm{MA}_{\aleph_1}$ starting from a model of ZFC, the statement that $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds for a ccc forcing $\mathbb{P}$ is *independent from* ZFC. It is natural to ask: When is the statement true and when is it not? We discuss this question in § 2.4.

From now on, we will restrict our attention to *definable forcings.* Let $n$ be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is *provably* $\Delta^1_n$ if there are $\Sigma^1_n$ formula $\phi$ and $\Pi^1_n$ formula $\psi$ such that the statement "$\phi$ and $\psi$ define the same partial order $(\mathbb{P}, \leq_{\mathbb{P}})$ with the incompatibility relation $\perp_{\mathbb{P}}$" is provable in ZFC. All the typical strongly arboreal forcings are provably $\Delta^1_2$. We will need this definability condition for forcings when we compute the complexity of $I_{\mathbb{P}}{}^*$.

In some of our results in § 2.4, we shall need a strengthening of the standard notion of properness for definable forcings. Let $\mathbb{P}$ be a provably $\Delta^1_n$ forcing for some $n \geq 1$. We say $\mathbb{P}$ is *strongly proper* if for any countable transitive model $M$ of a finite fragment of ZFC, if $\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}}$ are absolute between $M$ and $V$ respectively, (i.e., $P^M, \leq^M_{\mathbb{P}}, \perp^M_{\mathbb{P}}$ are the same as $P \cap M, \leq_{\mathbb{P}} \cap (M \times M), \perp_{\mathbb{P}} \cap (M \times M)$ respectively), then for any condition $p$ in $P^M$ (or $\mathbb{P} \cap M$), there is an $(M, \mathbb{P})$-generic condition $q$ below $p$, i.e., if $M \vDash$ "$A$ is a maximal antichain in $\mathbb{P}$", then $A \cap M$ is predense below $q$.[5] Let us compare strong properness with properness. (For the definition of properness, see § 1.9.) Here $(M, \mathbb{P})$-generic conditions are the same as $(X, \mathbb{P})$-generic conditions for a countable elementary substructure $X$ of $\mathcal{H}_\theta$: If $\mathbb{P}$ is provably $\Delta^1_n$ for some $n \geq 1$, $X$ is a countable elementary substructure of

[4] For the proof, see [5, Theorem 13]. For basic definitions and properties of forcing axioms, see [37].

[5] Although we will not explicitly mention the finite fragment of ZFC we will use for the definition of strong properness, it will be large enough that we can proceed all the arguments in this chapter within the fragment as usual. From now on, we say "countable transitive models of ZFC" instead of "countable transitive models of a finite fragment of ZFC" for simplicity.

$\mathcal{H}_\theta$ for some large enough regular $\theta$ and $M$ is the transitive collapse of $X$, then a condition $p$ is $(M, \mathbb{P})$-generic if and only if it is $(X, \mathbb{P})$-generic in the usual sense. In particular, if $\mathbb{P}$ is provably $\Delta^1_n$ for some $n \geq 1$ and strongly proper, then $\mathbb{P}$ is proper. All the typical examples of proper, provably $\Delta^1_2$ forcings are strongly proper. But there is a ccc, provably $\Delta^1_3$ forcing which is not strongly proper.[6]

We use strong properness instead of properness as it allows us to leave out the quantification "$\in \mathcal{H}_\theta$" which would increase the complexity of our statements in the relevant results (Proposition 2.3.3, Theorem 2.4.8, Theorem 2.4.9) beyond projective.

## 2.3 The transcendence properties over inner models

By "transcendence over an inner model $M$", we refer to properties that express that the universe is different from $M$ in some concrete sense. E.g., the property $\omega_1^M < \omega_1^V$ is such a transcendence property; another transcendence property would be "there are $\mathbb{P}$-generics over $M$" for some nontrivial forcing $\mathbb{P}$. (Here, by *inner models*, we mean proper class transitive models of ZFC.) In § 2.1, we have seen that the generic filters of any strongly arboreal forcing can been seen as generic reals of the forcing. All such generic reals cannot exist in a given ground model: A partial order $\mathbb{P}$ is called *non-trivial* if for any condition $p$ in $\mathbb{P}$ there are two extensions $q, r$ of $p$ such that they are incompatible ($q \perp r$). It is easy to see that if $\mathbb{P}$ is non-trivial and $G$ is a $\mathbb{P}$-generic filter over $V$, then $G$ does not belong to $V$. Since $G$ can be coded by a generic real over $V$ for each strongly arboreal forcing, such a generic real does not belong to $V$ either. Hence the existence of generic reals over an inner model $M$ can be seen as a transcendence property over $M$.

Although this transcendence property measures the difference of two models of set theory very well and often plays an important role in set theory of the reals, it is sometimes too strong when we consider some specific problems. We now introduce a weaker notion called *quasi-generic reals*, which are obvious generalization of Cohen reals and random reals. This notion will give us the right transcendence property to characterize the regularity properties for sets of reals.

**Definition 2.3.1** (Brendle, Halbeisen, and Löwe [19]). Let $\mathbb{P}$ be strongly arboreal and $M$ be a transitive model of ZFC. A real $x$ is *quasi-$\mathbb{P}$-generic over $M$* if for

[6]Assuming $\omega_1$ is not $\Sigma_1$-Mahlo in L, Bagaria and Bosch constructed a ccc, provably $\Delta^1_3$ forcing which adds a real $x$ such that $\mathrm{L}[x]$ correctly computes $\omega_1$ (see the proof of [7, Theorem 6.1]). This partial order is not strongly proper because every $\mathbf{\Sigma}^1_3$ strongly proper forcing preserves the statement "$\mathrm{L}(\mathbb{R})$ is a Solovay model over L" by [6, Theorem 1] and this statement implies $\omega_1 > \omega_1^{\mathrm{L}[a]}$ for every real $a$.

any Borel code $c$ in $M$ with $B_c \in I_{\mathbb{P}}{}^*$, $x$ is not in $B_c$, where $B_c$ is the decoded Borel set from $c$.

**Example 2.3.2.**

1. Cohen forcing $\mathbb{C}$: Quasi-$\mathbb{C}$-generic reals are the same as Cohen reals by definition. Hence quasi-$\mathbb{C}$-genericity coincides with $\mathbb{C}$-genericity.
2. Random forcing $\mathbb{B}$: Quasi-$\mathbb{B}$-generic reals are the same as random reals by definition. Hence quasi-$\mathbb{B}$-genericity coincides with $\mathbb{B}$-genericity.
3. Hechler forcing $\mathbb{D}$: Quasi-$\mathbb{D}$-generic reals are the same as Hechler reals. Hence quasi-$\mathbb{D}$-genericity coincides with $\mathbb{D}$-genericity.
4. Sacks forcing $\mathbb{S}$: If $M$ is an inner model of ZFC, quasi-$\mathbb{S}$-generic reals over $M$ are the reals which are not in $M$ because any Borel set in $I_{\mathbb{S}}{}^* = N_{\mathbb{S}}$ is countable and this is also true in $M$ if the code is in $M$ by Shoenfield absoluteness (the absoluteness we mentioned in the paragraph after Definition 2.2.1). Therefore, quasi-$\mathbb{S}$-genericity does not coincide with $\mathbb{S}$-genericity.

The last example explains the difference between genericity and quasi-genericity for non-ccc strongly arboreal forcings: There is a model of set theory where there is a quasi-Sacks-generic real over L but there is no Sacks real over L, e.g., add one Cohen real over L. As is expected, genericity implies quasi-genericity for all the typical strongly arboreal forcings and the converse is true for most ccc forcings:

**Proposition 2.3.3.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_2$ forcing. Then

1. The set $\{c \mid B_c \in I_{\mathbb{P}}{}^*\}$ is $\Pi^1_2$. Hence the statement "$c$ codes a Borel set in $I_{\mathbb{P}}{}^*$" is absolute between inner models of ZFC.
2. Suppose $\mathbb{P}$ is also $\mathbf{\Sigma}^1_1$ and provably ccc, i.e., there is a formula $\phi$ defining $\mathbb{P}$ and the statement "$\phi$ is ccc" is provable in ZFC. Then the set $\{c \mid B_c \in I_{\mathbb{P}}{}^*\}$ is also $\mathbf{\Sigma}^1_2$ and hence $\mathbf{\Delta}^1_2$.
3. If $M$ is a transitive model of ZFC and a real $x$ is $\mathbb{P}$-generic over $M$, then $x$ is quasi-$\mathbb{P}$-generic over $M$.
4. Suppose $\mathbb{P}$ is provably ccc. Then if $M$ is an inner model of ZFC and a real $x$ is quasi-$\mathbb{P}$-generic over $M$, then $x$ is $\mathbb{P}$-generic over $M$.

*Proof.* We show the first statement. By Lemma 2.1.15, a set of reals $A$ is in $I_{\mathbb{P}}{}^*$ if and only if $\pi^{-1}(A)$ is meager in $\mathrm{St}(\mathbb{P})$. Hence, it suffices to show that $\{c \mid \pi^{-1}(B_c) \text{ is meager}\} \in \Pi^1_2$.

We prove the following:

$$\begin{aligned} \pi^{-1}(B_c) \text{ is meager} \iff (\forall M \ni c) \big(&M\colon \text{a c.t.m. of ZFC} \\ &\Longrightarrow M \models \text{“}\pi^{-1}(B_c) \text{ is meager”}\big). \end{aligned} \tag{$\star$}$$

First note that the right hand side makes sense: The statement "$\mathbb{P}$ is a strongly arboreal forcing" is $\Pi^1_2$ by the assumption that $\mathbb{P}$ is provably $\Delta^1_2$, so by downward absoluteness, this is also true in $M$ and then we can define a $\mathbb{P}$-name for a $\mathbb{P}$-generic real and the function $\pi$ in $M$. Since the right hand side is $\Pi^1_2$, it suffices to show the above equivalence.

The following claim is the key point where we use the unfolded Banach-Mazur games essentially:

**Claim 2.3.4.** Let $M$ be a countable transitive model of ZFC with $c \in M$. If $M \vDash$ "$\pi^{-1}(B_c)$ is meager", then for any $T \in \mathbb{P} \cap M$, there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager in $V$.

*Proof of Claim 2.3.4.* Take any $T$ in $\mathbb{P} \cap M$. Since $\mathbb{P}$ is provably $\Delta^1_2$, $\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}}$ are absolute between $M$ and $V$. Hence $M$ satisfies the assumption in the definition of strong properness and we can take a $T' \leq T$ such that $T'$ is $(M, \mathbb{P})$-generic by strong properness of $\mathbb{P}$.

We show that $T'$ satisfies the desired property, i.e., $O_{T'} \cap \pi^{-1}(B_c)$ is meager in $V$. For that, we will use the unfolded Banach-Mazur games introduced in § 1.8. Let $U$ be a tree on $\omega \times \omega$, recursive in $c$ such that $B_c = \mathrm{p}[U]$ holds in any transitive model $N$ of ZFC with $c \in N$, where $\mathrm{p}[U]$ is the projection of $[U]$ to the first coordinate.[7] Since $\pi$ is continuous in $\mathrm{dom}(\pi)$ and $\pi^{-1}(B_c) = \exists^{\mathbb{R}}(\pi \times \mathrm{id})^{-1}([U])$, we can apply Theorem 1.8.5 for $A = \pi^{-1}(B_c)$, $F = (\pi \times \mathrm{id})^{-1}([U])$ and $X = \mathrm{dom}(\pi)$ (or $X = \mathrm{dom}(\pi) \cap O_{T'}$).

Since $\mathrm{dom}(\pi)$ is comeager in $\mathrm{St}(\mathbb{P})$, it suffices to show that player II has a winning strategy in the game $G^{**}_u\big((\pi \times \mathrm{id})^{-1}([U]), \mathrm{dom}(\pi) \cap O_{T'}\big)$ (call it $G'$), namely player I first chooses $({S_0}', y_0)$, where ${S_0}' \leq T'$. Since $M \vDash$ "$\pi^{-1}(B_c)$ is meager", by applying Theorem 1.8.5 in $M$, we can find a winning strategy $\sigma$ for player II in the game $G^{**}_u\big((\pi \times \mathrm{id})^{-1}([U]), \mathrm{dom}(\pi)\big)$ in $M$ (call it $G^M$). The idea is to transfer $\sigma$ to a winning strategy for player II in $G'$ in $V$. Instead of writing down a winning strategy for player II in $G'$, we describe how to win the game $G'$ for player II as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $G'$ | I | $\big({S_0}' \ (\leq T'), y_0\big)$ | | $({S_2}', y_1)$ | | $\dots$ |
| | II | | ${S_1}'$ | | ${S_3}'$ | $\dots$ |
| $G^M$ | I | $(S_0, y_0)$ | | $(S_2, y_1)$ | | $\dots$ |
| | II | | $S_1$ | | $S_3$ | $\dots$ |

We construct sequences $\langle S_n \mid n \in \omega \rangle$, $\langle {S_n}' \mid n \in \omega \rangle$, $\langle y_n \mid n \in \omega \rangle$ with the following properties:

- $\big(\langle {S_n}' \mid n \in \omega \rangle, \langle y_n \mid n \in \omega \rangle\big)$ is a run in the game $G'$ in $V$,
- $\big(\langle S_n \mid n \in \omega \rangle, \langle y_n \mid n \in \omega \rangle\big)$ is a run in the game $G^M$ in $V$,

[7]For the existence of such $U$, see, e.g., [66, Theorem 7B.5].

- ${S_{2n}}'$ and $y_n$ are arbitrarily chosen by player I for each $n$,
- player II follows $\sigma$ in $G^M$, and
- ${S_{2n+1}}' \leq S_{2n+1}$ for each $n$.

Assuming we have constructed the above sequences, we prove that player II wins in the game $G'$. First note that $G^M$ is a closed game for player II, hence the strategy $\sigma$ remains winning in $V$. Therefore, $(\pi(u), y) \notin [U]$ for any $u \in \bigcap_{n\in\omega} O_{S_n}$ in $V$. But since ${S_{2n+1}}' \leq S_{2n+1}$ for each $n$, $(\pi(u), y) \notin [U]$ for any $u \in \bigcap_{n\in\omega} O_{{S_n}'}$, hence player II wins the game $G'$.

We describe how to construct the above sequences. Suppose we have $\langle ({S_i}', S_i, y_i) \mid i < 2n\rangle$ for some $n$. We decide ${S_{2n}}'$, ${S_{2n+1}}'$, $S_{2n}$, $S_{2n+1}$ and $y_n$. By the above properties, ${S_{2n}}'$ and $y_n$ are arbitrarily chosen by player I and $S_{2n+1}$ will be decided by the rest and $\sigma$. So let's decide $S_{2n}$ and ${S_{2n+1}}'$.

Let $D$ be the set of all possible candidates for $S_{2n+1}$ by $\sigma$ and the previous play $\langle S_i \mid i < 2n\rangle, \langle y_i \mid i < n\rangle$. Then in $M$, $D$ is dense below $S_{2n-1}$ (if $n > 0$). Since ${S_{2n}}' \leq {S_{2n-1}}' \leq S_{2n-1}$ and $T'$ is $(M, \mathbb{P})$-generic, $D \cap M = D$ is predense below ${S_{2n}}'$. Take an element from $D$ which is compatible with ${S_{2n}}'$ and choose $S_{2n}$ so that the element we have taken becomes $S_{2n+1}$ by $\sigma$ and let ${S_{2n+1}}'$ be a common extension (in $V$) of ${S_{2n}}'$ and $S_{2n+1}$. This finishes the construction of the sequences. □ (Claim 2.3.4)

Now let us prove the equivalence ($\star$):

Suppose $\pi^{-1}(B_c)$ is meager and assume there is a countable transitive model $M$ of ZFC with $c \in M$ such that $M \vDash$ "$\pi^{-1}(B_c)$ is not meager". We will derive a contradiction. Since every Borel set is $\mathbb{P}$-Baire, $\pi^{-1}(B_c)$ has the Baire property in $M$. Hence if $\pi^{-1}(B_c)$ is not meager in $M$, then there is a $T \in \mathbb{P}^M$ such that $\pi^{-1}(B_c)$ is comeager in $O_T$ (i.e., $\pi^{-1}(B_c) \cap O_T$ is comeager in $O_T$) in $M$. By Claim 2.1.16 (b), $\pi^{-1}([T] \setminus B_c)$ is almost included in the meager set $O_T \setminus \pi^{-1}(B_c)$, hence, in $M$, $\pi^{-1}([T] \setminus B_c)$ is meager in $\mathrm{St}(\mathbb{P})$. Now apply the above claim for the Borel set $[T] \setminus B_c$. Then we get a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}([T] \setminus B_c)$ is meager in $V$. But this means that $O_{T'}$ is almost included in $\pi^{-1}(B_c)$. Since $O_{T'}$ is not meager by Lemma 2.1.1, $\pi^{-1}(B_c)$ is not meager, which contradicts the assumption that $\pi^{-1}(B_c)$ is meager.

For the other direction, suppose the right hand side holds for $\pi^{-1}(B_c)$ and we show that it is actually meager in $V$. By Fact 2.1.17, it suffices to show that for any $T$ in $\mathbb{P}$, there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager. So fix any $T$ and pick a countable transitive model $M$ with $c, T \in M$. Then by Claim 2.3.4, there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager, as desired.

We next show the second statement of this proposition. For that, it suffices to see the following by Lemma 2.1.15:

$$\begin{aligned}\pi^{-1}(B_c) \text{ is meager} &\iff (\exists M \ni c)\ \big(M\colon \text{a countable transitive model}\\ &\qquad \text{of ZFC and } M \vDash \text{“}\pi^{-1}(B_c) \text{ is meager”}\big)\\ &\iff (\forall M \ni c)\ \big(M\colon \text{a countable transitive model}\\ &\qquad \text{of ZFC} \implies M \vDash \text{“}\pi^{-1}(B_c) \text{ is meager”}\big),\end{aligned}$$

where $\pi = f_{\dot{x}_G}$ as before.

We only show the first equivalence. For left to right, if we take a countable elementary substructure $X$ of $\mathcal{H}_\theta$ for enough large $\theta$ such that $X$ has all the essential elements, then the transitive collapse of $X$ will do the job for $M$ in the right hand side.

For right to left, take an $M$ with the property in the right hand side. The idea is the same as the proof of Claim 2.3.4 in the first item of Lemma 2.3.3. This time, we use the unfolded Banach-Mazur game $G_u^{**}\big((\pi \times \mathrm{id})^{-1}([U])\mathrm{dom}(\pi)\big)$ both in $M$ and $V$ and translate a winning strategy in $G^M$ to the one in $G'$.

By the assumption, in $M$, player II has a winning strategy $\sigma'$ in $G^M$. The construction of a winning strategy for II in $G'$ in $V$ from $\sigma'$ is exactly the same as Claim 2.3.4. But instead of using the $(M, \mathbb{P})$-genericity for a condition $T'$, we use the following:

**Claim 2.3.5.** Let $D$ be a dense subset of $\mathbb{P}$ in $M$. Then $D$ is predense in $\mathbb{P}$ in $V$.

*Proof of Claim 2.3.5.* Let $D$ be a dense subset of $\mathbb{P}$ in $M$. Then since $\mathbb{P}$ is provably ccc, in $M$, there is a countable maximal antichain $A \subseteq D$. But since $\mathbb{P}$ is $\mathbf{\Sigma}^1_1$, the statement “a real codes a maximal antichain” is $\mathbf{\Sigma}^1_1 \wedge \mathbf{\Pi}^1_1$ and therefore $A$ remains a maximal antichain in $V$. Hence $D$ is predense in $\mathbb{P}$ in $V$. $\square$(*Claim* 2.3.5)

The rest is exactly the same as Claim 2.3.4.

We show the third statement of this proposition. Let $x$ be $\mathbb{P}$-generic over $M$. Then the set $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$ is a $\mathbb{P}^M$-generic filter over $M$. We show that $x \notin B_c$ when $c$ is a Borel code in $M$ with $B_c \in I_{\mathbb{P}}{}^*$.

First, we make a small observation about $x$ and Borel sets with their codes in $M$. Let $i^M$ be the dense embedding from $\mathbb{P}^M$ to $\Big(\big(\mathbf{B}/I_{\mathbb{P}}{}^*\big) \setminus \{0\}\Big)^M$ defined in Proposition 2.1.18 applied in $M$ and $i_*^M(G_x)$ be the $\big(\mathbf{B}/I_{\mathbb{P}}{}^*\big)$-generic filter over $M$ induced by $i^M$ and $G_x$. Using the fact that $I_{\mathbb{P}}{}^*$ is a $\sigma$-ideal, it is routine to check that $B \in i_*^M(G_x)$ if and only if $x \in B$ for any Borel set $B$ with a code in $M$.

Now let $c$ be a Borel code in $M$ with $B_c \in I_{\mathbb{P}}{}^*$ in $V$. By the first item of in this proposition and the downward absoluteness for $\Pi^1_2$ formulas, $M \vDash$ “$B_c \in I_{\mathbb{P}}{}^*$”. Suppose that $x$ does belong to $B_c$. Then by the above observation, $B_c \in i_*^M(G_x)$. But this implies that $M \vDash$ “$B_c \notin I_{\mathbb{P}}{}^*$”, hence by upward absoluteness for $\Sigma^1_2$ formulas, $B_c \notin I_{\mathbb{P}}{}^*$. Contradiction!

We show the last statement of this proposition. Let $x$ be a quasi-$\mathbb{P}$-generic real over $M$ and put $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$. We show that $G_x$ is a $\mathbb{P}^M$-generic filter over $M$.

We first show that $G_x$ meets every maximal antichain of $\mathbb{P}^M$ in $M$. Take any maximal antichain $A$ of $\mathbb{P}^M$ in $M$. Since $\mathbb{P}$ is provably ccc, $A$ is countable in $M$. Now consider $B = \bigcup\{[T] \mid T \in A\}$. Then $B$ is a Borel set with a code in $M$ and $M \models$ "${}^\omega\omega \setminus B \in I_{\mathbb{P}}{}^*$". By the first item of this proposition, this is also true in $V$. Since $x$ is quasi-$\mathbb{P}$-generic over $M$, $x \notin {}^\omega\omega \setminus B$, i.e., $x$ is in $B$. Hence $G_x$ meets $A$.

Now we show that $G_x$ is a filter. Take any two elements $T_1, T_2$ in $G_x$. We will find a common extension of $T_1$, $T_2$ in $G_x$. Consider $D = \{S \in \mathbb{P} \mid ([S] \cap [T_1] = \emptyset$ and $[S] \cap [T_2] = \emptyset)$ or $(S \leq T_1$ and $[S] \cap [T_2] = \emptyset)$ or $(S \leq T_2$ and $[S] \cap [T_2] = \emptyset)$ or $(S \leq T_1, T_2)\}$ in $M$. Then by strong arborealness of $\mathbb{P}$, $D$ is dense in $M$. Hence $G_x$ meets $D$. Take a condition $S$ from $G_x \cap D$. Then only the last case in $D$ happens because $S \in G_x \iff x \in [S]$. Hence $S \leq T_1, T_2$. Therefore, $G_x$ is a $\mathbb{P}^M$-generic filter over $M$. $\square$

## 2.4 The equivalence results

In § 1.8 and § 2.2, we have asked when every $\mathbf{\Delta}^1_2$ set of reals has the Baire property and when $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds for a strongly arboreal forcing $\mathbb{P}$. In fact, the answer to the first question is exactly the same as the one for the second question, for Cohen forcing: Bagaria [4] and Woodin [89] showed that every $\mathbf{\Delta}^1_2$ set of reals has the Baire property if and only if $\mathbf{\Sigma}^1_3$-$\mathbb{C}$-absoluteness holds where $\mathbb{C}$ is Cohen forcing. They also proved the same equivalence holds for Lebesgue measurability and random forcing and the same holds for the Baire property for dominating topology ($\mathbb{D}$-measurability) and Hechler forcing (see [42, 20]). These are the typical cases for ccc, strongly arboreal forcings. How about non-ccc forcings? Halbeisen and Judah [30] showed the same equivalence for completely Ramseyness ($\mathbb{R}$-measurability) and Mathias forcing and the author [34] proved it for the property not being a Bernstein set ($\mathbb{S}$-measurability) and Sacks forcing. Therefore, the regularity properties for $\mathbf{\Delta}^1_2$ sets and $\mathbf{\Sigma}^1_3$ forcing absoluteness are closely related. We can further connect the transcendence property over L with these two properties: E.g., Judah and Shelah [43] proved that every $\mathbf{\Delta}^1_2$ set of reals has the Baire property if and only if for any real $x$, there is a Cohen real over $\mathrm{L}[x]$. They also proved the same equivalence for Lebesgue measurability and random reals. Similarly Brendle and Löwe [20] showed that there is no $\mathbf{\Delta}^1_2$ Bernstein set if and only if for any real $x$ there is a real not in $\mathrm{L}[x]$. As we have seen in § 2.3, these latter statements can be seen as the existence of quasi-generic reals over $\mathrm{L}[x]$ for reals $x$ while the existence of generic reals might not work, e.g., for the last statement, there is a model of set theory where for any real $x$ there is a real not in $\mathrm{L}[x]$ but there is no Sacks real over L.[8]

[8]For example, start with L and add $\omega_1$ many Cohen reals.

In this section, we prove the above equivalence results for a wide class of strongly arboreal forcings in a uniform way and explore the equivalence between regularity properties for $\mathbf{\Delta}^1_3$ sets (or $\mathbf{\Sigma}^1_3$ sets), $\mathbf{\Sigma}^1_4$ forcing absoluteness, and the transcendence properties over the core model K.

Now we are ready to state our main theorems in this chapter:

**Theorem 2.4.1.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then the following are equivalent:
1. Every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-measurable, and
2. $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds.

**Theorem 2.4.2.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, $\mathbf{\Sigma}^1_2$ forcing. Assume the following:

$$\{c \mid \ c \text{ is a Borel code and } B_c \in I_{\mathbb{P}}{}^*\} \in \mathbf{\Sigma}^1_2. \tag{$*$}$$

Then the following are equivalent:
1. Every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-measurable,
2. $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds, and
3. For any real $a$ and $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over $\mathrm{L}[a]$.

Before going to the proofs of these theorems, let us see the general equivalence theorem between $\mathbb{P}$-Baireness and the forcing absoluteness via $\mathbb{P}$:

**Theorem 2.4.3** (Castells)**.** Let $\mathbb{P}$ be a partial order. Then the following are equivalent:
1. Every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-Baire, and
2. $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds.

*Proof.* The idea for this argument goes back to [25, Theorem 3.1].[9]

We first show the direction from $\mathbb{P}$-Baireness to forcing absoluteness. We assume every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-Baire and we show that $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness. To derive a contradiction, suppose it fails. Then there are a $\Sigma^1_3$ formula $\phi$, a real $a$, and a $\mathbb{P}$-generic filter $G$ over $V$ such that $V[G] \models \phi(a)$ but $V \not\models \phi(a)$. This is because any $\Sigma^1_3$ formula is upward absolute from $V$ to $V[G]$ by Shoenfield absoluteness.

Let $\psi$ be the $\Sigma^1_1$ formula such that $\phi = (\exists x)\ (\forall y)\ \psi$. Then there are $p \in G$ and a $\mathbb{P}$-name $\tau$ for a real such that $p \Vdash (\forall y)\ \psi(\tau, y, \check{a})$. By the assumption, in $V$, $(\forall x)\ (\exists y)\ \neg\psi(x, y, a)$. Since $\psi$ is a $\Sigma^1_1$ formula, by the Kondô-Addison theorem [51], there is a $\mathbf{\Sigma}^1_2$ function $g\colon {}^\omega\omega \to {}^\omega\omega$ such that $V \models$ "$(\forall x)\ \neg\psi(x, g(x), a)$".[10]

[9] We would like to thank Neus Castells for providing her notes with a proof of Theorem 2.4.3. Our statement of Theorem 2.4.3 and presentation of the proof differ slightly from Castells's note.

[10] Actually, $g$ can be taken as a $\mathbf{\Pi}^1_1$ function in this case. But for the analogous argument for Theorem 2.4.7, we write $\mathbf{\Sigma}^1_2$.

Now we claim $g\circ f_\tau$ is Baire measurable, where $f_\tau$ is defined in Lemma 2.1.2. It suffices to check that $(g\circ f_\tau)^{-1}([s])$ has the Baire property for each finite sequence of natural numbers $s$, where $[s]=\{x\in {}^\omega\omega \mid x\supseteq s\}$, the basic open set from $s$ in the Baire space. Take any such $s$. Since $g$ is $\mathbf{\Sigma}^1_2$, the set $g^{-1}([s])$ is $\mathbf{\Delta}^1_2$. By the first assumption, it is $\mathbb{P}$-Baire, in particular, $f_\tau^{-1}g^{-1}([s])=(g\circ f_\tau)^{-1}([s])$ has the Baire property in $St(\mathbb{P})$.

The idea is to approximate $\tau,\tau_{g\circ f_\tau}$ and a witness for $\psi$ by a tree and using the absoluteness of the wellfoundedness of the tree between $V$ and $V[G]$, we will derive a contradiction. Let $T$ be a tree on $\omega\times\omega\times\omega$ such that $(\forall x,y)\ \big(\psi(x,y,a)\iff(\exists z)\ (x,y,z)\in[T]\big)$.

Since $f_{\tau_{g\circ f_\tau}}(u)=g\circ f_\tau(u)$ for comeager many $u$, there is a sequence $\langle D_n\mid n\in\omega\rangle$ of dense open sets in $\mathbb{P}$ such that $f_{\tau_{g\circ f_\tau}}(u)=g\circ f_\tau(u)$ for each $u\in\bigcap_{n\in\omega}\bigcup\{O_p\mid p\in D_n\}$. Consider the following tree $U$ on $\mathbb{P}\times\omega\times\omega\times\omega$ in $V$:

$$\begin{aligned}(\vec{p},s,t,v)\in U\iff{}&\vec{p}\text{ is a decreasing sequence in }\mathbb{P},\\&\text{if }\vec{p}=\langle p_i\mid i<n\rangle,\ \text{then }\mathrm{lh}(s)=\mathrm{lh}(t)=\mathrm{lh}(v)=n,\\&(s,t,v)\in T,p_0=p,(\forall i<n)\ p_i\in\bigcap_{j<i}D_j,\ \text{and}\\&(\forall i<n)\ p_i\Vdash\text{“}s\restriction i\subseteq\tau,\ t\restriction i\subseteq\tau_{g\circ f_\tau}\text{”}\end{aligned}$$

We claim that $U$ is wellfounded in $V$ but ill-founded in $V[G]$. Suppose there is an infinite path through $U$ in $V$ and call it $(\vec{p},x,y,z)$. Take any $u\in St(\mathbb{P})$ containing each element in $\vec{p}$ (i.e., any ultrafilter on $\mathbb{P}$ extending the set $\vec{p}$). Then $u\in\bigcap_{n\in\omega}\bigcup\{O_p\mid p\in D_n\}$ and hence $f_{\tau_{g\circ f_\tau}}(u)=g\circ f_\tau(u)$. Furthermore, by the definition of $f_\tau$ and $f_{\tau_{g\circ f_\tau}}$, $f_\tau(u)=x$, $f_{\tau_{g\circ f_\tau}}(u)=g\circ f_\tau(u)=y$ and $(x,y,z)\in[T]$. But this implies $\psi(x,g(x),a)$, contradicting $(\forall x)\ \neg\psi(x,g(x),a)$ in $V$. Hence $U$ is wellfounded in $V$. On the other hand, $U$ is certainly ill-founded in $V[G]$ because $G,\tau^G=f_\tau(G),\tau^G_{g\circ f_\tau}=g\circ f_\tau(G)$ and a witness for $\psi(\tau^G,\tau^G_{g\circ f_\tau},a)$ easily give an infinite path through $U$. Contradiction!

Next we show the direction from forcing absoluteness to $\mathbb{P}$-Baireness. Take any $\mathbf{\Delta}^1_2$ set $A$ and a Baire measurable function $f$ from $St(\mathbb{P})$ to the reals. We show that $f^{-1}(A)$ has the Baire property in $St(\mathbb{P})$.

Since $A$ is $\mathbf{\Delta}^1_2$, there are $\Sigma^1_2$ formulas $\phi$ and $\psi$ defining $A$ with a real parameter $a$, in particular,

$$(\forall x)\ \phi(x,a)\iff\psi(x,a).\tag{$\dagger$}$$

Note that this statement is $\Pi^1_3(a)$. Hence by $\mathbf{\Sigma}^1_3$-absoluteness for $\mathbb{P}$, the statement ($\dagger$) remains true in $V^{\mathbb{P}}$. This is the only part we use the second assumption.

Now we use Shoenfield trees to get the absolute tree representation for $A$ and ${}^\omega\omega\setminus A$ between $V$ and $V^{\mathbb{P}}$. Let $\kappa$ be sufficiently large so that $\kappa$ remains uncountable in $V^{\mathbb{P}}$. Let $U_1$, $U_2$ be Shoenfield trees on $\omega\times\kappa$ for $\phi$ and $\psi$. Since $\kappa$ remains uncountable in $V^{\mathbb{P}}$, the Shoenfield trees for $\phi$ and $\psi$ up to $\kappa$ constructed in $V^{\mathbb{P}}$ are the same as $U_1$, $U_2$ respectively. Moreover, since (†) remains true in $V^{\mathbb{P}}$, we have the following:

$$A = \mathrm{p}[U_1], \quad {}^{\omega}\omega \setminus A = \mathrm{p}[U_2],$$
$$\Vdash_{\mathbb{P}} \text{“}\mathrm{p}[U_1] \cup \mathrm{p}[U_2] = {}^{\omega}\omega,\ \mathrm{p}[U_1] \cap \mathrm{p}[U_2] = \emptyset\text{”}.$$

Let $D_i = \{p \mid p \Vdash \text{“}\tau_f \in \mathrm{p}[U_i]\text{”}\}$ and $O_i = \bigcup\{O_p \mid p \in D_i\}$ for $i = 1, 2$, where $\tau_f$ is from Lemma 2.1.2. Then $D_1 \cup D_2$ is dense in $\mathbb{P}$ and any two elements $p_i$ of $D_i$ are incompatible for $i = 1, 2$. Hence $O_1 \cup O_2$ is dense open in $St(\mathbb{P})$ and $O_1 \cap O_2 = \emptyset$. So it suffices to show that $O_i \setminus f^{-1}(\mathrm{p}[U_i])$ is meager in $St(\mathbb{P})$ for $i = 1, 2$.

We only show that $O_1 \setminus f^{-1}(\mathrm{p}[U_1])$ is meager in $St(\mathbb{P})$. By Fact 2.1.17, it suffices to show that $O_p \setminus f^{-1}(\mathrm{p}[U_1])$ is meager for each $p$ in $D_1$. The following claim is the key point, where we use Banach-Mazur games essentially. Let $\theta$ be sufficiently large regular cardinal.

**Claim 2.4.4.** Let $a$ be any set in $\mathcal{H}_\theta$. Then the set $A$ of all $G \in St(\mathbb{P})$ such that there is a countable elementary substructure $X$ of $\mathcal{H}_\theta$ with $a \in X$ such that $G \cap X$ is $\mathbb{P}$-generic over $X$ is comeager in $St(\mathbb{P})$.

*Proof of Claim 2.4.4.* Fix a set $a$ in $\mathcal{H}_\theta$. We prove the claim by using the Banach-Mazur game $G^{**}\big(A, \mathrm{St}(\mathbb{P})\big)$. By Theorem 1.8.3, it suffices to show that player II has a winning strategy in this game. Since $\{O_p \mid p \in \mathbb{P}\}$ forms a basis in $\mathrm{St}(\mathbb{P})$, we may assume that two players will pick elements of $\mathbb{P}$ instead of nonempty open sets in $\mathrm{St}(\mathbb{P})$.

Instead of specifying a winning strategy for player II, we describe how to win the game for player II. We will also construct a $\le$-decreasing sequence $\langle p_n \mid n \in \omega\rangle$ and an $\subseteq$-increasing sequence $\langle X_n \mid n \in \omega\rangle$ of countable elementary substructures of $\mathcal{H}_\theta$ such that

- $a, \mathbb{P} \in X_0$, $p_{2n-1}, p_{2n} \in X_n$,
- $p_{2n}$ is arbitrarily chosen by player I, and
- any dense set of $\mathbb{P}$ in $X_n$ contains $p_m$ for some $m$.

We can easily arrange this construction by a standard book-keeping argument. Now we are done: Let $X$ be the union of all $X_n$. Then for any $G$ containing each $p_n$, $G \cap X$ is $\mathbb{P}$-generic over $X$ because $G \cap X \supseteq \{p_n \mid n \in \omega\}$ and any dense set of $\mathbb{P}$ in $X$ must contain $p_m$ for some $m$. □ (Claim 2.4.4)

We now prove that $O_p \setminus f^{-1}(\mathrm{p}[U_1])$ is meager if $p \in D_1$. By the claim, it is enough to see that $f(G) \in \mathrm{p}[U_1]$ for $G$ satisfying the property in the claim for some suitable $a$ and $p \in G$. Also we may assume $f(G) = f_{\tau_f}(G)$ because it is true for comeager many $G$ by Lemma 2.1.2.

Take a countable elementary substructure $X$ of $\mathcal{H}_\theta$ for $G$ as in the claim for $a = (\mathbb{P}, U_1, p, f, \tau_f)$. Then $G \cap X$ is $\mathbb{P}$-generic over $X$. Since $p \in D_1$, $p \Vdash \tau_f \in \mathrm{p}[U_1]$ and hence $X \models$ "$p \Vdash \tau_f \in \mathrm{p}[U_1]$". Since $G \cap X \subseteq X$, we can apply forcing theorem to $X$ and $G \cap X$ and get $X[G \cap X] \models \tau_f^{G \cap X} \in \mathrm{p}[U_1]$. By upward absoluteness, $\tau_f^{G \cap X} \in \mathrm{p}[U_1]$ in $V$. Note that $\tau_f^{G \cap X} = f_{\tau_f}(G)$ because for any natural numbers $m$ and $n$,

$$\begin{aligned} \tau_f^{G \cap X}(m) = n &\iff (\exists p \in G \cap X)\ p \Vdash \tau_f(\check{m}) = \check{n} \\ &\iff (\exists p \in G)\ p \Vdash \tau_f(\check{m}) = \check{n} \\ &\iff f_{\tau_f}(G)(m) = n. \end{aligned}$$

Hence $f(G) = f_{\tau_f}(G) \in \mathrm{p}[U_1]$ as desired. □

Now we prove Theorem 2.4.1 and Theorem 2.4.2:

*Proof of Theorem 2.4.1.* By Theorem 2.4.3, it suffices to show that every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-measurable if and only if every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-Baire. By Lemma 2.1.15, it is enough to see that every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-Baire assuming every $\mathbf{\Delta}^1_2$ set of reals is $\mathbb{P}$-measurable.

The following claim is the key point:

**Claim 2.4.5.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing and $\tau$ be a $\mathbb{P}$-name for a real. Then for any $T$ in $\mathbb{P}$, there is a $T' \leq T$ and a Borel function $g \colon [T'] \to {}^\omega\omega$ such that $T' \Vdash \tau = g(\dot{x_G})$.

*Proof of Claim 2.4.5.* This is a combination of Proposition 2.1.18 in this thesis and [90, Proposition 2.3.1]. □

Now take any $\mathbf{\Delta}^1_2$-set $A$ and a Baire measurable function $f$ from $\mathrm{St}(\mathbb{P})$ to the reals. We show that $f^{-1}(A)$ has the Baire property. It suffices to show that $\{T \mid O_T \cap f^{-1}(A)$ is meager or $O_T \setminus f^{-1}(A)$ is meager$\}$ is dense in $\mathbb{P}$.

So take any $T$ in $\mathbb{P}$ and we will find an extension $S$ of $T$ with the above property. By the above claim, there is a $T' \leq T$ and a Borel function $g \colon [T'] \to {}^\omega\omega$ such that $T' \Vdash \tau_f = g(\dot{x_G})$, where $\tau_f$ is the $\mathbb{P}$-name for a real defined in Lemma 2.1.2. Hence, by Lemma 2.1.2, $f = g \circ f_{\dot{x_G}}$ almost everywhere in $O_{T'}$. Since $g^{-1}(A)$ is $\mathbf{\Delta}^1_2$, it is $\mathbb{P}$-measurable by the assumption. By Lemma 2.1.15, $f_{\dot{x_G}}^{-1}(g^{-1}(A)) = (g \circ f_{\dot{x_G}})^{-1}(A)$ has the Baire property. Hence $f^{-1}(A)$ has the Baire property in $O_{T'}$. In particular, there is an $S \leq T'$ such that either $O_S \cap f^{-1}(A)$ is meager or $O_S \setminus f^{-1}(A)$ is meager, as desired. □

*Proof of Theorem 2.4.2.* We have seen the equivalence between the regularity property and forcing absoluteness. We will show the direction from forcing absoluteness to the transcendence property and the direction from the transcendence property to the regularity property.

We first show the direction from forcing absoluteness to the transcendence property. Take a real $a$ and $T$ in $\mathbb{P}$. We will find a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a]$ with $x \in [T]$. But by the assumption $(*)$, the statement "There is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a]$ with $x \in [T]$" is $\boldsymbol{\Sigma}^1_3$ and this is true in a generic extension $V[G]$ with $T \in G$ by the same argument as in Proposition 2.3.3. (Although $\mathbb{P}$ might not be provably $\Delta^1_2$ as we assumed in Proposition 2.3.3, we used it only to see $M \models B_c \in I_{\mathbb{P}}{}^*$ when $B_c \in I_{\mathbb{P}}{}^*$ in $V$ and this is ensured by the assumption $(*)$ and Shoenfield absoluteness without using $\mathbb{P}$ being provably $\Delta^1_2$.) Hence by $\boldsymbol{\Sigma}^1_3$-forcing absoluteness, the statement is also true in $V$.

We show the direction from the transcendence property to the regularity property. Take any $\boldsymbol{\Delta}^1_2$ set $A$ and we will show that $A$ is $\mathbb{P}$-measurable. Take any $T$ in $\mathbb{P}$.

**Case 1**: $\omega_1^{\mathrm{L}[a]} < \omega_1^V$ for every real $a$.

In this case, we can actually show that every $\boldsymbol{\Sigma}^1_2$ set of reals is $\mathbb{P}$-measurable as follows (now we assume $A$ is $\boldsymbol{\Sigma}^1_2$ instead of $\boldsymbol{\Delta}^1_2$): Pick a real $a$ such that $T \in \mathrm{L}[a]$ and $A$ is $\Sigma^1_2(a)$ and $\mathrm{L}[a]$ contains a parameter of the $\Sigma^1_2$ definition of $\mathbb{P}$. Take a Shoenfield tree $U$ for $A$ in $\mathrm{L}[a]$, i.e., $A = \mathrm{p}[U]$ Then there is an extension $T' \leq T$ in $\mathbb{P}^{\mathrm{L}[a]}$ such that either $\mathrm{L}[a] \models$ "$T' \Vdash \dot{x_G} \in \mathrm{p}[U]$" or $\mathrm{L}[a] \models$ "$T' \Vdash \dot{x_G} \notin \mathrm{p}[U]$", where $\dot{x_G}$ is a canonical $\mathbb{P}$-name for a generic real. We may assume that $\mathrm{L}[a] \models$ "$T' \Vdash \dot{x_G} \in \mathrm{p}[U]$". (The other case is similar.)

By the assumption, the set of all dense sets of $\mathbb{P}^{\mathrm{L}[a]}$ in $\mathrm{L}[a]$ is countable. Hence there is a countable transitive model $M \subseteq \mathrm{L}[a]$ of ZFC such that $M$ contains all the reals and all the dense subsets of $\mathbb{P}^{\mathrm{L}[a]}$ in $\mathrm{L}[a]$. (E.g., take a countable elementary submodel of $\mathrm{L}_\theta[a]$ containing all the reals and the dense subsets in $\mathrm{L}[a]$ and collapse it.) Since $\mathbb{P}$ is $\boldsymbol{\Sigma}^1_2$, $\mathrm{L}[a]$ computes $\mathbb{P}$ correctly, $M$ also computes $\mathbb{P}$ correctly. Now we apply the strong properness of $\mathbb{P}$ and get an extension $T'' \leq T$ such that $T''$ is $(M, \mathbb{P})$-generic condition and hence also $(\mathrm{L}[a], \mathbb{P})$-generic. Therefore maximal antichains in $\mathbb{P}^{\mathrm{L}[a]}$ stay maximal in $V$ below $T''$. Together with the condition that the set of all dense sets in $\mathrm{L}[a]$ is countable, we can conclude that almost all the reals are $\mathbb{P}$-generic over $\mathrm{L}[a]$ below $T''$. Since we have $\mathrm{L}[a] \models$ "$T' \Vdash \dot{x_G} \in \mathrm{p}[U]$", almost all the reals below $T''$ belong to $\mathrm{p}[U] = A$, as desired.

**Case 2**: $\omega_1^{\mathrm{L}[a]} = \omega_1^V$ for some real $a$.

The idea for this argument goes back to [19, Proposition 2.1]. Pick a real $a$ with $T \in \mathrm{L}[a]$ such that $\omega_1^{\mathrm{L}[a]} = \omega_1^V$ and $A$ is $\Delta^1_2(a)$. The idea is to decompose $[T] \cap A$ and $[T] \setminus A$ into Borel sets in an absolute way between $\mathrm{L}[a]$ and $V$, then a Borel set containing a quasi-$\mathbb{P}$-generic real over $\mathrm{L}[a]$ must be $I_{\mathbb{P}}{}^*$-positive and below that Borel set we will find an extension of $T$ as a witness for the $\mathbb{P}$-measurability of $A$.

Since $[T] \cap A$ and $[T] \setminus A$ are $\Sigma^1_2(a)$ sets, there are Shoenfield trees $U_1$ and $U_2$ in $\mathrm{L}[a]$ for $[T] \cap A$ and $[T] \setminus A$ respectively. From these trees, we can naturally decompose $[T] \cap A$ and $[T] \setminus A$ into $\omega_1$ many Borel sets as in [66, 2F.1–2F.3], i.e., there are sequences $\langle c_\alpha \mid \alpha < \omega_1 \rangle$, $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ of Borel codes in $\mathrm{L}[a]$ such that $[T] \cap A = \bigcup_{\alpha<\omega_1} B_{c_\alpha}$ and $[T] \setminus A = \bigcup_{\alpha<\omega_1} B_{d_\alpha}$. The point is that the above equations are absolute between $\mathrm{L}[a]$ and $V$ because those two sequences only depend on $U_1, U_2$, and $\omega_1$, and we have $\omega_1^{\mathrm{L}[a]} = \omega_1^V$ as we assumed.

By assumption, there is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a]$ with $x \in [T]$. Hence there is an $\alpha < \omega_1$ such that either $x \in B_{c_\alpha}$ or $x \in B_{d_\alpha}$. Without loss of generality, we may assume $x \in B_{c_\alpha}$. Since $c_\alpha$ is in $\mathrm{L}[a]$, by the definition of quasi-$\mathbb{P}$-genericity, $B_{c_\alpha}$ is not in ${I_\mathbb{P}}^*$. Since every Borel set is $\mathbb{P}$-measurable, there is a condition $T'$ such that $[T'] \setminus B_{c_\alpha} \in I_\mathbb{P}$. Since $B_{c_\alpha} \subseteq [T] \cap A$, we have $T' \leq T$ and $[T'] \setminus A \in I_\mathbb{P}$, as desired. □

We do not know whether we could eliminate the condition $(*)$ in Theorem 2.4.2 under some reasonable assumptions for $\mathbb{P}$. For further discussions about this issue, see § 2.6.

So far we have investigated the connection between $\mathbb{P}$-measurability for $\boldsymbol{\Delta}^1_2$ sets, $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-absoluteness, and the transcendence property over L. How about $\mathbb{P}$-measurability for $\boldsymbol{\Sigma}^1_2$ sets? Is there any such equivalence? Solovay proved that every $\boldsymbol{\Sigma}^1_2$ set has the Baire property if and only if for any real $a$, the set of all Cohen reals over $\mathrm{L}[a]$ is comeager. He also proved the same equivalence for Lebesgue measurability and random reals. Similar equivalences have been obtained for other forcings (see, e.g., [20, Proposition 5.12]). We now give a general equivalence result for this phenomenon:

**Theorem 2.4.6.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, $\boldsymbol{\Sigma}^1_2$ forcing. Assume

$$\{c \mid \ c \text{ is a Borel code and } B_c \in {I_\mathbb{P}}^*\} \in \boldsymbol{\Sigma}^1_2, \tag{$*$}$$

and

$$I_\mathbb{P} \text{ is Borel-generated or } I_\mathbb{P} = N_\mathbb{P}, \tag{$**$}$$

where $I_\mathbb{P}$ is *Borel-generated* if any element of $I_\mathbb{P}$ is a subset of an element of $I_\mathbb{P}$ which is Borel.

Then the following are equivalent:
1. Every $\boldsymbol{\Sigma}^1_2$ set of reals is $\mathbb{P}$-measurable, and
2. For any real $a$, ${}^\omega\omega \setminus \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{L}[a]\} \in {I_\mathbb{P}}^*$.

The ideal $I_\mathbb{P}$ is Borel-generated if $\mathbb{P}$ is ccc and $I_\mathbb{P} = N_\mathbb{P}$ for all the typical non-ccc forcings admitting a fusion argument as we discussed in Lemma 2.1.12. Hence the condition $(**)$ is always true for typical strongly arboreal forcings.

*Proof.* We show the direction from the regularity property to the transcendence property. Take any real $a$ and we show that the set $A = \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{L}[a]\}$ is of measure one with respect to ${I_\mathbb{P}}^*$. Suppose not. Then ${}^\omega\omega \setminus A \notin {I_\mathbb{P}}^*$.

By the assumption $(*)$, ${}^\omega\omega \setminus A$ is $\mathbf{\Sigma}^1_2$. So by the assumption 1, it is $\mathbb{P}$-measurable. Hence there is a $T$ in $\mathbb{P}$ such that $[T] \setminus ({}^\omega\omega \setminus A) = [T] \cap A \in I_{\mathbb{P}}$. We show that this cannot happen.

**Case 1**: $I_{\mathbb{P}}$ is Borel-generated.

Since $[T] \cap A \in I_{\mathbb{P}}$, there is a Borel set $B \subseteq [T]$ in $I_{\mathbb{P}}$ such that $[T] \cap A \subseteq B$. Let $c$ be a Borel code for $B$. By Theorem 2.4.2, there is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a, c]$ with $x \in [T]$. Since $B \in I_{\mathbb{P}}$, $x \notin B$. But this is impossible because $x$ is also quasi-$\mathbb{P}$-generic over $\mathrm{L}[a]$ and hence $x \in [T] \cap A \subseteq B$.

**Case 2**: $I_{\mathbb{P}} = N_{\mathbb{P}}$.

In this case, $[T] \cap A$ is $\mathbb{P}$-null, hence there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. By Theorem 2.4.2, there is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a]$ with $x \in [T']$. Hence $x \in [T'] \cap A$, a contradiction.

We now show the direction from the transcendence property to the regularity property. Take any $\mathbf{\Sigma}^1_2$ set $A$. We show that $A$ is $\mathbb{P}$-measurable. Let $T$ be in $\mathbb{P}$. We will find an extension $T'$ of $T$ approximating $A$ as in the definition of $\mathbb{P}$-measurability. If $[T] \cap A \in {I_{\mathbb{P}}}^*$, we are done. So we assume $[T] \cap A \notin {I_{\mathbb{P}}}^*$.

**Case 1**: $\omega_1^{\mathrm{L}[a]} < \omega_1^V$ for every real $a$.

The same as Case 1 in Theorem 2.4.2.

**Case 2**: $\omega_1^{\mathrm{L}[a]} = \omega_1^V$ for some real $a$.

Let $a$ be a real such that $[T] \cap A$ is $\Sigma^1_2(a)$ and $\omega_1^{\mathrm{L}[a]} = \omega_1^V$. Then we have a Shoenfield tree in $\mathrm{L}[a]$ for $[T] \cap A$ and we get an $\omega_1$ many Borel decomposition of $[T] \cap A$ into Borel sets $\{B_{c_\alpha} \mid \alpha < \omega_1\}$ with $c_\alpha \in \mathrm{L}[a]$ for each $\alpha$ as in the proof of Theorem 2.4.2. Since $[T] \cap A \notin {I_{\mathbb{P}}}^*$ and the set of quasi-$\mathbb{P}$-generic reals over $\mathrm{L}[a]$ is of measure one with respect to ${I_{\mathbb{P}}}^*$ by the assumption 2, there is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{L}[a]$ with $x \in [T] \cap A$, so there is an $\alpha$ such that $x \in B_{c_\alpha}$.

The rest is the same as in the proof from the transcendence property to the regularity property in Theorem 2.4.2. Since $c_\alpha \in \mathrm{L}[a]$ and $x$ is quasi-$\mathbb{P}$-generic over $\mathrm{L}[a]$, $B_{c_\alpha} \notin {I_{\mathbb{P}}}^*$. Since any Borel set is $\mathbb{P}$-measurable, there is a $T'$ in $\mathbb{P}$ such that $[T'] \setminus B_{c_\alpha} \in I_{\mathbb{P}}$. But $B_{c_\alpha} \subseteq [T] \cap A$. Hence $T' \leq T$ and $[T'] \setminus A \in I_{\mathbb{P}}$, as desired. $\square$

We do not know if there is a forcing absoluteness statement corresponding to $\mathbb{P}$-measurability for $\mathbf{\Sigma}^1_2$ sets in general. For some forcings, it is true, e.g., Judah [42] proved that $\mathbf{\Sigma}^1_3$-$\mathbb{D}$-absoluteness is equivalent to the Baire property (in the usual topology in the Baire space) for all $\mathbf{\Sigma}^1_2$ sets. (The same equivalence holds for amoeba forcing and Lebesgue measurability.) But we do not know how to uniformly find a forcing corresponding to $\mathbb{P}$-measurability for $\mathbf{\Sigma}^1_2$ sets given $\mathbb{P}$.

We have linked $\mathbb{P}$-measurability for $\mathbf{\Delta}^1_2$ sets and $\mathbf{\Sigma}^1_2$ sets with forcing absoluteness and the transcendence properties over L. How about $\mathbb{P}$-measurability for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_3$ sets? Unfortunately we cannot prove the equivalence between $\mathbb{P}$-measurability for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness in ZFC in general, e.g., start from L and add $\omega_1$ many Cohen reals, then in this model, $\mathbf{\Sigma}^1_4$-forcing absoluteness for Cohen forcing holds but there is a $\mathbf{\Sigma}^1_2$ set of reals without the Baire property. With an additional assumption (sharps for sets), we will establish the analogues of the equivalence results we have obtained for $\mathbb{P}$-measurability for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_3$ sets, $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness, and the transcendence property over the core model K:

**Theorem 2.4.7.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing.

1. Assume that every real has a sharp and that $\mathbf{\Delta}^1_2$-determinacy fails. Then if every $\mathbf{\Delta}^1_3$ set of reals is $\mathbb{P}$-measurable, then $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness holds.
2. Suppose that every set has a sharp. Then if $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness holds, then every $\mathbf{\Delta}^1_3$ set of reals is $\mathbb{P}$-measurable.

In particular, if every set has a sharp, then either $\mathbf{\Delta}^1_2$-determinacy holds or every $\mathbf{\Delta}^1_3$ set of reals is $\mathbb{P}$-measurable if and only if $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness holds.

**Theorem 2.4.8.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_2$ forcing. Suppose every real has a sharp. Then either $\mathbf{\Delta}^1_2$-determinacy holds or the following are equivalent:

1. Every $\mathbf{\Delta}^1_3$ set of reals is $\mathbb{P}$-measurable,
2. $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness holds, and
3. For any real $a$ and any $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over $\mathrm{K}_a$, where $\mathrm{K}_a$ is the core model constructed from $a$-mice.

**Theorem 2.4.9.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_2$ forcing. Suppose every real has a sharp. Assume

$$I_{\mathbb{P}} \text{ is Borel-generated or } I_{\mathbb{P}} = N_{\mathbb{P}}. \tag{$**$}$$

Then either $\mathbf{\Delta}^1_2$-determinacy holds or the following are equivalent:

1. Every $\mathbf{\Sigma}^1_3$ set of reals is $\mathbb{P}$-measurable, and
2. For any real $a$, ${}^{\omega}\omega \setminus \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{K}_a\} \in {I_{\mathbb{P}}}^*$, where $\mathrm{K}_a$ is the core model constructed from $a$-mice.

Note that the additional assumption "Every set has a sharp" is equivalent to every $\mathbf{\Sigma}^1_2$ set of reals being $\mathbb{P}$-Baire for any $\mathbb{P}$ (or universally Baire). So our setting is that, assuming that $\mathbf{\Sigma}^1_2$ sets of reals behave nicely for any forcing $\mathbb{P}$, we consider the equivalence mentioned above.

Also note that we do not need the analogue of the assumption $(*)$ in Theorem 2.4.2 in the above theorems because the set of all Borel codes whose decodes are in ${I_{\mathbb{P}}}^*$ is $\Pi^1_2$ as we proved in Proposition 2.3.3.

*Proof of Theorem 2.4.7.* We start with proving the first item of this theorem, i.e., we show the direction from the regularity property to forcing absoluteness assuming that every real has a sharp and that $\mathbf{\Delta}^1_2$-determinacy fails. First note that we may assume that every $\mathbf{\Delta}^1_3$ set is $\mathbb{P}$-Baire by the same argument for the same direction in Theorem 2.4.1. The argument is basically the same as in

Theorem 2.4.3. What we need is to uniformize a $\mathbf{\Pi}^1_2$ relation by a $\mathbf{\Sigma}^1_3$ function as we uniformized a $\mathbf{\Pi}^1_1$ relation by a $\mathbf{\Sigma}^1_2$ function in Theorem 2.4.3. The rest is exactly the same. But such uniformization is possible assuming the failure of $\mathbf{\Delta}^1_2$-determinacy.

**Theorem 2.4.10** (Folklore[11])**.** Suppose every real has a sharp. Then either $\mathbf{\Delta}^1_2$-determinacy holds or $\mathbf{\Sigma}^1_3$ has the uniformization property, i.e., any $\mathbf{\Sigma}^1_3$ relation can be uniformized by a $\mathbf{\Sigma}^1_3$ function.[12]

*Proof of Theorem 2.4.10.* It suffices to show that every $\mathbf{\Pi}^1_2$ relation can be uniformized by a $\mathbf{\Sigma}^1_3$ function. Suppose $\mathbf{\Delta}^1_2$-determinacy fails. By Theorem 1.12.5, there is a real $a_0$ such that for any $a \geq_{\mathrm{T}} a_0$, the $a$-relativized version of the core model $\mathrm{K}_a$ exists and every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_a$ and $V$. (Recall that $\leq_{\mathrm{T}}$ is the Turing order on the reals.) For each $a \geq_{\mathrm{T}} a_0$, let $<_a$ be the canonical good $\Delta^1_3(a)$ well-ordering on the reals in $\mathrm{K}_a$ ensured by Theorem 1.11.2. Given a real $b$ and a $\Pi^1_2(b)$ relation $R$, define the uniformization function $f$ as follows:

$$f(x) = y \iff y \text{ is the } <_{\langle x, a_0, b\rangle}\text{-least element with } (x,y) \in R,$$

where $\langle x, a_0, b\rangle$ is the real coding $x, a_0$ and $b$. For each $x \in \mathrm{dom}(R)$, such a $y$ always exists because every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_{\langle x, a_0, b\rangle}$ and $V$. So $f$ uniformizes $R$ and considering the fact that $<_a$ is a good $\Delta^1_3(a)$ well-ordering in $\mathrm{K}_a$ for each $a \geq_{\mathrm{T}} a_0$ in a uniform way, it is easy to see that $f$ is $\mathbf{\Sigma}^1_3$. □

Now we show the second item of Theorem 2.4.7, i.e., the direction from forcing absoluteness to the regularity property assuming sharps for sets. The argument is the same as for the implication in Theorem 2.4.3. By Theorem 1.12.4, it suffices to check that every real has a sharp in $V[G]$ and $u_2^V = u_2^{V[G]}$ for any $\mathbb{P}$-generic filter $G$ over $V$.

We first show that every real has a sharp in $V[G]$ whenever $G$ is a $\mathbb{P}$-generic filter over $V$ assuming sharps for sets in $V$. Take any $\mathbb{P}$-generic filter $G$ over $V$ and a real $x$ in $V[G]$. Let $\tau$ be a $\mathbb{P}$-name with $\tau^G = x$. Since we have a sharp for $(\tau, \mathbb{P})$ in $V$, we have an elementary embedding $j$ from $\mathrm{L}(\tau, \mathbb{P})$ to itself with critical point above the ranks of $\tau$ and $\mathbb{P}$ in $V$. Since the critical point of $j$ is above the ranks of $\tau$ and $\mathbb{P}$, $j$ preserves $\tau$ and $\mathbb{P}$ and we can lift $j$ to $\bar{\jmath}\colon \mathrm{L}(\tau,\mathbb{P})[G] \to \mathrm{L}(\tau,\mathbb{P})[G]$ in $V[G]$ in the following standard way:

$$\bar{\jmath}(\sigma^G) = j(\sigma)^G,$$

for any $\mathbb{P}$-name $\sigma$ in $\mathrm{L}(\tau,\mathbb{P})$. Since $x = \tau^G \in \mathrm{L}(\tau,\mathbb{P})[G]$, $\bar{\jmath}\restriction \mathrm{L}[x]$ gives us a non-trivial elementary embedding from $\mathrm{L}[x]$ to itself, hence $x^{\#}$ exists as desired.

[11] The author would like to thank Hugh Woodin for pointing out this fact to him.

[12] Since $\mathbf{\Delta}^1_2$-determinacy implies that $\mathbf{\Pi}^1_3$ has the uniformization property, this fact states the dichotomy of the uniformization property for $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$.

We now show that $u_2^V = u_2^{V[G]}$ for any $\mathbb{P}$-generic filter $G$ over $V$ which will complete the proof. First note that $u_2$ is the length of a provably $\Delta^1_3$ prewellordering given in Schlicht [75, Example 3.2.7] assuming sharps for reals. But by a result of Schlicht [75, Theorem 2.1.9], the length of the prewellordering is the same between in $V$ and $V[G]$, assuming sharps for sets and $\mathbb{P}$ being proper. Therefore, $u_2^V = u_2^{V[G]}$. □

*Proof of Theorem 2.4.8.* In Theorem 2.4.7, we have seen the equivalence between the regularity property and forcing absoluteness. We show the direction from forcing absoluteness to the transcendence property and the one from the transcendence property to the regularity property. (Note that the assumption, the existence of sharps for reals, is weaker than the existence of sharps for sets. But we used sharps for sets only for (2) in Theorem 2.4.7, i.e., for the direction from forcing absoluteness to the regularity property, which we will not use here. We will prove the equivalence of the three statements just from sharps for reals.)

We show the direction from forcing absoluteness to the transcendence property. All we need is that the statement "there is a quasi-$\mathbb{P}$-generic real $x$ over $\mathrm{K}_a$ with $x \in [T]$" is $\mathbf{\Sigma}^1_4$ for each real $a$ and each $T \in \mathbb{P}$. But this is true by Proposition 2.3.3 and the fact that the set of reals in $\mathrm{K}_a$ is $\Sigma^1_3(a)$ in $V$.

We now show the direction from the transcendence property to the regularity property. The argument is basically the same as the one in Theorem 2.4.2. Assume the failure of $\mathbf{\Delta}^1_2$-determinacy. By Theorem 1.12.5, there is a real $a_0$ such that $\mathrm{K}_a$ exists and every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_a$ and $V$ for any $a \geq_{\mathrm{T}} a_0$.

**Case 1.** $\omega_1^{\mathrm{K}_a} < \omega_1^V$ for every real $a \geq_{\mathrm{T}} a_0$.

As in Theorem 2.4.2, we can conclude that every $\mathbf{\Delta}^1_3$ set of reals (even $\mathbf{\Sigma}^1_3$ set of reals) is $\mathbb{P}$-measurable by using the fact that every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_a$ for $a \geq_{\mathrm{T}} a_0$.

**Case 2.** $\omega_1^{\mathrm{K}_a} = \omega_1^V$ for some real $a$.

We need the absolute decomposition of $\mathbf{\Sigma}^1_3$ sets into Borel sets between $\mathrm{K}_a$ and $V$ for some real $a \geq_{\mathrm{T}} a_0$. The following result is essential; its proof was communicated to us by Ralf Schindler:

**Theorem 2.4.11** (Schindler)**.** If $u_2^{\mathrm{K}_a} < u_2^V$ for every real $a \geq_{\mathrm{T}} a_0$, then $\omega_1^{\mathrm{K}_a} < \omega_1^V$ for every real $a \geq_{\mathrm{T}} a_0$.

*Proof.* Here we use the machinery of inner model theory.

For simplicity, we assume $\mathrm{K}_{a_0} = \mathrm{K}$ and only prove $\omega_1^{\mathrm{K}} < \omega_1^V$ assuming $u_2^{\mathrm{K}_a} < u_2^V$ for each real $a$. The general case will be proved in the same way.

Toward a contradiction, we assume $\omega_1^{\mathrm{K}} = \omega_1^V$. The following is the first point:

**Claim 2.4.12.** Let $a$ be a real. The mouse $\mathrm{K}_a|\omega_1$ is universal for countable $a$-mice, i.e., $M \leq^* \mathrm{K}_a|\omega_1$ for any countable $a$-mouse $M$, where $\leq^*$ is the mouse order.

*Proof of Claim 2.4.12.* Suppose there is a countable $a$-mouse $M$ with $M >^* \mathrm{K}_a|\omega_1$. Coiterate $M$ and $\mathrm{K}_a|\omega_1$ and let $\mathcal{T},\mathcal{U}$ be the resulting trees for $M$ and $\mathrm{K}_a|\omega_1$ respectively.

**Case 1**: $\mathrm{lh}(\mathcal{T})$ is countable.

Since $M >^* \mathrm{K}_a|\omega_1$, $\mathcal{U}$ does not have a drop. But then the last model of $\mathcal{U}$ cannot be an initial segment of the last model of $\mathcal{T}$ since the length of $\mathcal{T}$ is countable, a contradiction.

**Case 2**: $\mathrm{lh}(\mathcal{T})$ is uncountable.

Since $M >^* \mathrm{K}_a|\omega_1$, $\mathcal{U}$ does not have a drop. If $\mathcal{U}$ was non-trivial, then the final model of $\mathcal{U}$ would be non-sound and could not be a proper initial segment of the final model of $\mathcal{T}$. Hence $\mathcal{U}$ is trivial and $\mathrm{K}_a|\omega_1$ is an initial segment of the final model of $\mathcal{T}$. But this means $\omega_1$ is a limit of critical points of embeddings via $\mathcal{T}$, hence $\omega_1$ is inaccessible in $\mathrm{K}_a$, contradicting the assumption $\omega_1^{\mathrm{K}_a} = \omega_1^{\mathrm{K}} = \omega_1^V$. □ (Claim 2.4.12)

**Case 1**: There is a real $a$ such that $a^{¶}$ does not exist.

This case was taken care of by Steel and Welch. In [81, Lemma 3.6], they assumed $u_2 = \omega_2$, which is stronger than $u_2^{\mathrm{K}_a} < u_2^V$ for each real $a$, and proved there is a countable mouse stronger than $\mathrm{K}|\omega_1$ with respect to mouse order. But assuming $\omega_1^{\mathrm{K}} = \omega_1^V$ and the non-existence of $0^{¶}$, we can run the same argument only assuming $u_2^{\mathrm{K}} < u_2^V$ and get the same conclusion. Furthermore, we can easily relativize this argument to $\mathrm{K}_a$. Hence assuming $\omega_1^{\mathrm{K}} = \omega_1^V$ (even $\omega_1^{\mathrm{K}_a} = \omega_1^V$) and the non-existence of $a^{¶}$, if $u_2^{\mathrm{K}_a} < u_2^V$, then there is an $a$-mouse stronger than $\mathrm{K}_a|\omega_1$ with respect to mouse order, which contradicts the $a$-relativized version of Claim 2.4.12.

**Case 2**: For every real $a$, $a^{¶}$ exists.

This case is new. Since $u_2^{\mathrm{K}} < u_2^V$, there is a real $a$ such that $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$. The idea is to use $a^{\dagger}$ (which exists since $a^{¶}$ exists) and linearly iterate it with the lower measure in $a^{\dagger}$ with length $\omega_1$. Then the height of the last model is bigger than $u_2^{\mathrm{K}}$ since $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$. Now we restrict this linear iteration map to K in $a^{\dagger}$ constructed up to the point with the top measure. The point is this is an iteration map on it and the final model of this iteration has height bigger than $u_2^{\mathrm{K}}$. Since it is a countable mouse, by Claim 2.4.12, we get a countable mouse in K with the same property, which yields a contradiction by a standard boundedness argument.

We discuss this idea in detail. Let $i$ be the linear iteration map of $a^{\dagger}$ derived from the iterated ultrapower starting from the lower measure in it with length $\omega_1$. Then the target $N$ of $i$ has height bigger than $u_2^{\mathrm{K}}$ since $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$, the critical point of $i$ goes to $\omega_1$, and $N$ has a cardinal bigger than $\omega_1$ and $a \in N$. Let $\mathrm{K}^{a^{\dagger}|\Omega}$ be the K in $a^{\dagger}|\Omega$, where $\Omega$ is the critical point of the top measure in

$a^\dagger$.[13] Then $\mathrm{K}^{a^\dagger|\Omega}$ is a mouse and we call it $M$.

We claim that if we restrict $i$ to $M$, then it is an iteration map on $M$. Since $i$ is from a linear iteration of ultrapowers via measures, by applying the result of Schindler [72, Corollary 3.1] in each ultrapower in the iteration, we can prove that the restriction of $i$ to $M$ is an iteration with length $\omega_1$ (which itself might be very complicated). Moreover, the final model of this iteration has height greater than $u_2^{\mathrm{K}}$ because $i$ maps $\Omega$ greater than or equal to $(\omega_1^+)^{\mathrm{L}[a]}$. Let us call the tree of this iteration $\mathcal{T}$ and let $M_\alpha$ be the $\alpha$th iterate via $\mathcal{T}$ and $i^{\mathcal{T}}_{\alpha,\beta}\colon M_\alpha \to M_\beta$ be the induced maps for $\alpha \leq \beta \leq \omega_1$.

Since $M$ is a countable mouse, by Claim 2.4.12, there is an $\alpha_0 < \omega_1$ such that $M \leq^* \mathrm{K}|\alpha_0$. We will show that $\mathrm{K}|\alpha_0$ has the same property, i.e., there is an iteration from $\mathrm{K}|\alpha_0$ with length $\omega_1$ such that the height of the final model is greater than $u_2^{\mathrm{K}}$. (Note that there might be a drop.) Coiterate $\mathrm{K}|\alpha_0$ and $M$ and let $\pi\colon M \to N$ be the resulting map. Note that there is no drop from the $M$-side because $M \leq^* \mathrm{K}|\alpha_0$.

We will construct $\langle N_\alpha \mid \alpha \leq \omega_1\rangle$, $\langle \pi_\alpha\colon M_\alpha \to N_\alpha \mid \alpha \leq \omega_1\rangle$, and $\langle i^{\mathcal{U}}_{\alpha,\beta}\colon N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1\rangle$ with the following properties:

(1) The diagrams below all commute,

(2) $M_\alpha \sim^* N_\alpha \sim^* M_{\alpha+1}$ for each $\alpha$, i.e., they are equal with respect to mouse order,

(3) $N_\alpha$ is the direct limit of $N_\beta$ ($\beta < \alpha$) for limit $\alpha$, and

(4) $i^{\mathcal{U}}_{\alpha,\alpha+1}$ and $\pi_{\alpha+1}$ are the maps resulting from the comparison between $N_\alpha$ and $M_{\alpha+1}$ for each $\alpha$.

$$\begin{tikzcd}
\mathrm{K}|\alpha_0 \arrow[r, squiggly] & N = N_0 \arrow[r, "i^{\mathcal{U}}_{0,1}"] & N_1 \arrow[r, "i^{\mathcal{U}}_{1,2}"] & \cdots \arrow[r] & N_\alpha \arrow[r, "i^{\mathcal{U}}_{\alpha,\alpha+1}"] & \cdots \arrow[r] & N_{\omega_1} \\
 & M = M_0 \arrow[u, "\pi=\pi_0"'] \arrow[r, "i^{\mathcal{T}}_{0,1}"'] & M_1 \arrow[u, "\pi_1"'] \arrow[r, "i^{\mathcal{T}}_{1,2}"'] & \cdots \arrow[r] & M_\alpha \arrow[u, "\pi_\alpha"'] \arrow[r, "i^{\mathcal{T}}_{\alpha,\alpha+1}"'] & \cdots \arrow[r] & M_{\omega_1} \arrow[u, "\pi_{\omega_1}"']
\end{tikzcd}$$

The above properties uniquely specify $\langle N_\alpha \mid \alpha \leq \omega_1\rangle$, $\langle \pi_\alpha\colon M_\alpha \to N_\alpha \mid \alpha \leq \omega_1\rangle$, and $\langle i^{\mathcal{U}}_{\alpha,\beta}\colon N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1\rangle$. Hence it suffices to check (1) and (2) above for this construction.

For (1), it suffices to show that $i^{\mathcal{U}}_{\alpha,\alpha+1} \circ \pi_\alpha = \pi_{\alpha+1} \circ i^{\mathcal{T}}_{\alpha,\alpha+1}$ for each $\alpha$. By the Dodd-Jensen Lemma (Theorem 1.11.4), any two iteration maps without drops from a mouse to a mouse are the same. By (2) for $\alpha$, $\pi_\alpha$, $\pi_{\alpha+1}$, $i^{\mathcal{T}}_{\alpha,\alpha+1}$, and $i^{\mathcal{U}}_{\alpha,\alpha+1}$ are all iteration maps without drops. Hence we get the desired commutativity. (2) follows from the fact that all the maps constructed before are simple iteration maps.

Since the height of $N_{\omega_1}$ is greater than or equal to that of $M_{\omega_1}$, there is an iteration from $\mathrm{K}|\alpha_0$ with length $\omega_1$ whose final model has height greater than $u_2^{\mathrm{K}}$,

[13] Note that $a^\dagger|\Omega$ is a transitive model of ZFC and obviously there is no inner model with a Woodin cardinal in that model. Hence by Theorem 1.11.2, one can construct K in $a^\dagger|\Omega$.

as desired.

Since $\mathrm{K}|\alpha_0$ is in K and $\alpha_0$ is countable in K, there is a real $x$ in K coding $\mathrm{K}|\alpha_0$. We show that the height of $N_{\omega_1}$ is less than $(\omega_1^+)^{\mathrm{L}[x]}$. In $\mathrm{L}[x]$, we collapse $\omega_1^V$ with the forcing $\mathrm{Coll}(\omega, \omega_1^V)$. Let $g\colon \omega \to \omega_1^V$ be a generic surjection over $\mathrm{L}[x]$. Since $K|\alpha_0$ is coded by $x$ and the length of iteration is $\omega_1^V$ which is countable in $\mathrm{L}[x][g]$ witnessed by $g$, by the boundedness lemma in $\mathrm{L}[x][g]$, the height of $N_{\omega_1}$ is less than $\omega_1^{\mathrm{L}[x][g]} = (\omega_1^+)^{\mathrm{L}[x]}$, as desired. Since $x$ is in $K$, $(\omega_1^+)^{\mathrm{L}[x]} < u_2^K$ and hence the height of $N_{\omega_1}$ is less than $u_2^K$. But the height was greater than $u_2^K$. Contradiction! □

Now by the assumption in Case 2 and Theorem 2.4.11, there is a real $a$ such that $\omega_1^{\mathrm{K}_a} = \omega_1^V$ and $u_2^{\mathrm{K}_a} = u_2^V$. By Theorem 1.12.4, the Martin-Solovay trees for $\boldsymbol{\Sigma}^1_3$ sets are absolute between $\mathrm{K}_a$ and $V$. Hence we get the absolute decomposition of $\boldsymbol{\Sigma}^1_3$ sets into Borel sets between $\mathrm{K}_a$ and $V$, as desired. The rest is exactly the same as in Theorem 2.4.2. □

*Proof of Theorem 2.4.9.* The argument is exactly the same as Theorem 2.4.6 by replacing $\mathrm{L}[a]$ with $\mathrm{K}_a$ and using the analogous facts about $\mathrm{K}_a$ stated in Theorem 1.11.2 and Theorem 1.12.5. □

## 2.5 Applications

We now use our theorems to answer some open questions in set theory of the reals.

The first one is about Silver forcing $\mathbb{V}$, whose conditions are uniform perfect trees on 2 ordered by inclusion, where a tree $T$ on 2 is *uniform* if for any two nodes $s$ and $t$ of $T$ with the same length, $s^\frown\langle i\rangle \in T \iff t^\frown\langle i\rangle \in T$ for $i = 0, 1$. In [19], Brendle, Halbeisen, and Löwe proved that every $\boldsymbol{\Delta}^1_2$ set of reals is $\mathbb{V}$-measurable assuming that for any real $a$ there is a quasi-$\mathbb{V}$-generic real over $\mathrm{L}[a]$. Then they asked whether the converse is true. We answer this question positively:

**Proposition 2.5.1.** Assume every $\boldsymbol{\Delta}^1_2$ set of reals is $\mathbb{V}$-measurable. Then for any real $a$, there is a quasi-$\mathbb{V}$-generic real over $\mathrm{L}[a]$.

*Proof.* Since Silver forcing is strongly arboreal and proper, by Theorem 2.4.2, it suffices to show that the set of Borel codes with $B_c \in {I_{\mathbb{V}}}^*$ is $\boldsymbol{\Sigma}^1_2$. We use the following fact:

**Fact 2.5.2** (Zapletal)**.** Let $G$ be the graph on ${}^\omega 2$ connecting two binary sequences if they differ in exactly one place. Let $I$ be the $\sigma$-ideal generated by Borel $G$-independent sets (i.e., Borel sets in ${}^\omega 2$ such that any two distinct elements of them are not connected by $G$). Then every analytic set is either in $I$ or contains $[T]$ for some $T \in \mathbb{V}$.

*Proof.* See [90, Lemma 2.3.37]. □

We show how to use Fact 2.5.2 to prove Proposition 2.5.1. We first show that $I \subseteq I_{\mathbb{V}}$. It suffices to see that every Borel $G$-independent set is in $N_{\mathbb{V}}$. Take such Borel set $B$. Since every Borel set is $\mathbb{V}$-measurable and $I_{\mathbb{V}} = N_{\mathbb{V}}$, for each $T \in \mathbb{V}$, there is a $T' \leq T$ such that either $[T'] \subseteq B$ or $[T'] \cap B = \emptyset$. But the former case cannot happen because $[T']$ contains many $G$-dependent elements. Hence $[T'] \cap B = \emptyset$. Therefore $B$ is $\mathbb{V}$-null.

With the above fact, this means every Borel set is either in ${I_{\mathbb{V}}}^*$ or contains $[T]$ for some $T \in \mathbb{V}$. Since sets in ${I_{\mathbb{V}}}^*$ cannot contain $[T]$ for some $T$ in $\mathbb{V}$, $B_c \in {I_{\mathbb{V}}}^*$ if and only if $B_c$ is in $I$, i.e., it is the union of a countable set of $G$-invariant Borel sets. This is easily seen to be $\mathbf{\Sigma}^1_2$, as desired. □

Regarding $I_{\mathbb{V}} = N_{\mathbb{V}}$, the following is a direct consequence of Theorem 2.4.6 and Proposition 2.5.1 (or an easy consequence of [19, Lemma 3.1]):[14]

**Corollary 2.5.3.** The following are equivalent:

1. Every $\mathbf{\Sigma}^1_2$ set of reals is $\mathbb{V}$-measurable, and
2. For any real $a$, the set of quasi-$\mathbb{V}$-generic reals over $\mathrm{L}[a]$ is of measure one with respect to $N_{\mathbb{V}}$.

Another application is for eventually different forcing $\mathbb{E}$ by Brendle and Löwe [21]. They used Theorem 2.4.6 to prove that the Baire property in eventually different topology for every $\mathbf{\Sigma}^1_2$ set of reals is equivalent to the statement "$\omega_1$ is inaccessible by reals", i.e., for every real $a$, $\omega_1^V$ is inaccessible in $\mathrm{L}[a]$, which is the strongest regularity property for $\mathbf{\Sigma}^1_2$ sets (Hechler forcing also has this feature).

We state their results and their proofs here. Recall the definition of eventually different forcing from § 1.9 and the definition of the eventually different topology $\mathcal{E}$ from Example 2.1.6. Also, the meager ideal in the topology $\mathcal{E}$ is the same as $I_{\mathbb{E}}$ by Example 2.1.6. As mentioned in 1.9, eventually different forcing is ccc. Hence by Lemma 1.9 3., ${I_{\mathbb{E}}}^* = I_{\mathbb{E}}$. By Proposition 2.1.8, the Baire property in the topology $\mathcal{E}$ coincides with $\mathbb{E}$-measurability. Since $\mathbb{E}$ is provably ccc and simply definable, by Proposition 2.3.3 (3), quasi-$\mathbb{E}$-genericity is the same as $\mathbb{E}$-genericity.

**Theorem 2.5.4** (Brendle and Löwe [21])**.** The following are equivalent:

1. Every $\mathbf{\Delta}^1_2$ set of reals has the Baire property in the eventually different topology $\mathcal{E}$,
2. $\mathbf{\Sigma}^1_3$-$\mathbb{E}$-absoluteness holds, and
3. For any real $a$, there is an $\mathbb{E}$-generic real $x$ over $\mathrm{L}[a]$.

*Proof.* By Theorem 2.4.2, it suffices to check the condition $(*)$ in Theorem 2.4.2. But since $\mathbb{E}$ is provably ccc and simply definable, the condition $(*)$ follows from Proposition 2.3.3 (2). □

[14] This answers [19, Question 3] positively.

**Theorem 2.5.5** (Brendle and Löwe [21]). The following are equivalent:

1. Every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property in the eventually different topology $\mathcal{E}$,
2. For any real $a$, the set of $\mathbb{E}$-generic reals over $\mathrm{L}[a]$ is comeager in the eventually different topology $\mathcal{E}$, and
3. For any real $a$, $\omega_1^{\mathrm{L}[a]} < \omega_1$.

*Proof.* For the equivalence between (1) and (2), by Theorem 2.4.6 it suffices to see that $I_{\mathbb{E}}$ is Borel-generated. But as mentioned in the paragraph after Theorem 2.4.6, $I_{\mathbb{P}}$ is Borel-generated if $\mathbb{P}$ is ccc.

We show the direction from (3) to (2). Let $a$ be a real. Since $\mathbb{E}$-generic reals over $\mathrm{L}[a]$ are the same as quasi-$\mathbb{E}$-generic reals over $\mathrm{L}[a]$, it suffices to show that the set of quasi-$\mathbb{E}$-generic reals over $\mathrm{L}[a]$ is comeager in the topology $\mathcal{E}$. Since $\omega_1^{\mathrm{L}[a]}$ and CH holds in $\mathrm{L}[a]$, the set of Borel codes in $\mathrm{L}[a]$ is countable in $V$. Hence the union of Borel meager sets in the topology $\mathcal{E}$ with a Borel code in $\mathrm{L}[a]$ is also meager in the topology $\mathcal{E}$. Therefore the set of quasi-$\mathbb{E}$-generic reals over $\mathrm{L}[a]$ is comeager in the topology $\mathcal{E}$.

Next, we show the direction from 2. to 3. Toward a contradiction, assume there is a real $a$ such that $\omega_1^{\mathrm{L}[a]} = \omega_1$. Then, in $\mathrm{L}[a]$, there is a sequence $\langle f_\alpha \in {}^\omega\omega \mid \alpha < \omega_1\rangle$ of pairwise eventually different functions, i.e., for any $\alpha < \beta < \omega_1$, there is a natural number $n_0$ such that $f_\alpha(n) \neq f_\beta(n)$ for all $n \geq n_0$. For each $\alpha < \omega_1$, let $E_\alpha$ be the set of reals not eventually different from $f_\alpha$. It is easy to see that $E_\alpha$ is meager in the topology $\mathcal{E}$. The following is the key point:

**Theorem 2.5.6** (Brendle). If $A$ is meager in the topology $\mathcal{E}$, then the set $\{\alpha < \omega_1 E_\alpha \subseteq A\}$ is countable.

*Proof.* See [54, Theorem 4.7]. □

Since $E_\alpha$ is meager in the topology $\mathcal{E}$ with a Borel code in $\mathrm{L}[a]$, by 3., $\bigcup_{\alpha<\omega_1} E_\alpha$ must be meager in the topology $\mathcal{E}$. But this contradicts Theorem 2.5.6. □

Brendle and Löwe have also investigated the relation between the Baire property in the eventually different topology and other regularity properties. Here are the relations they listed in their paper [21] as in Figure 2.1:

In Figure 2.1, the letters $\mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{L}, \mathbb{M}, \mathbb{R}, \mathbb{S}$, and $\mathbb{V}$ stand for random, Cohen, Hechler, eventually different, Laver, Miller, Mathias, Sacks, and Silver forcing, respectively. $\mathbf{\Sigma}^1_2(\mathbb{P})$ stands for the statement that every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable and the same for $\mathbf{\Delta}^1_2(\mathbb{P})$. All the non-existence of implications means "one statement does not imply the other in ZFC", e.g., $\mathbf{\Delta}^1_2(\mathbb{R})$ does not imply $\mathbf{\Delta}^1_2(\mathbb{C})$ in ZFC, except for the non-implications from $\mathbf{\Delta}^1_2(\mathbb{L})$ to and $\mathbf{\Sigma}^1_2(\mathbb{V})$ and from $\mathbf{\Delta}^1_2(\mathbb{L})$ to $\mathbf{\Delta}^1_2(\mathbb{V})$ (it is currently not known whether $\mathbf{\Delta}^1_2(\mathbb{L})$ does not imply $\mathbf{\Sigma}^1_2(\mathbb{V})$ and whether $\mathbf{\Delta}^1_2(\mathbb{L})$ does not imply $\mathbf{\Delta}^1_2(\mathbb{V})$). All the implications and

$$\begin{tikzcd}[column sep=small]
 & & & \mathbf{\Sigma}^1_2(\mathbb{E}) = \mathbf{\Sigma}^1_2(\mathbb{D}) \arrow[ddll, Rightarrow] \arrow[drr, Rightarrow] & & \\
 & & & & & \mathbf{\Sigma}^1_2(\mathbb{B}) \arrow[dll, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
 & \mathbf{\Sigma}^1_2(\mathbb{R}) = \mathbf{\Delta}^1_2(\mathbb{R}) \arrow[d, Rightarrow] \arrow[ddrr, Rightarrow] & & \mathbf{\Sigma}^1_2(\mathbb{C}) = \mathbf{\Delta}^1_2(\mathbb{D}) \arrow[dll, Rightarrow] \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{E}) \arrow[dl, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{B}) \arrow[dddll, Rightarrow, bend left] \\
 & \mathbf{\Sigma}^1_2(\mathbb{L}) = \mathbf{\Delta}^1_2(\mathbb{L}) \arrow[dd, Rightarrow] \arrow[drr, dotted, "?"] \arrow[ddrr, dotted, "?"'] & & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & & & \mathbf{\Sigma}^1_2(\mathbb{V}) \arrow[dll, Rightarrow] \arrow[d, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{M}) = \mathbf{\Delta}^1_2(\mathbb{M}) \arrow[d, Rightarrow] & & \mathbf{\Delta}^1_2(\mathbb{V}) \arrow[dll, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{S}) = \mathbf{\Delta}^1_2(\mathbb{S}) & & & &
\end{tikzcd}$$

Figure 2.1: Regularity properties for $\mathbf{\Delta}^1_2$ sets and $\mathbf{\Sigma}^1_2$ sets

the non-implications not involving $\mathbb{E}$ have been known before their work and they have established the implications and the non-implications involving $\mathbb{E}$ using Theorem 2.4.2 and Theorem 2.4.6 in this chapter. Here characterizing the regularity properties in terms of the transcendence properties over L (rather forcing absoluteness) is essential, which was not known for eventually different forcing before our work.

Using Theorem 2.4.8 and Theorem 2.4.9, we can establish the same implications and non-implications for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_3$ sets assuming sharps for reals as in Figure 2.2:

Again, we do not know whether $\mathbf{\Delta}^1_3(\mathbb{L})$ does not imply $\mathbf{\Sigma}^1_3(\mathbb{V})$ and whether $\mathbf{\Delta}^1_3(\mathbb{L})$ does not imply $\mathbf{\Delta}^1_3(\mathbb{V})$ assuming sharps for reals. The proofs of the implications and non-implications are exactly the same as for $\mathbf{\Delta}^1_2$ sets and $\mathbf{\Sigma}^1_2$ sets by replacing L with K. We suspect many of the implications and the non-implications for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_3$ sets we have shown above are also well-known to experts in this area.

## 2.6 Conclusion and Questions

We introduced two general regularity properties, $\mathbb{P}$-Baireness and $\mathbb{P}$-measurability, and reduced the problems of $\mathbb{P}$-measurability to ones of $\mathbb{P}$-Baireness with the flavor of Baire category and used Banach-Mazur games and their variants. Then we proved general equivalence theorems between the regularity properties, forcing

$$\begin{tikzcd}
 & \mathbf{\Sigma}^1_3(\mathbb{E}) = \mathbf{\Sigma}^1_3(\mathbb{D}) \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & \mathbf{\Sigma}^1_3(\mathbb{B}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{\Sigma}^1_3(\mathbb{R}) = \mathbf{\Delta}^1_3(\mathbb{R}) \arrow[d, Rightarrow] \arrow[ddr, Rightarrow] & \mathbf{\Sigma}^1_3(\mathbb{C}) = \mathbf{\Delta}^1_3(\mathbb{D}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{E}) \arrow[dl, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{B}) \arrow[dddll, Rightarrow, bend left] \\
\mathbf{\Sigma}^1_3(\mathbb{L}) = \mathbf{\Delta}^1_3(\mathbb{L}) \arrow[dd, Rightarrow] \arrow[dr, dotted, "?"] \arrow[ddr, dotted, "?"] & \mathbf{\Delta}^1_3(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_3(\mathbb{V}) \arrow[d, Rightarrow] \arrow[dl, Rightarrow] & & \\
\mathbf{\Sigma}^1_3(\mathbb{M}) = \mathbf{\Delta}^1_3(\mathbb{M}) \arrow[d, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{V}) \arrow[dl, Rightarrow] & & \\
\mathbf{\Sigma}^1_3(\mathbb{S}) = \mathbf{\Delta}^1_3(\mathbb{S}) & & &
\end{tikzcd}$$

Figure 2.2: Regularity properties for $\mathbf{\Delta}^1_3$ sets and $\mathbf{\Sigma}^1_3$ sets

absoluteness, and transcendence properties over some canonical inner models in a uniform way and applied the theorems to answer some open questions in set theory of the reals. This is one of the instances where reducing problems to the ones of infinite games gives us clear intuition and solutions.

We close this chapter by raising several questions and discussing them.

**On $I_{\mathbb{P}}$ and $I_{\mathbb{P}}{}^*$.** Although $I_{\mathbb{P}}{}^*$ is the same as $I_{\mathbb{P}}$ for most cases as we have seen in Lemma 2.1.12, as in Question 2.1.11, we still do not know whether this is true in general. What we could hope is that this is true at least for Borel sets:

**Question 2.6.1.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then can we prove $B \in I_{\mathbb{P}}$ if and only if $B \in I_{\mathbb{P}}{}^*$ for any Borel set $B$?

If this is true, we do not have to mention $I_{\mathbb{P}}{}^*$ in our theorems.

**On the condition $(*)$ in Theorem 2.4.2.** It is interesting to give sufficient conditions for $\mathbb{P}$ satisfying $(*)$ in Theorem 2.4.2, i.e., the set of all Borel codes with $B_c \in I_{\mathbb{P}}{}^*$ is $\mathbf{\Sigma}^1_2$. These conditions could be definability conditions on $I_{\mathbb{P}}{}^*$ or directly on $\mathbb{P}$. For the first case, we have a useful sufficient condition: We say that a $\sigma$-ideal $I$ on the reals is $\mathbf{\Sigma}^1_2$ *on* $\mathbf{\Sigma}^1_1$ if for any analytic set $A \subseteq {}^\omega 2 \times {}^\omega\omega$, the set $\{c \mid A_c \in I\}$ is $\mathbf{\Sigma}^1_2$. It is easy to check that if $I_{\mathbb{P}}{}^*$ is $\mathbf{\Sigma}^1_2$ on $\mathbf{\Sigma}^1_1$, then $(*)$ holds. Since $I_{\mathbb{P}}$ is $\mathbf{\Sigma}^1_2$ on $\mathbf{\Sigma}^1_1$ and $I_{\mathbb{P}} = I_{\mathbb{P}}{}^*$ for most cases, $(*)$ is true for most $\mathbb{P}$. For the second case, we ask the following:

**Question 2.6.2.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_2$-forcing. Then can we prove $(*)$?

Although the condition $(*)$ is true for most typical forcings, we have one example, namely Mathias forcing, that we do not about the answer of the above question. Solving this particular question might give us another intuition of this problem.

**$\boldsymbol{\Delta}^1_2$-determinacy and $\boldsymbol{\Sigma}^1_4$-forcing absoluteness.** In Theorem 2.4.7, we use the failure of $\boldsymbol{\Delta}^1_2$-determinacy to prove the equivalence between the regularity property and forcing absoluteness. But it could be that both are consequences of $\boldsymbol{\Delta}^1_2$-determinacy. Since we have not used the failure of $\boldsymbol{\Delta}^1_2$-determinacy for the direction from forcing absoluteness to the regularity property, it is enough to ask whether $\boldsymbol{\Delta}^1_2$-determinacy implies $\boldsymbol{\Sigma}^1_4$-forcing absoluteness:

**Question 2.6.3.** Suppose $\boldsymbol{\Delta}^1_2$-determinacy holds. Then can we prove $\boldsymbol{\Sigma}^1_4$-$\mathbb{P}$-absoluteness for each strongly arboreal, proper, provably $\Delta^1_2$-forcing $\mathbb{P}$?

**Sharps for sets vs. sharps for reals.** In Theorem 2.4.7, we have assumed the existence of sharps for sets. Since the result is about reals, it is natural to ask whether we can reduce this assumption to sharps for reals. The obstacle is whether proper forcings preserve the statement "every real has a sharp" and $u_2$:

**Question 2.6.4.** Suppose every real has a sharp. Let $\mathbb{P}$ be a strongly arboreal, proper, provably $\Delta^1_2$-forcing. Then can we prove that every real has a sharp in $V^{\mathbb{P}}$ and $u_2^V = u_2^{V^{\mathbb{P}}}$?

Finally, we show that in the case of provably ccc, $\boldsymbol{\Sigma}^1_1$-forcings, things work perfectly:

**Proposition 2.6.5.** Let $\mathbb{P}$ be a strongly arboreal, provably ccc, $\boldsymbol{\Sigma}^1_1$-forcing. Then:

1. $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$.
2. $I_{\mathbb{P}}$ is Borel-generated.
3. The condition $(*)$ holds. Moreover, $\{c \mid B_c \in {I_{\mathbb{P}}}^*\} \in \boldsymbol{\Delta}^1_2$.
4. Let $M$ be a transitive model of ZFC. Then a real $x$ is $\mathbb{P}$-generic over $M$ if and only if $x$ is quasi-$\mathbb{P}$-generic over $M$.
5. If $\boldsymbol{\Delta}^1_2$-determinacy holds, then so does $\boldsymbol{\Sigma}^1_4$-$\mathbb{P}$-absoluteness.
6. If every real has a sharp, then every real has a sharp also in $V^{\mathbb{P}}$ and $u_2^V = u_2^{V^{\mathbb{P}}}$.

*Proof.* (1) is already mentioned in Lemma 2.1.12, (2) is already mentioned in the paragraph after Theorem 2.4.6, and (3) is already shown in Proposition 2.3.3.

The argument for (4) is exactly the same as for Lemma 2.3.3. For (5), see [75, Lemma 2.2.4]. For (6), see [75, Lemma 2.2.2, Theorem 2.2.7, Example 3.2.7]. □

Note that the assumption of Proposition 2.6.5 is true for all the typical ccc, strongly arboreal forcings.

# Chapter 3

# Games themselves

In this chapter, we compare the stronger versions of determinacy of Gale-Stewart games and Blackwell games, i.e., the Axiom of Real Determinacy $\mathsf{AD}_{\mathbb{R}}$ and the Axiom of Real Blackwell Determinacy Bl-$\mathsf{AD}_{\mathbb{R}}$. In § 3.1, we show that Bl-$\mathsf{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists and that the consistency of Bl-$\mathsf{AD}_{\mathbb{R}}$ is strictly stronger than that of AD. In § 3.2, we show that Bl-$\mathsf{AD}_{\mathbb{R}}$ implies that every set of reals is $\infty$-Borel. From this, we can derive almost all the regularity properties for every set of reals. In § 3.3, we discuss the possibility of the equivalence between $\mathsf{AD}_{\mathbb{R}}$ and Bl-$\mathsf{AD}_{\mathbb{R}}$ under ZF+DC. In § 3.4, we discuss the possibility of the equiconsistency between $\mathsf{AD}_{\mathbb{R}}$ and Bl-$\mathsf{AD}_{\mathbb{R}}$.

Throughout this chapter, we use standard notations from set theory and assume familiarity with descriptive set theory. By *reals*, we mean elements of the Cantor space and we use $\mathbb{R}$ to denote the Cantor space.

## 3.1 Real Blackwell Determinacy and $\mathbb{R}^{\#}$

In this section, we prove that Bl-$\mathsf{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists and that the consistency of Bl-$\mathsf{AD}_{\mathbb{R}}$ is strictly stronger than that of AD.

Solovay [77] proved that $\mathsf{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists. Our plan is to mimic Solovay's proof using Blackwell games. In order to do so, we analyze his proof which has two main components:

**Theorem 3.1.1** (Solovay)**.** The axiom $\mathsf{AD}_{\mathbb{R}}$ implies that there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, where $\mathcal{P}_{\omega_1}(\mathbb{R})$ is the set of all countable subsets of $\mathbb{R}$.

*Proof.* See [77, Lemma 3.1]. □

**Theorem 3.1.2** (Solovay)**.** Suppose there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and every real has a sharp. Then $\mathbb{R}^{\#}$ exists.

*Proof.* See [77, Lemma 4.1 & Theorem 4.4]. □

Hence it suffices to show that there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ from Bl-$\mathrm{AD}_{\mathbb{R}}$ because Bl-$\mathrm{AD}_{\mathbb{R}}$ implies AD in $\mathrm{L}(\mathbb{R})$, which implies that every real has a sharp by the result of Harrington [31].

**Theorem 3.1.3.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Let us first see what is a fine normal measure. Let $X$ be a set and $\kappa$ be an uncountable cardinal. As usual, we denote by $\mathcal{P}_\kappa(X)$ the set of all subsets of $X$ with cardinality less than $\kappa$, i.e., subsets $A$ of $X$ such that there are an $\alpha < \kappa$ and a surjection from $\alpha$ to $A$. Let $U$ be a set of subsets of $\mathcal{P}_\kappa(X)$. We say that $U$ is *$\kappa$-complete* if $U$ is closed under intersections with $<\kappa$-many elements; we say it is *fine* if for any $x \in X$, $\{a \in \mathcal{P}_\kappa(X) \mid x \in a\} \in U$; we say that $U$ is *normal* if for any family $\{A_x \in U \mid x \in X\}$, the diagonal intersection $\triangle_{x\in X} A_x$ is in $U$ (where $\triangle_{x\in X} A_x = \{a \in \mathcal{P}_\kappa(X) \mid (\forall x \in a)\ a \in A_x\}$). We say that $U$ is a *fine measure* if it is a fine $\kappa$-complete ultrafilter, and we say that it is a *fine normal measure* if it is a fine normal $\kappa$-complete ultrafilter.

*Proof of Theorem 3.1.3.* The following is the key point: A subset $A$ of ${}^\omega\mathbb{R}$ is *range-invariant* if for any $\vec{x}$ and $\vec{y}$ in ${}^\omega\mathbb{R}$ with $\mathrm{ran}(\vec{x}) = \mathrm{ran}(\vec{y})$, $\vec{x} \in A$ if and only if $\vec{y} \in A$.

**Lemma 3.1.4.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every range-invariant subset of ${}^\omega\mathbb{R}$ is determined.

*Proof of Lemma 3.1.4.* Let $A$ be a range-invariant subset of ${}^\omega\mathbb{R}$. We show that if there is an optimal strategy for player I in $A$, then so is a winning strategy for player I in $A$. The case for player II is similar and we will skip it.

Let us first introduce some notations. Given a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$, a countable set of reals $a$ is *closed under $f$* if for any finite sequence $s$ of elements in $a$, $f(s)$ is in $a$. For a strategy $\sigma\colon \mathbb{R}^{\mathrm{Even}} \to \mathbb{R}$ for player I, where $\mathbb{R}^{\mathrm{Even}}$ is the set of all finite sequences of reals with even length, a countable set of reals $a$ is *closed under $\sigma$* if for any finite sequence $s$ of elements in $a$ with even length, $\sigma(s)$ is in $a$. For a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$, a countable set of reals $a$ is *closed under $F$* if for any finite sequence $s$ of elements in $a$, $F(s)$ is a subset of $a$.

The following two claims are basic:

**Claim 3.1.5.** There is a winning strategy for player I in $A$ if and only if there is a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if $a$ is a countable set of reals and closed under $f$, then any enumeration of $a$ belongs to $A$.

*Proof of Claim 3.1.5.* We first show the direction from left to right. Given a winning strategy $\sigma$ for player I in $A$, let $f$ be such that if $a$ is closed under $f$, then $a$ is closed under $\sigma$. (Since $\sigma$ is a function from $\mathbb{R}^{\mathrm{Even}}$ to $\mathbb{R}$, any function from ${}^{<\omega}\mathbb{R}$ to $\mathbb{R}$ extending $\sigma$ will do.) We see this $f$ works for our purpose. Let $a$ be a countable set of reals closed under $f$. Then since $a$ is closed under $\sigma$ and countable, there is a run $x$ of the game following $\sigma$ such that its range is equal to $a$. Since $\sigma$ is winning for player I, $x$ is in $A$ and by the range-invariance of $A$, any enumeration of $a$ is also in $A$.

We now show the direction from right to left. Given such an $f$, we can arrange a strategy $\sigma$ for player I such that if $x$ is a run of the game following $\sigma$, then the range of $x$ is closed under $f$: Given a finite sequence of reals $(a_0, \cdots, a_{2n-1})$, consider the set of all finite sequences $s$ from elements of $\{a_0, \cdots a_{2n-1}\}$ and all the values $f(s)$ from this set. What we should arrange is to choose $\sigma(a_0, \cdots, a_{2n-1})$ in such a way that the range of any run of the game via $\sigma$ will cover all such values $f(s)$ when $(a_0, \cdots, a_{2n-1})$ is a finite initial segment of the run for any $n$ in $\omega$ moves. But this is possible by a standard book-keeping argument. By the property of $f$, this implies that $x$ is in $A$ and hence $\sigma$ is winning for player I. $\square$ (Claim 3.1.5)

**Claim 3.1.6.** There is a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if $a$ is a countable set of reals and closed under $f$, then any enumeration of $a$ belongs to $A$ if and only if there is a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if $a$ is a countable set of reals and closed under $F$, then any enumeration of $a$ belongs to $A$.

*Proof of Claim 3.1.6.* We first show the direction from left to right: Given such an $f$, let $F(s) = \{f(s)\}$. Then it is easy to check that this $F$ works.

We show the direction from right to left: Given such an $F$, it suffices to show that there is an $f$ such that if $a$ is closed under $f$ then $a$ is also closed under $F$. We may assume that $F(s) \neq$ for each $s$. Fix a bijection $\pi\colon \mathbb{R} \to {}^{\omega}\mathbb{R}$. Let $g\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ be such that $\mathrm{ran}\big(\pi(g(s))\big) = F(s)$ for each $s$ (this is possible because every relation on the reals can be uniformized by a function by Theorem 1.14.9). Let $h\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ be such that $h(s) = \pi\big(s(0)\big)(\mathrm{lh}(s) - 1)$, where $\mathrm{lh}(s)$ is the length of $s$ when $s \neq \emptyset$, if $s = \emptyset$ let $h(s)$ be an arbitrary real.

It is easy to see that if $a$ is closed under $g$ and $h$, then so is under $F$: Fix a finite sequence $s$ of reals in $a$. We have to show that each $x$ in $F(s)$ is in $a$. Consider $g(s)$. By the closure under $g$, $g(s)$ is in $a$. By choice of $g$, we know that $\mathrm{ran}(\pi(g(s))) = F(s)$, so it is enough to show that $x$ is in $a$ for any $x$ in $\mathrm{ran}(\pi(g(s))$. Suppose $x$ is the $n$th bit of $\pi(g(s))$. Consider the finite sequence $t = \big(g(s), ..., g(s)\big)$ of length $n+1$. Then $h(t) = \pi(t(0))(\mathrm{lh}(t) - 1) = \pi(g(s))(n) = x$. But $g(s)$ is in $a$ and $a$ was closed under $h$, so $x$ is in $a$.

Now it is easy to construct an $f$ such that if $a$ is closed under $f$, then so is under $g$ and $h$. $\square$ (Claim 3.1.6)

By the above two claims, it suffices to show that there is a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if $a$ is a countable set of reals and closed under $F$, then any enumeration of $a$ belongs to $A$.

Let $\sigma$ be an optimal strategy for player I in $A$. Let $F$ be as follows:

$$F(s) = \begin{cases} \emptyset & \text{if } \mathrm{lh}(s) \text{ is odd,} \\ \{y \in \mathbb{R} \mid \sigma(s)(y) \neq 0\} & \text{otherwise.} \end{cases}$$

Then $F$ is as desired: If $a$ is closed under $F$, then enumerate $a$ to be $\langle a_n \mid n \in \omega\rangle$ and let player I follow $\sigma$ and let player II play the Dirac measure for $a_n$ at her $n$th move. Then the probability of the set $\{x \in {}^{\omega}\mathbb{R} \mid \mathrm{ran}(x) = a\}$ is 1 and since $\sigma$ is optimal for player I in $A$, there is an $x$ such that the range of $x$ is $a$ and $x$ is in $A$. But by the range-invariance of $A$, any enumeration of $a$ belongs to $A$. □ (Lemma 3.1.4)

We shall be closely following Solovay's original idea. We define a family $U \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ as follows: Fix $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and consider the following game $\tilde{G}_A$: Players alternately play reals; say that they produce an infinite sequence $\vec{x} = (x_i \mid i \in \omega)$. Then player II wins the game $\tilde{G}_A$ if $\mathrm{ran}(\vec{x}) \in A$, otherwise player I wins. Since the payoff set of this game is range-invariant as a Gale-Stewart game, by Lemma 3.1.4, it is determined.

We say that $A \in U$ if and only if player II has a winning strategy in $\tilde{G}_A$. We shall show that it is a fine normal measure under the assumption of Bl-$\mathrm{AD}_{\mathbb{R}}$, thus finishing the proof of Theorem 3.1.3.

A few properties of $U$ are obvious: For instance, we see readily that $\emptyset \notin U$ and that $\mathcal{P}_{\omega_1}(\mathbb{R}) \in U$, as well as the fact that $U$ is closed under taking supersets. In order to see that $U$ is a fine family, fix a real $x$, and let player II play $x$ in her first move: This is a winning strategy for player II in $\tilde{G}_{\{a \mid x \in a\}}$.

We next show that for any set $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$, either $A$ or the complement of $A$ is in $U$. Given any such set $A$, suppose $A$ is not in $U$. We show that the complement of $A$ is in $U$. Since the game $\tilde{G}_A$ is determined, by the assumption, there is a winning strategy $\sigma$ for I in $\tilde{G}_A$. Setting $\tau(s) = \sigma\big(s\restriction(\mathrm{lh}(s) - 1)\big)$ for $s \in \mathbb{R}^{\mathrm{Odd}}$, it is easy to see that $\tau$ is a winning strategy for player II in the game $\tilde{G}_{A^c}$.

We show that $U$ is closed under finite intersections. Let $A_1$ and $A_2$ be in $U$. Since the payoff sets in the games $\tilde{G}_{A_1}$ and $\tilde{G}_{A_2}$ are range-invariant, by the analogue of Claim 3.1.5, there are functions $f_1\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and $f_2\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if $a$ is closed under $f_i$, then $a$ is in $A_i$ for $i = 1, 2$. Then it is easy to find an $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if $a$ is closed under $f$, then $a$ is closed under both $f_1$ and $f_2$. By the analogue of Claim 3.1.5 again, this $f$ witnesses the existence of a winning strategy for player II in the game $\tilde{G}_{A_1 \cap A_2}$.

We have shown that $U$ is an ultrafilter on subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. We show the $\omega_1$-completeness of $U$ as follows: By Theorem 1.14.8, every set of reals is Lebesgue measurable assuming Bl-AD. If there is a non-principal ultrafilter on $\omega$, then there is a set of reals which is not Lebesgue measurable. Hence there is no non-principal ultrafilter on $\omega$, which implies that any ultrafilter is $\omega_1$-complete. In particular, $U$ is $\omega_1$-complete.

The last to show is that $U$ is normal. Let $\{A_x \mid x \in \mathbb{R}\}$ be a family of sets in $U$. We show that $\triangle_{x\in\mathbb{R}} A_x$ is in $U$. Consider the following game $\tilde{G}$: Player I moves $x$, then player II passes. After that, they play the game $\tilde{G}_{A_x}$. This is Blackwell determined and player II has an optimal strategy $\tau$ since each $A_x$ is in $U$. Let $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ be as follows:

$$F(s) = \begin{cases} \emptyset & \text{if } \mathrm{lh}(s) \text{ is even,} \\ \{y \in \mathbb{R} \mid \tau(s)(y) \neq 0\} & \text{otherwise.} \end{cases}$$

We claim that if $a$ is closed under $F$, then $a$ is in $\triangle_{x\in\mathbb{R}} A_x$. Then, by the analogues of Claim 3.1.5 and Claim 3.1.6, $F$ will witness the existence of a winning strategy for player II in the game $\tilde{G}_{\triangle_{x\in\mathbb{R}} A_x}$ and we will have proved that $\triangle_{x\in\mathbb{R}} A_x \in U$.

Suppose $a$ is closed under $F$. We show that $a \in A_x$ for each $x \in a$. Fix an $x$ in $a$ and enumerate $a$ to be $(x_n \mid n \in \omega)$. In the game $\tilde{G}$, let player I first move $x$ and then they play the game $\tilde{G}_{A_x}$. Let player II follow $\tau$ and player I play the Dirac measure concentrating on $x_n$ at the $n$th move. Then the probability of the set $\{\vec{x} \in {}^{\omega}\mathbb{R} \mid x_0 = x \text{ and } \mathrm{ran}(\vec{x}) = a\}$ is 1 and since $\tau$ is optimal for player II in the game $\tilde{G}$, there is an $\vec{x}$ such that the range of $\vec{x}$ is $a$ and $\vec{x}$ is a winning run for player II in $\tilde{G}$, hence $a$ is in $A_x$. □ (Theorem 3.1.3)

**Corollary 3.1.7.** The consistency of Bl-$\mathrm{AD}_{\mathbb{R}}$ is strictly stronger than that of AD.

*Proof.* Since Bl-$\mathrm{AD}_{\mathbb{R}}$ implies Bl-AD by the first item of Proposition 1.14.2 and Bl-AD implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ by Corollary 1.14.7, Bl-$\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$. By Theorem 3.1.3, Bl-$\mathrm{AD}_{\mathbb{R}}$ also implies the existence of $\mathbb{R}^{\#}$. By the property of $\mathbb{R}^{\#}$, one can construct a set-size elementary substructure of $\mathrm{L}(\mathbb{R})$. Hence $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ and the existence of $\mathbb{R}^{\#}$ imply the consistency of AD. Therefore, Bl-$\mathrm{AD}_{\mathbb{R}}$ implies the consistency of AD and by Gödel's Incompleteness Theorem, the consistency of Bl-$\mathrm{AD}_{\mathbb{R}}$ is strictly stronger than that of AD. □

## 3.2 Real Blackwell Determinacy and regularity properties

In this section, we show that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies almost all the regularity properties for every set of reals. Note that $\mathrm{DC}_{\mathbb{R}}$ follows from the uniformization for every relation on the reals. Hence by Theorem 1.14.9, Bl-$\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{DC}_{\mathbb{R}}$. For the rest of the sections in this chapter, we freely use $\mathrm{DC}_{\mathbb{R}}$ when we assume Bl-$\mathrm{AD}_{\mathbb{R}}$ and we fix a fine normal measure $U$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$, which exists by Theorem 3.1.3.

We start with proving the perfect set property for every set of reals. Recall that a set of reals $A$ has the *perfect set property* if either $A$ is countable or $A$ contains a perfect subset, where a perfect set of reals is a closed set without isolated points.

**Theorem 3.2.1.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals has the perfect set property.

*Proof.* The theorem follows from the following two lemmas:

**Lemma 3.2.2.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set, i.e., for any relation $R$ on the reals, there is a Borel function $f$ such that the set $\{x \mid (x, f(x)) \in R$ or there is no real $y$ with $(x, y) \in R\}$ is of Lebesgue measure one.

*Proof of Lemma 3.2.2.* The conclusion follows by a folklore argument from Lebesgue measurability and uniformization for any relation on the reals both of which are consequences of Bl-$\mathrm{AD}_{\mathbb{R}}$ by Theorem 1.14.8 and Theorem 1.14.9).

Let $R$ be an arbitrary relation on the reals. We may assume the domain of $R$ is the whole space, i.e., for any real $x$, there is a real $y$ such that $(x, y) \in R$. We will find a Borel function uniformizing $R$ almost everywhere.

By the uniformization principle, there is a function $g$ uniformizing $R$. For each finite binary sequence $s$, the set $g^{-1}([s])$ is Lebesgue measurable by Theorem 1.14.8. Hence for each $s$ there is a Borel set $B_s$ such that $g^{-1}([s]) \triangle B_s$ is Lebesgue null. Now define $f$ so that the following holds: For each finite binary sequence $s$,

$$f(x) \in [s] \iff x \in B_s.$$

Then by the property of $B_s$, $f$ is defined almost everywhere, Borel, and is equal to $g$ almost everywhere. Hence any Borel extension of $f$ will be the one we desired. □ (Lemma 3.2.2)

**Lemma 3.2.3** (Raisonnier and Stern)**.** Suppose every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set. Then every set of reals has the perfect set property.

*Proof of Lemma 3.2.3.* See [70, Theorem 5]. □

□ (Theorem 3.2.1)

Next, we show that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that every set of reals has the Baire property. We first introduce the Blackwell meager ideal as an analogue of the meager ideal. A set $A$ of reals is *Blackwell meager* if player II has an optimal strategy in the Banach-Mazur game $G^{**}(A)$. Let $I_{\mathrm{BM}}$ denote the set of all Blackwell meager sets of reals.

**Lemma 3.2.4.** Assume Bl-AD. Then any meager set is in $I_{\mathrm{BM}}$, $[s] \notin I_{\mathrm{BM}}$ for each finite binary sequence $s$, and $I_{\mathrm{BM}}$ is a $\sigma$-ideal. Moreover, every set of reals is measurable with respect to $I_{\mathrm{BM}}$, i.e., for any set $A$ of reals and finite binary sequence $s$, there is a finite binary sequence $t$ extending $s$ such that either $[t] \cap A$ or $[t] \setminus A$ is in $I_{\mathrm{BM}}$.

*Proof.* By Theorem 1.8.3, if a set $A$ of reals is meager, then player II has a winning strategy in the Banach-Mazur game $G^{**}(A)$ and in particular player II has an optimal strategy in $G^{**}(A)$ by Theorem 1.14.3. Hence $A$ is Blackwell meager.

It is easy to see that $[s] \notin I_{\mathrm{BM}}$ for each finite binary sequence $s$ by letting player I first play the Dirac measure concentrating on $s$ in the game $G^{**}([s])$.

We show that $I_{\mathrm{BM}}$ is a $\sigma$-ideal. The closure of $I_{\mathrm{BM}}$ under subsets is immediate. We prove that it is closed under countable unions.

In order to prove this, we need to develop the appropriate *transfer technique* (as discussed and applied in [55]) for the present context. Let $\pi \subseteq \omega$ be an infinite and co-infinite set. We think of $\pi$ as the set of rounds in which player I moves. We identify $\pi$ with the increasing enumeration of its members, i.e., $\pi = \{\pi_i \mid i \in \omega\}$. Similarly, we write $\bar{\pi}$ for the increasing enumeration of $\omega \backslash \pi$, i.e., $\omega \backslash \pi = \{\bar{\pi}_i \mid i \in \omega\}$. For notational ease, we call $\pi$ a **I-coding** if no two consecutive numbers are in $\pi$ and $0 \in \pi$ (i.e., the first move is played by I). We call $\pi$ a **II-coding** if no two consecutive numbers are in $\omega \backslash \pi$ and $0 \in \pi$.

Fix $A \subseteq {}^{\omega}\omega$ and define two variants of $G_A^{**}$ with alternative orders of play as determined by $\pi$. If $\pi$ is a I-coding, the game $G_A^{**\pi,\mathrm{I}}$ is played as follows:

$$\begin{array}{lllllll} \mathrm{I} & s_{\pi_0} = s_0 & & s_{\pi_1} & & \dots \\ \mathrm{II} & & s_{\pi_0+1}, \dots, s_{\pi_0-1} & & s_{\pi_1+1}, \dots, s_{\pi_2-1} & \dots \end{array}$$

If $\pi$ is a II-coding, then the game $G_A^{**\pi,\mathrm{II}}$ is played as follows:

$$\begin{array}{llllll} \mathrm{I} & s_0, \dots, s_{\bar{\pi}_0-1} & & s_{\bar{\pi}_0+1}, \dots, s_{\bar{\pi}_1-1} & & \dots \\ \mathrm{II} & & s_{\bar{\pi}_0} & & s_{\bar{\pi}_1} & \dots \end{array}$$

In both cases, player II wins the game if $s_0^\frown s_1^\frown \dots^\frown s_n^\frown \dots \notin A$. Obviously, we have

$$G_A^{**} = G_A^{**\mathrm{Even},\mathrm{II}}$$

where Even is the set of even numbers.

**Lemma 3.2.5.** Let $A$ be a subset of the Baire space and $\pi$ be a I-coding. Then there is a translation $\sigma \mapsto \sigma_\pi$ of mixed strategies for player I such that if $\sigma$ is an optimal strategy for player I in $G_A^{**}$, then $\sigma_\pi$ is an optimal strategy for player I in $G_A^{**\pi,\mathrm{I}}$.

Similarly, if $\pi$ is a II-coding, there is a translation $\tau \mapsto \tau_\pi$ of mixed strategies for player II such that if $\tau$ is an optimal strategy for player II in $G_A^{**}$, then $\tau_\pi$ is an optimal strategy for player II in $G_A^{**\pi,\mathrm{II}}$.

*Proof of Lemma 3.2.5.* We prove only the lemma for the games $G_A^{**\pi,\mathrm{I}}$, the other proof being similar. If $\vec{s} = \langle s_i \mid i \in \omega \rangle$ is an infinite sequence of finite binary sequences, we define

$$b_i^{\vec{s}} = s_{\pi_i+1}^\frown \dots^\frown s_{\pi_{i+1}-1}.$$

Note that in order to compute $b_i^{\vec{s}}$, we only need the first $\pi_{i+1}$ bits of $\vec{s}$. The idea is that now the $G_A^{**}$-run

$$\begin{array}{lccccccc} \mathrm{I} & s_{\pi_0} & & s_{\pi_1} & & s_{\pi_2} & & \ldots \\ \mathrm{II} & & b_0^{\vec{s}} & & b_1^{\vec{s}} & & b_2^{\vec{s}} & \ldots \end{array} \qquad (*)$$

yields the same output in terms of the concatenation of all played finite sets as the run $\vec{s}$ in the game $G_A^{**\pi,\mathrm{I}}$. We can define a map $\pi^*$ on infinite sequences of finite binary sequences by

$$(\pi^*(\vec{s}))_i = \begin{cases} s_{\pi_k} & \text{if } i = 2k, \\ b_k^{\vec{s}} & \text{if } i = 2k+1, \end{cases}$$

and see that $s_0{}^\frown s_1{}^\frown \ldots = (\pi^*(\vec{s}))_0{}^\frown(\pi^*(\vec{s}))_1{}^\frown \ldots$.

Now, given a mixed strategy $\sigma$ for player I in $G_A^{**}$ and a run $\vec{s}$ of the game $G_A^{**\pi,\mathrm{I}}$, we define $\sigma_\pi$ via $\pi^*$ as follows:

$$\sigma_\pi(s_0, \ldots, s_{\pi_m - 1}) = \sigma(s_{\pi_0}, b_0^{\vec{s}}, \ldots, s_{\pi_i}, b_i^{\vec{s}}, \ldots, s_{\pi_{m-1}}, b_{m-1}^{\vec{s}}).$$

Assume that $\sigma$ is an optimal strategy for player I in $G_A^{**}$ and fix an arbitrary mixed strategy $\tau$ in the game $G_A^{**\pi,\mathrm{I}}$. We show that the payoff set for $A$ in $G_A^{**\pi,\mathrm{I}}$ is $\mu_{\sigma_\pi,\tau}$-measurable and $\mu_{\sigma_\pi,\tau}(A) = 1$. In order to do so, we construct a mixed strategy $\tau_{\pi^{-1}}$ for player II in $G_A^{**}$ so that the game played by $\sigma_\pi$ and $\tau$ is essentially the same as the game played by $\sigma$ and $\tau_{\pi^{-1}}$.

Given a sequence $\vec{b}$ of moves in $G_A^{**}$, we need to unravel it into a sequence of moves in $G_A^{**\pi,\mathrm{I}}$ in an inverse of the maps $\vec{s} \mapsto b_i^{\vec{s}}$ according to $(*)$, i.e., $b_{2i+1} = b_i^{\vec{s}}$. Thus, we define

$$\begin{aligned} A_{2i+1}^{\vec{b}} &= \{\vec{s} \mid b_i^{\vec{s}} = b_{2i+1}\}, \\ A_{\leq 2i+1}^{\vec{b}} &= \bigcap_{j \leq i} A_{2j+1}^{\vec{b}}. \end{aligned}$$

Note that only a finite fragment of $\vec{s}$ is needed to check whether $b_i^{\vec{s}} = b_{2i+1}$, and thus we think of $A_{\leq 2i+1}^{\vec{b}}$ as a set of $(\pi_{i+1} - (i+1))$-tuples of finite binary sequences. In the following, when we quantify over all "$\vec{s} \in A_{\leq i}^{\vec{b}}$", we think of this as a collection of finite strings of finite binary sequences. In order to pad the moves made in $G_A^{**\pi,\mathrm{I}}$, we define the following notation: For infinite sequences $\vec{s}$ and $\vec{b}$, we write

$$x_i^{\vec{s},\vec{b}} = (b_{2i}, s_{\pi_i+1}, ..., s_{\pi_{i+1}-1}).$$

Compare $(*)$ to see that if $\vec{s}$ corresponds to moves in $G_A^{**\pi,\mathrm{I}}$ and $\vec{b}$ to the moves in $G_A^{**}$, then these are exactly the finite sequences that player II will have to respond to in $G_A^{**\pi,\mathrm{I}}$. Moreover, for a given sequence $\vec{z}$ of finite binary sequences, we let

$$P_\tau(z_0, ..., z_n) = \prod_{i \leq n, i \notin \pi} \tau(z_0, ..., z_{i-1})(z_i).$$

Fix a sequence $\vec{b}$ of finite binary sequences with even length and define $\tau_{\pi^{-1}}$ as follows:

$$\tau_{\pi^{-1}}(b_0,\ldots,b_{2m})(b_{2m+1}) \;=\; \frac{\sum_{\vec{s}\in A^{\vec{b}}_{\leq 2m+1}} P_\tau(x_0^{\vec{s},\vec{b}\frown}\ldots{}^\frown x_m^{\vec{s},\vec{b}})}{\prod_{i=1}^m \tau_{\pi^{-1}}(b_0,\ldots,b_{2i-2})(b_{2i-1})}.$$

Using the two operations $\sigma\mapsto\sigma_\pi$ and $\tau\mapsto\tau_{\pi^{-1}}$, since the payoff set for $G^{**}_A$ is invariant under $\pi^*$, it now suffices to prove for all basic open sets $[t]$ induced by a finite sequence $t=(b_0,...,b_{\mathrm{lh}(t)-1})$ that $\mu_{\sigma,\tau_{\pi^{-1}}}([t])=\mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([t]))$. We prove this by induction on the length of $t$, and have to consider three different cases:

**Case 1.** $\mathrm{lh}(t)=0$. This is immediate.

**Case 2.** $\mathrm{lh}(t)=2m+1$ with $m\geq 0$. By induction hypothesis, we have that $X=\mu_{\sigma,\tau_{\pi^{-1}}}([b_0,\ldots,b_{2m-1}])=\mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([b_0,\ldots,b_{2m-1}]))$. Thus,

$$\begin{aligned}\mu_{\sigma,\tau_{\pi^{-1}}}([b_0,\ldots,b_{2m}]) &= X\cdot\sigma(b_0,\ldots,b_{2m-1})(b_{2m})\\ &= \mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([b_0,\ldots,b_{2m}])).\end{aligned}$$

**Case 3.** $\mathrm{lh}(t)=2m+2$ with $m\geq 0$.

$$\begin{aligned}\mu_{\sigma,\tau_{\pi^{-1}}}(t) &= \prod_{i=0}^m \sigma(b_0,\ldots,b_{2i-1})(b_{2i})\cdot\sum_{\vec{s}\in A^{\vec{b}}_{\leq 2m+1}} P_\tau(x_0^{\vec{s},\vec{b}\frown}\ldots{}^\frown x_m^{\vec{s},\vec{b}})\\ &= \mu_{\sigma_\pi,\tau}\big((\pi^*)^{-1}([b_0,\ldots,b_{2m+1}])\big).\end{aligned}$$

This calculation finishes the proof of this lemma. □ (Lemma 3.2.5)

We now show that $I_{\mathrm{BM}}$ is closed under countable unions. Let $\{A_n\mid n\in\omega\}$ be a family of sets in $I_{\mathrm{BM}}$. Take an optimal strategy $\tau_n$ in the game $G^{**}(A_n)$ for each $n$. We prove that $\bigcup_{n\in\omega}A_n$ is also in $I_{\mathrm{BM}}$.

Fix a bookkeeping bijection $\rho$ from $\omega\times\omega$ to $\omega$ such that $\rho(n,m)<\rho(n,m+1)$ and $\rho(n,0)\geq n$. We are playing infinitely many games in a diagram where the first coordinate is for the index of the game we are playing, and the second coordinate is for the number of moves. Hence the pair $(n,m)$ stands for "$m$th move in the $n$th game". Define a II-coding $\pi_n=\omega\backslash\{2\rho(n,i)+1\mid i\in\omega\}$ corresponding to the following game diagram:

| | | | | | |
|---|---|---|---|---|---|
| I | $s_0,\ldots,s_{2\rho(n,0)}$ | | $s_{2\rho(n,0)+2},\ldots,s_{2\rho(n,1)}$ | | $\ldots$ |
| II | | $s_{2\rho(n,0)+1}$ | | $s_{2\rho(n,1)+1}$ | $\ldots$ |

By Lemma 3.2.5, we know that for each $n\in\omega$, we get an optimal strategy $(\tau_n)_{\pi_n}$ for the game $G^{**\pi_n,\mathrm{II}}_{A_n}$. Let $\tau$ be the following mixed strategy

$$\tau(s_0,\ldots,s_{2\rho(n,m)})=(\tau_n)_{\pi_n}(s_0,\ldots,s_{2\rho(n,m)}).$$

The properties of $\rho$ make sure that this strategy is well-defined. We shall now prove that $\tau$ is an optimal strategy for player II in $G^{**}_{\bigcup_{n\in\omega} A_n}$.

Pick any mixed strategy $\sigma$ for player I in $G^{**}_{\bigcup_{n\in\omega} A_n}$ and define strategies $\sigma_n$ for $G^{**\pi_n,\mathrm{II}}_{A_n}$. Let $m = \rho(k,\ell)$, then

$$\begin{aligned}\sigma_n(s_0,\dots,s_{2m-1}) &= \sigma(s_0,\dots,s_{2m-1}), \text{ and}\\ \sigma_n(s_0,\dots,s_{2m}) &= (\tau_k)_{\pi_k}(s_0,\dots,s_{2m}) \text{ (if } k\neq n).\end{aligned}$$

Note that for each $n\in\omega$, $\mu_{\sigma,\tau} = \mu_{\sigma_n,(\tau_n)_{\pi_n}}$.

The payoff set (for player II) in $G^{**}_{\bigcup_{n\in\omega} A_n}$ is $A = \{\vec{s} \mid s_0^\frown s_1^\frown \dots \notin \bigcup_{n\in\omega} A_n\}$. We show that $\mu_{\sigma,\tau}(A) = 1$. Since $A = \bigcap_{n\in\omega}\{\vec{s} \mid s_0^\frown s_1^\frown \dots \notin A_n\}$, it suffices to check that the sets $B_n = \{\vec{s} \mid s_0^\frown s_1^\frown \dots \notin A_n\}$ has $\mu_{\sigma,\tau}$-measure 1. But $\mu_{\sigma,\tau}(B_n) = \mu_{\sigma_n,(\tau_n)_{\pi_n}}(B_n) = 1$. Thus we have shown that $I_{\mathrm{BM}}$ is a $\sigma$-ideal.

We finally show that every set $A$ of reals is measurable with respect to $I_{\mathrm{BM}}$, i.e., for any finite binary sequence $s$, there is a finite binary sequence $t$ extending $s$ such that either $[t]\cap A$ or $[t]\setminus A$ is in $I_{\mathrm{BM}}$. Fix such $A$ and $s$. If $[s]\cap A$ is in $I_{\mathrm{BM}}$, we are done. So suppose not. Then player II does not have an optimal strategy in the game $G^{**}([s]\cap A)$. By Bl-AD, there is an optimal strategy $\sigma$ for player I in the game $G^{**}([s]\cap A)$. Let $t$ be any $s'$ with $\sigma(\emptyset)(s')\neq 0$. Then since $\sigma$ is optimal, $t$ extends $s$ and the strategy $\sigma$ easily gives us an optimal strategy for player II in the game $G^{**}([t]\setminus A)$. Hence $[t]\setminus A$ is in $I_{\mathrm{BM}}$. $\square$ (Lemma 3.2.4)

Recall the notions of Stone space $\mathrm{St}(\mathbb{P})$ and $\mathbb{P}$-Baireness for a partial order $\mathbb{P}$ from chapter 2. The based set of $\mathrm{St}(\mathbb{P})$ was the set of all ultrafilters on $B_{\mathbb{P}}$ where $B_{\mathbb{P}}$ is a completion of $\mathbb{P}$. Without the Axiom of Choice, it might be empty if $\mathbb{P}$ is big. But in this chapter, we only consider partial orders $\mathbb{P}$ which are elements of $\mathcal{H}_{\omega_1}$ in $V$, i.e., the transitive closure of $\mathbb{P}$ is countable in $V$. If $\mathbb{P}$ is an element of $\mathcal{H}_{\omega_1}$, then $\mathrm{St}(\mathbb{P})$ is essentially the same as $\mathrm{St}(\mathbb{C})$ where $\mathbb{C}$ is Cohen forcing, hence the Cantor space ${}^{\omega}\omega$

Since every meager set is Blackwell meager as we have seen in Lemma 3.2.4, if $\mathbb{P}$ is in $\mathcal{H}_{\omega_1}$, then one can consider Blackwell meagerness for subsets of $\mathrm{St}(\mathbb{P})$ by identifying $\mathrm{St}(\mathbb{P})$ with the Cantor space.

We are now ready to prove the Baire property for every set of reals from $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

**Theorem 3.2.6.** Assume $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. Then every set of reals has the Baire property.

*Proof.* Take any set $A$ of reals. We show that $A$ has the Baire property. Let $\mathcal{A}^2_A$ be the second-order arithmetic structure with $A$ as a unary predicate. Since any relation on the reals can be uniformized by a function by Theorem 1.14.9, we can construct a Skolem function $F$ for $\mathcal{A}^2_A$ and by a simple coding of finite sequences of reals and formulas via reals, we regard it as a function from the reals to themselves. Let $\Gamma_F = \{(x,s)\in\mathbb{R}\times{}^{<\omega}2 \mid F(x)\supseteq s\}$. The following are the key objects for the proof (they are called *term relations*): Recall from

Lemma 2.1.2 that for a $\mathbb{P}$-name $\tau$ for a real, $f_\tau$ is the Baire measurable function (which is continuous on a comeager set) corresponding to $\tau$.

$$\begin{aligned}
\tau_A = \{(\mathbb{P},p,\sigma) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
& \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A\}, \\
\tau_{A^{\mathrm{c}}} = \{(\mathbb{P},p,\sigma) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
& \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A^{\mathrm{c}}\}, \\
\tau_{\Gamma_F} = \{(\mathbb{P},p,\sigma,s) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
& \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G),s\big) \in \Gamma_F\}, \\
\tau_{\Gamma_F{}^{\mathrm{c}}} = \{(\mathbb{P},p,\sigma,s) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
& \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G),s\big) \in \Gamma_{F^{\mathrm{c}}}\},
\end{aligned}$$

where $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$ means "for all $G$ modulo a Blackwell meager set in $\mathrm{St}(\mathbb{P})\ldots$". Let $M = \mathrm{HOD}^{\mathrm{L}[\tau_A,\tau_{A^{\mathrm{c}}},\tau_{\Gamma_F},\tau_{\Gamma_F{}^{\mathrm{c}}}]}_{\tau_A,\tau_{A^{\mathrm{c}}},\tau_{\Gamma_F},\tau_{\Gamma_F{}^{\mathrm{c}}}}$ and for $G \in \mathrm{St}(\mathbb{P})$, let $A_G = \{f_\sigma(G) \mid (\exists p \in G)\ (\mathbb{P},p,\sigma) \in \tau_A \cap M\}$. Note that for any countable ordinal $\alpha$, $\mathcal{P}(\alpha) \cap M$ is countable: Since $M$ is a transitive model of ZFC, if $\mathcal{P}(\alpha) \cap M$ was uncountable, then there would be an uncountable sequence of distinct reals which would contradict Lebesgue measurability for every set of reals. Hence for any $\mathbb{P} \in \mathcal{H}_{\omega_1} \cap M$, the set of $\mathbb{P}$-generic filters over $M$ is comeager, in particular Blackwell comeager (i.e., its complement is Blackwell meager). Therefore, when we discuss statements starting from $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$, we may assume that $G$ is $\mathbb{P}$-generic over $M$.

**Claim 3.2.7.**

1. Let $\mathbb{P}$ be a partial order in $M$. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ A_G = A \cap M[G] \in M[G]$ and $M[G]$ is closed under $F$.

2. Let $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega)^M$, where $\mathrm{Coll}(\omega, 2^\omega)$ is the forcing collapsing the cardinal $2^\omega$ into countable with finite conditions. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ A_G$ has the Baire property in $M[G]$.

*Proof.* We first show that $A_G = A \cap M[G]$ for Blackwell comeager many $G$. Since $I_{\mathrm{BM}}$ is a $\sigma$-ideal, for Blackwell comeager many $G$, $G$ is $\mathbb{P}$-generic over $M$ and if $(\mathbb{P},p,\sigma) \in \tau_A \cap M$ (resp., $\tau_{A^{\mathrm{c}}} \cap M$) and $p \in G$, then $f_\sigma(G) = \sigma^G \in A$ (resp., $A^{\mathrm{c}}$). We show that $A_G = A \cap M[G]$ for any such $G$.

Fix such a $G$. We first prove that $A_G \subseteq A \cap M[G]$. Take any real $x$ in $A_G$. Then there is a $p \in G$ and a $\sigma$ such that $(\mathbb{P},p,\sigma) \in \tau_A \cap M$ and $\sigma^G = x$. Then by the property of $G$, $x = \sigma^G = f_\sigma(G) \in A$, as desired. We show that $A \cap M[G] \subseteq A_G$. Let $x$ be a real in $M[G]$ which is not in $A_G$. We prove that $x$ is also not in $A$. Since $x$ is in $M[G]$, there is a $\mathbb{P}$-name $\sigma$ for a real in $M$ such that $\sigma^G = x$. Since $A$ is measurable with respect to $I_{\mathrm{BM}}$ by Lemma 3.2.4, the set $\{p \in \mathbb{P} \mid \text{ either } (\mathbb{P},p,\sigma) \in \tau_A \cap M \text{ or } (\mathbb{P},p,\sigma) \in \tau_{A^{\mathrm{c}}} \cap M\}$ is dense and it is in $M$. Since $G$ is $\mathbb{P}$-generic over $M$, there is a $p \in G$ such that either $(\mathbb{P},p,\sigma) \in \tau_A$ or $(\mathbb{P},p,\sigma) \in \tau_{A^{\mathrm{c}}}$. But $(\mathbb{P},p,\sigma) \in \tau_A$ cannot hold because it would imply $x = \sigma^G \in A_G$. Hence $(\mathbb{P}, p, \sigma) \in \tau_{A^c}$ and $x = \sigma^G = f_\sigma(G) \in A^c$ by the property of $G$, as desired.

Let $\rho_A = \{(\sigma, p) \mid (\mathbb{P}, p, \sigma) \in \tau_A \cap M\}$. Since the comprehension axioms with $\tau_A$ as a unary predicate hold in $M$, $\rho_A$ is a $\mathbb{P}$-name for a set of reals in $M$ and $\rho_A^G = A_G \in M[G]$. Hence $A_G = A \cap M[G] \in M[G]$ for Blackwell comeager many $G$, as desired.

Next, we show that $M[G]$ is closed under $F$ for Blackwell comeager many $G$. We prove this for any $G$ which is $\mathbb{P}$-generic over $M$ such that if $(\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_F}$ (resp., $\tau_{\Gamma_{F^c}}$) and $p$ is in $G$, then $F(\sigma^G) \supseteq s$ (resp., $F(\sigma^G) \not\supseteq s$). Fix such a $G$ and let $x$ be a real in $M[G]$. We show that $F(x)$ is also in $M[G]$. Since $x$ is in $M[G]$, there is a $\mathbb{P}$-name $\sigma$ for a real in $M$ such that $\sigma^G = x$. Since every subset of $\mathrm{St}(\mathbb{P})$ is measurable with respect to $I_{\mathrm{BM}}$, the function $G' \mapsto F\bigl(f_\sigma(G')\bigr)$ is continuous modulo a Blackwell meager set in $\mathrm{St}(\mathbb{P})$. Hence for any finite binary sequence $s$, the set of all $p \in \mathbb{P}$ such that either $\bigl(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\bigr)\ p \in G' \implies F\bigl(f_\sigma(G')\bigr) \supseteq s$ or $\bigl(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\bigr)\ p \in G' \implies F\bigl(f_\sigma(G')\bigr) \not\supseteq s$ is dense and is in $M$. By the genericity and the property of $G$, for any $s$, there is a $p \in G$ such that $F(\sigma^G) \supseteq s$ if and only if $\bigl(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\bigr)\ p \in G' \implies F\bigl(f_\sigma(G')\bigr) \supseteq s$ if and only if $(\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_F} \cap M$. Hence $F(x) = F(\sigma^G) = \bigcup\{s \mid (\exists p \in G)\ (\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_f} \cap M\}$, which is in $M[G]$, as desired.

Finally, we show that $A_G$ has the Baire property in $M[G]$ for Blackwell comeager many $G$ when $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega)^M$. Actually, we show that $A_G$ has the Baire property in $M[G]$ for any $\mathbb{P}$-generic $G$ over $M$. Let $s$ be a finite binary sequence. We show that there is a $t$ extending $s$ such that either $[t] \cap A_G$ or $[t] \setminus A_G$ is meager in $M[G]$. Let $\dot{c}$ be a canonical name for a Cohen real. Since one can embed Cohen forcing into $\mathrm{Coll}(\omega, 2^\omega)^M$ in a natural way in $M$, we may regard $\dot{c}$ as a $\mathbb{P}$-name for a Cohen real. Since $2^\omega$ in $M$ is countable in $M[G]$, the set of Cohen reals over $M$ is comeager in $M[G]$. Take any Cohen real $c$ over $M$ with $s \subseteq c$ in $M[G]$. We may assume $c$ is in $A_G$ (the case $c \notin A_G$ can be dealt with in the same way). Recall that $\rho^G = A_G$ and hence by the forcing theorem, there is a $p \in G$ and a $\sigma$ such that $M \vDash p \Vdash$ "$\dot{c} = \sigma \supseteq \check{s}$" and $(\mathbb{P}, p, \sigma) \in \tau_A \cap M$, which implies $(\mathbb{P}, p, \dot{c}) \in \tau_A \cap M$, namely $(\dot{c}, p) \in \rho_A$. But the value of $\dot{c}$ will be decided within Cohen forcing and by the definition of $\tau_A$, we may assume that $p$ is a condition of Cohen forcing extending $s$. Hence for any Cohen real $c'$ over $M$ with $p \subseteq c$ in $M[G]$, $c$ is in $A_G$. Since the set of all Cohen reals over $M$ is comeager in $M[G]$, this is what we desired. $\square$ (Claim 3.2.7)

We now finish the proof of Theorem 3.2.6 by showing that $A$ has the Baire property. Let $G$ be such that the conclusions of Claim 3.2.7 hold. By the first item of Claim 3.2.7, the structure $(\omega, {}^\omega\omega \cap M[G], \mathrm{app}, +, \cdot, =, 0, 1, A_G)$ is an elementary substructure of $\mathcal{A}_A^2$. Since the Baire property for $A$ can be described in the structure $\mathcal{A}_A^2$ in this language and $A_G$ has the Baire property in $M[G]$, $A$ also has the Baire property, as desired. $\square$ (Theorem 3.2.6)

Next, we show that every set of reals is $\infty$-Borel assuming $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. For that purpose, we introduce the Vopěnka algebra and its variant, which is a main tool for our argument. The original motivation for the Vopěnka algebra is to make every set to be generic over HOD, the class of all the hereditarily ordinal definable sets, i.e., any element of the transitive closure of a given set is ordinal definable. HOD is an important inner model of ZFC containing all the (possible) important inner models with large cardinals and it is close to $V$ in the sense that any set in $V$ can be generic over HOD via the Vopěnka algebra.

We define the Vopěnka algebra and its variant for $\mathrm{HOD}_X$, where $X$ is an arbitrary set, $\mathrm{OD}_X$ is the class of all sets ordinal definable with a parameter $X$, and $\mathrm{HOD}_X$ is the class of sets $a$ where any element of the transitive closure of $a$ is in $\mathrm{OD}_X$.

Take any arbitrary set $X$ and fix an ordinal definable injection $i_X \colon \mathrm{OD}_X \to \mathrm{HOD}_X$. Then consider the *Vopěnka algebra* $\mathbb{P}_{V,X}$ in $\mathrm{HOD}_X$ as follows: $\mathbb{P}_{V,X} = \{i_X(A) \mid A \in \mathrm{OD}_X \text{ and } A \subseteq \mathcal{P}(\omega)\}$. For $p, q \in \mathbb{P}_{V,X}$, $p \leq q$ if $i_X^{-1}(p) \subseteq i_X^{-1}(q)$. It is easy to see that the definition of $\mathbb{P}_{V,X}$ does not depend on the choice of $i_X$, i.e., if there are two such injections, then the corresponding two partial orders are isomorphic in $\mathrm{HOD}_X$. Vopěnka [87] proved that $\mathbb{P}_{V,\emptyset}$ is a complete Boolean algebra in HOD (when $X = \emptyset$) and each real in $V$ can be seen as a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD in the following way: For each real $x$ in $V$, the set $G_x = \{p \in \mathbb{P}_{V,\emptyset} \mid x \in i_\emptyset^{-1}(p)\}$ is a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD and $\mathrm{HOD}[x] = \mathrm{HOD}[G_x]$. Conversely, if $G$ is a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD, then the set $\bigcap\{i_\emptyset^{-1}(p) \mid p \in G\}$ is a singleton. We call the element of the singleton a *Vopěnka real over HOD* and denote it $y_G$. Then $y_{G_x} = x$ for each real $x$ in $V$. The analogue of the above results holds for $\mathrm{HOD}_X$ for arbitrary set $X$.

We now introduce a variant of the Vopěnka algebra, namely the *Vopěnka algebra with $\infty$-Borel codes.* Given a set $X$, consider the following partial order $\mathbb{P}^*_{V,X}$ in $\mathrm{HOD}_X$: Conditions of $\mathbb{P}^*_{V,X}$ are $\infty$-Borel codes in $\mathrm{HOD}_X$ where the ordinals used in their trees are below $\Theta$ in $\mathrm{HOD}_X$ and for $\phi, \psi$ in $\mathbb{P}^*_{V,X}$, $\phi \leq \psi$ if $B_\phi \subseteq B_\psi$.[1] Then we can prove the analogue of Vopěnka's theorem in exactly the same way:

**Theorem 3.2.8** (ZF)**.** (Folklore) Let $X$ be an arbitrary set.

1. $\mathbb{P}^*_{V,X}$ is a complete Boolean algebra in $\mathrm{HOD}_X$.
2. For each real $x$ in $V$, the set $G_x = \{\phi \in \mathbb{P}^*_{V,X} \mid x \in B_\phi\}$ is $\mathbb{P}^*_{V,X}$-generic over $\mathrm{HOD}_X$ and $\mathrm{HOD}_X[x] = \mathrm{HOD}_X[G_x]$. Conversely, if $G$ is a $\mathbb{P}^*_{V,X}$-generic filter over $\mathrm{HOD}_X$, then the set $\bigcap\{B_\phi \mid \phi \in G\}$ is a singleton and we call the real in the singleton a *Vopěnka real over* $\mathrm{HOD}_X$ and denote it $y_G$. Then $\mathrm{HOD}_X[y_G] = \mathrm{HOD}_X[G]$ and $y_{G_x} = x$ for each $G$ and $x$.

*Proof.* The proof is exactly the same as for the Vopěnka algebra which can be found, e.g., in Jech's textbook [37, Theorem 15.46]. □

[1] For any $\infty$-Borel code $\phi$ in $\mathrm{HOD}_X$, there is an $\infty$-Borel code $\psi$ where the ordinals used in the tree of $\psi$ is less than $\Theta$ in $\mathrm{HOD}_X$ such that $\phi \leq \psi$ and $\psi \leq \phi$. Hence the restriction of ordinals for $\infty$-Borel codes will not affect the structure of this partial order.

The difference between $\mathbb{P}_{V,X}$ and $\mathbb{P}^*_{V,X}$ is that $y_G$ might not recover $G$ from $\mathrm{HOD}_X$ for $\mathbb{P}_{V,X}$ while $\mathrm{HOD}_X[y_G] = \mathrm{HOD}_X[G]$ for $\mathbb{P}^*_{V,X}$. This is because the injection $i_X$ is not in $\mathrm{HOD}_X$ in general while the definition of $\mathbb{P}^*_{V,X}$ does not refer to OD. For our purpose, we will use $\mathbb{P}^*_{V,X}$.

**Theorem 3.2.9.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals is $\infty$-Borel.

*Proof.* We modify the argument for the following theorem by Woodin:

**Theorem 3.2.10** (Woodin)**.** Assume AD and that every relation on the reals can be uniformized. Then every set of reals is $\infty$-Borel.

Let $A$ be an arbitrary set of reals. We show that $A$ is $\infty$-Borel.

By Theorem 3.2.6, every set of reals has the Baire property. Hence every subset of $\mathrm{St}(\mathbb{P})$ has the Baire property for any $\mathbb{P} \in \mathcal{H}_{\omega_1}$. We freely use this fact later. We fix a simple coding of elements of $\mathcal{H}_{\omega_1}$ by reals and if we say "a real $x$ codes…", then we refer to this coding.

Let $\tau_A$ and $R_A$ be as follows:

$$\begin{aligned}\tau_A = \{(\mathbb{P}, p, \sigma) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\ & \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A\}, \\ R_A = \{(x, y) \mid\ & \text{if } x \text{ codes a } (\mathbb{P}, p, \sigma) \in \tau_A, \text{ then } y \text{ codes a } (D_i \mid i < \omega) \\ & \text{such that } (\forall i)\ D_i \text{ is dense in } \mathbb{P} \text{ and} \\ & \big(\forall G \in \mathrm{St}(\mathbb{P})\big)\ \big(p \in G, (\forall i)\ G \cap D_i \neq \emptyset \implies f_\sigma(G) \in A\big)\},\end{aligned}$$

where "$\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big) \ldots$" means "For comeager many $G$ in $\mathrm{St}(\mathbb{P}) \ldots$". Note that the term relation $\tau_A$ defined here is different from the one in Theorem 3.2.6 in the sense that now we use comeagerness for the quantifier $\forall^\infty$ instead of Blackwell comeagerness.

Let $F_A$ uniformize $R_A$ and $\Gamma_A$ be the graph of $F_A$, i.e., $\Gamma_A = \{(x, s) \mid s \in {}^{<\omega}\omega, F_A(x) \supseteq s\}$. Define $\tau_{\Gamma_A}$ as follows:

$$\begin{aligned}\tau_{\Gamma_A} = \{(\mathbb{P}, p, \sigma, s) \in \mathcal{H}_{\omega_1} \mid\ & \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\ & \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G), s\big) \in \Gamma_A\},\end{aligned}$$

here we also use comeagerness for the quantifier $\forall^\infty$.

Let $A^{\mathrm{c}}$ be the complement of $A$ and define and construct $\tau_{A^{\mathrm{c}}}, R_{A^{\mathrm{c}}}, F_{A^{\mathrm{c}}}, \Gamma_{A^{\mathrm{c}}}$, and $\tau_{\Gamma_{A^{\mathrm{c}}}}$ as above.

The following is the key point:

**Claim 3.2.11** (Woodin)**.** Let $M$ be a transitive subset of $\mathcal{H}_{\omega_1}$ and $(M, \in, \tau_A, \tau_{\Gamma_A})$ is a model of ZFC.[2] Let $(\mathbb{P}, p, \sigma) \in M \cap \tau_A$. Then for every $\mathbb{P}$-generic filter $G$ over $M$, if $p$ is in $G$, then $\sigma^G \in A$. The same holds for $A^{\mathrm{c}}$.

[2]Here it satisfies Comprehension scheme and Replacement scheme for formulas in the language of set theory with predicates for $\tau_A$ and $\tau_{\Gamma_A}$.

*Proof of Claim 3.2.11.* Let $\mathbb{Q} = \mathrm{Coll}\big(\omega, \mathrm{TC}(\mathbb{P})\big)$, where $\mathrm{Coll}\big(\omega, \mathrm{TC}(\mathbb{P})\big)$ is the standard forcing collapsing $\mathrm{TC}(\mathbb{P})$ into a countable set with finite sets as conditions. Since $\mathbb{P}, p, \sigma$ are countable in $M^{\mathbb{Q}}$, there is a $\mathbb{Q}$-name $\sigma'$ for a real in $M$ coding the triple $(\mathbb{P}, p, \sigma)$.

**Subclaim 3.2.12.** There is a $\mathbb{Q}$-name $\rho$ for a real in $M$ such that in $V$, for comeager many $H$ in $\mathrm{St}(\mathbb{Q})$, $f_\rho(H) = F_A(f_{\sigma'}(H))$.

*Proof of Subclaim 3.2.12.* First note that the map $f\colon H \mapsto F_A\big(f_{\sigma'}(H)\big)$ is continuous on a comeager set in $\mathrm{St}(\mathbb{Q})$, i.e., Baire measurable. This is because every subset of $\mathrm{St}(\mathbb{Q})$ has the Baire property in $\mathrm{St}(\mathbb{Q})$ and we can do the same argument as the one in Proposition 3.2.2 to uniformize a relation almost everywhere (since we use open sets in $\mathrm{St}(\mathbb{Q})$ to approximate subsets in $\mathrm{St}(\mathbb{Q})$ in this case, we get a continuous function instead of a Borel function).

Let $\rho = \tau_f$ where the notation $\tau_f$ is from Lemma 2.1.2. Then $\rho$ is a $\mathbb{Q}$-name for a real because the map $f$ is Baire measurable as we observed. Moreover, $\rho$ is in $M$ because

$$\big((m,n)\check{}, q\big) \in \rho \iff (\exists s \in {}^{<\omega}2)\ \big(s(m) = n \text{ and } \big(\mathbb{Q}, q, (\sigma, s)\big) \in \tau_{\Gamma_A}\big)$$

and the right hand side of the equivalence is definable in $(M, \tau_A, \tau_{\Gamma_A})$, which is a model of ZFC by assumption. Finally, by Lemma 2.1.2, it is easy to see that for comeager many $H$ in $\mathrm{St}(\mathbb{Q})$, $f_\rho(H) = F_A(f_{\sigma'}(H))$. □ (Subclaim 3.2.12)

Now let $G$ be a $\mathbb{P}$-generic filter over $M$ with $p \in G$. We show that $f_\sigma(G) \in A$. Take a $\mathbb{Q}$-generic filter $H$ over $M[G]$ with $\rho^H = F_A(\sigma'^H)$. This is possible by Subclaim 3.2.12 and that $M[G] \subseteq \mathcal{H}_{\omega_1}$. Then $G$ is also a $\mathbb{P}$-generic filter over $M[H]$ and $F_A(\sigma'^H) = \rho^H \in M[H]$. But by the definition of $F_A$, $F_A(\sigma'^H)$ codes a sequence $(D_i \mid i \in \omega)$ such that $D_i$ is a dense subset of $\mathbb{P}$ in $M[H]$ for each $i \in \omega$ and for any $G'$ in $\mathrm{St}(\mathbb{P})$, if $G' \cap D_i \neq$ for each $i$, then $f_\sigma(G') \in A$. But $G$ is a $\mathbb{P}$-generic filter over $M[H]$ and each $D_i$ is in $M[H]$. Hence $G \cap D_i \neq \emptyset$ for each $i \in \omega$ and $f_\sigma(G) \in A$, as desired. □ (Claim 3.2.11)

Let $X = (A, \tau_A, \tau_{\Gamma_A}, \tau_{A^c}, \tau_{\Gamma_{A^c}})$. Recall that $U$ is the fine normal measure on $\mathcal{P}_{\omega_1}$ we fixed at the beginning of this section. Let $M = \mathrm{L}(X, \mathbb{R})[U]$. Since the statement "a real is in the decode of an $\infty$-Borel code" is absolute between transitive models of ZF as in § 1.13 and $M$ contains all the reals, if $A$ is $\infty$-Borel in $M$, so is in $V$.

From now on, we work in $M$ and prove that $A$ is $\infty$-Borel in $M$, which completes the proof of this theorem. The benefit of working in $M$ is that we have DC in $M$ because $\mathrm{DC}_{\mathbb{R}}$ implies DC in $M$ while DC might fail in $V$ in general. Note that $U \cap M$ is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in $M$ and we use $U$ to denote $U \cap M$ from now on.

We find a set of ordinals $S$ and a formula $\phi$ such that for any real $x$,

$$x \in A \iff \mathrm{L}[S, x] \vDash \phi(x). \tag{3.1}$$

By Fact 1.13.2, this implies that $A$ is $\infty$-Borel.

For $a$ in $\mathcal{P}_{\omega_1}(\mathbb{R})$, let $M_a, \mathbb{Q}^*_a$, and $b_a$ be as follows:

$$\begin{aligned} M_a &= \mathrm{HOD}_X^{\mathrm{L}_{\omega_1}[X](a)}, \\ \mathbb{Q}^*_a &= \mathbb{P}^*_{V,X} \text{ in } M_a, \\ b_a &= \sup \{q \in \mathbb{Q}^*_a \mid (\mathbb{Q}^*_a, q, \dot{y_G}) \in \tau_A\} \text{ in } M_a, \end{aligned}$$

where $\dot{y_G}$ is a canonical $\mathbb{Q}^*_a$-name for a Vopěnka real given in Theorem 3.2.8.

Note that $M_a$ is a transitive subset of $\mathcal{H}_{\omega_1}$ and $(M_a, \tau_A, \tau_{\Gamma_A})$ and $(M_a, \tau_{A^c}, \tau_{\Gamma_{A^c}})$ are models of ZFC because $\mathrm{L}_{\omega_1}[X](a)$ is a transitive model of ZF (to check the power set axiom, we use the condition that there is no uncountable sequence of distinct reals ensured by Lebesgue measurability). Note also that $b_a$ is well-defined because $\mathbb{Q}^*_a$ is a complete Boolean algebra in $M_a$ by Theorem 3.2.8.

Then we claim that for each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and real $x$ which induces the filter $G_x$ that is $\mathbb{P}^*_{V,X}$-generic filter over $M_a$, $x \in A \iff b_a \in G_x$. Fix $a$ and $x$. Assume $b_a \in G_x$. We show that $x \in A$. If we apply Claim 3.2.11 to $M = M_a, (\mathbb{P}, p, \tau) = (\mathbb{Q}^*_a, b_a, \dot{y_G})$, and $G = G_x$, then we get $x \in A$ because $y_{G_x} = x$ as in Theorem 3.2.8. For the converse, we assume $b_a$ is not in $G_x$ and prove that $x$ is not in $A$. Let ${b_a}'$ be the one corresponding to $b_a$ for $A^c$ instead of for $A$, i.e.,

$${b_a}' = \sup \{q \in \mathbb{Q}^*_a \mid (\mathbb{Q}^*_a, q, \dot{y_G}) \in \tau_{A^c}\}.$$

Then $b_a \vee {b_a}' = \mathbf{1}$. This is because $f^{-1}_{\dot{y_G}}(A)$ has the Baire property in $\mathrm{St}(\mathbb{Q}^*_a)$. Since $b_a \notin G_x$ and $G_x$ is $\mathbb{P}^*_{V,X}$-generic over $M_a$, ${b_a}'$ is in $G_x$. Hence we can apply Claim 3.2.11 to $M_a, A^c, (\mathbb{Q}^*_a, {b_a}', \dot{y_G})$, and $G_x$ and we get $x \in A^c$, i.e., $x$ is not in $A$, as desired.

Fix an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$. Note that since $\mathbb{P}^*_{V,X}$ is the Vopěnka algebra with $\infty$-Borel codes defined in $M_a$, any real in $\mathrm{L}_{\omega_1}[X](a)$ is $\mathbb{P}^*_{V,X}$-generic over $M_a$. Hence for any real $x$ in $\mathrm{L}_{\omega_1}[X](a)$, $x \in A \iff b_a \in G_x$.

Now we use this local equivalence in $\mathrm{L}_{\omega_1}[X](a)$ to get the global equivalence (3.1) by taking the ultraproduct of $M_a$ via $U$. Let $M_\infty, \mathbb{Q}_\infty, b_\infty$ be as follows:

$$M_\infty = \prod_U M_a, \ \mathbb{Q}_\infty = \prod_U \mathbb{Q}^*_a, \ b_\infty = \prod_U b_a.$$

Note that Łoś's theorem holds for $M_\infty$ because there is a canonical function mapping $a$ to a well-order on $M_a$.[3] By DC (in $M$), $M_\infty$ is wellfounded. So we may assume $M_\infty$ is transitive. Hence, $M_\infty$ is a transitive model of ZFC, $\mathbb{Q}_\infty$ is a partial order consisting of $\infty$-Borel codes, and $b_\infty \in \mathbb{Q}_\infty$.

We claim that for each real $x$, $x \in A \iff x \in B_{b_\infty}$. This will establish the equivalence (3.1) because the pair $(\mathbb{Q}_\infty, b_\infty)$ can be seen as a set of ordinals since they are objects in the transitive model $M_\infty$ of ZFC.

[3]Łoś's theorem fails for $\prod_U \mathrm{L}_{\omega_1}[X](a)$. This is because $\mathrm{L}_{\omega_1}[X](a)$ is not a model of ZFC for almost all $a$ and we cannot assign a well-order on $\mathrm{L}_{\omega_1}[X](a)$ to each $a$ as we did for $\prod_U M_a$.

Let us fix a real $x$. By the fineness of $U$, $x \in a$ for almost all $a$ w.r.t. $U$. Then

$$\begin{aligned} x \in A &\iff b_a \in G_x \text{ for almost all } a \\ &\iff x \in B_{b_a} \text{ for almost all } a \\ &\iff x \in B_{b_\infty}, \end{aligned}$$

where the first equivalence is by the local equivalence we have seen and the third equivalence follows from Łoś's theorem for $\prod_U M_a[x]$ (note that $M_a[x]$ is a generic extension of $M_a$ given by $G_x$ and we can prove Łoś's theorem for $\prod_U M_a[x]$ in the same way as for $\prod_U M_a$). This completes the proof. □

Together with the non-existence of uncountable sequences of distinct reals, the $\infty$-Borelness for every set of reals gives us almost all the regularity properties we introduced in chapter 2 for every set of reals. Recall that $\mathbb{P}$-measurability for a strongly arboreal forcing $\mathbb{P}$ was the regularity property we introduced in Definition 2.1.7. Also recall that strongly proper forcings are strengthening of proper forcings for projective forcings.

**Proposition 3.2.13.** Assume that there is no uncountable sequence of distinct reals and every set of reals is $\infty$-Borel. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

*Proof.* The results for Cohen forcing, random forcing, and Mathias forcing are well-known and the proof is the same as the one in Case 1 in Theorem 2.4.2. We just replace $\mathrm{L}[a]$ in Theorem 2.4.2 with $\mathrm{L}[S]$, where $S$ codes a given set of reals and a given partial order $\mathbb{P}$. The fact that the set of all dense subsets of $\mathbb{P}$ in $\mathrm{L}[S]$ is countable follows from the non-existence of uncountable sequences of distinct reals (because $\mathrm{L}[S]$ is a ZFC model) and the fact that $\mathrm{L}[S]$ correctly computes $\mathbb{P}$ follows from that $S$ codes $\mathbb{P}$. The rest is exactly the same as in Case 1 in Theorem 2.4.2. □

**Corollary 3.2.14.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

## 3.3 Toward $\mathrm{AD}_{\mathbb{R}}$ from Bl-$\mathrm{AD}_{\mathbb{R}}$

In this section, we discuss the following conjecture:

**Conjecture 3.3.1** (DC). $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equivalent.

Since $\mathrm{AD}_{\mathbb{R}}$ implies Bl-$\mathrm{AD}_{\mathbb{R}}$ by Theorem 1.14.3, the question is whether Bl-$\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}_{\mathbb{R}}$ in ZF+DC. Woodin proved the following:

**Theorem 3.3.2** (Woodin). Assume AD and DC. Then the following are equivalent:

1. Every set of reals is Suslin,
2. The axiom $\mathsf{AD}_{\mathbb{R}}$ holds, and
3. Every relation on the reals can be uniformized.

Hence, to prove Conjecture 3.3.1, it suffices to show that every set of reals is Suslin from Bl-$\mathsf{AD}_{\mathbb{R}}$: If every set of reals is Suslin, then by Theorem 1.14.5, AD holds. Now by Theorem 3.3.2 and Theorem 1.14.9, $\mathsf{AD}_{\mathbb{R}}$ holds assuming Bl-$\mathsf{AD}_{\mathbb{R}}$ and DC. Note that Martin's Conjecture (i.e., Bl-AD implies AD) implies Conjecture 3.3.1 by Theorem 3.3.2. Hence it is interesting to see whether this is Conjecture is true or not.

We try to mimic the arguments for the implication from uniformization to Suslinness in Theorem 3.3.2 and reduce Conjecture 3.3.1 to a small conjecture. Throughout this section, we fix $U$ as a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, which exists by Theorem 3.1.3.

First, we show that every set of reals is strong $\infty$-Borel assuming Bl-$\mathsf{AD}_{\mathbb{R}}$. Before giving a definition of strong $\infty$-Borel codes, we start with a small lemma:

**Lemma 3.3.3.** Assume Bl-$\mathsf{AD}_{\mathbb{R}}$ and DC. Let $j\colon V \to \mathrm{Ult}(V,U)$ be the ultrapower map via $U$. Then $j(\omega_1) = \Theta$.

*Proof.* We first show that $j(\omega_1) \geq \Theta$. Let $\alpha$ be an ordinal less than $\Theta$ and $R$ be a prewellorder on the reals with length $\alpha$. Define $f\colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to \omega_1$ be as follows: For $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, $f(a)$ is the length of the prewellorder $R \cap (a \times a)$ on $a$. Since $a$ is countable, $f(a)$ is also countable. Hence $f \in_U c_{\omega_1}$, where $\in_U$ is the membership relation for $\mathrm{Ult}(V,U)$ and $c_{\omega_1}$ is the constant function on $\mathcal{P}_{\omega_1}(\mathbb{R})$ with value $\omega_1$.

We show that the structure $([f]_U, \in)$ is isomorphic to $(\alpha, \in)$ and hence $[f]_U = \alpha$, which implies $\alpha < j(\omega_1)$ because $f \in_U c_{\omega_1}$. For any $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let $\pi(a)$ be the transitive collapse of $\big(a, R \cap (a \times a)\big)$ into $\big(f(a), \in\big)$. Then by Łoś's Theorem for simple formulas, $[\pi]_U$ is an isomorphism between $\big([\mathrm{id}]_U, j(R) \cap ([\mathrm{id}]_U \times [\mathrm{id}]_U)\big)$ and $([f]_U, \in)$, where id is the identity function on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

**Claim 3.3.4.** The identity function id represents $\mathbb{R}$, i.e., $[\mathrm{id}]_U = \mathbb{R}$.

*Proof of Claim 3.3.4.* By the fineness of $U$, for any real $x$, $\{a \mid x \in a\} \in U$. Hence $[c_x]_U \in [\mathrm{id}]_U$. By the countable completeness of $U$, $[c_x]_U = x$ and hence $x \in [\mathrm{id}]_U$ for any real $x$. Suppose $f$ is a function on $\mathcal{P}_{\omega_1}(\mathbb{R})$ with $f \in_U \mathrm{id}$. Then by the normality of $U$, there is a real $x$ such that $\{a \mid x = f(a)\} \in U$, i.e., $c_x =_U f$. Hence $[f]_U = x$ and $[f]_U$ is a real, which finishes the proof. □ (Claim 3.3.4)

By Claim 3.3.4, we have $[\mathrm{id}]_U = \mathbb{R}$ and $j(R) \cap \big([\mathrm{id}]_U \times [\mathrm{id}]_U\big) = R$. Since $\big([\mathrm{id}]_U, j(R) \cap ([\mathrm{id}]_U \times [\mathrm{id}]_U)\big)$ and $([f]_U, \in)$ are isomorphic, $([f]_U, \in)$ is isomorphic to $(\mathbb{R}, R)$, which is isomorphic to $(\alpha, \in)$, as desired. Hence $\alpha < j(\omega_1)$ and $j(\omega_1) \geq \Theta$.

Next, we show that $j(\omega_1) \leq \Theta$. Let $f$ be a function from $\mathcal{P}_{\omega_1}(\mathbb{R})$ to $\omega_1$. We show that $[f]_U < \Theta$. By uniformization for every set of reals, there is a function

$e$ from the reals to themselves such that if a real $x$ codes an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, then $e(x)$ codes $f(a)$. Let $S$ be an $\infty$-Borel code for the graph $\Gamma_e$ of $e$ which exists by Theorem 3.2.9.

**Claim 3.3.5.** For all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, $f(a) < \Theta^{\mathrm{L}[S](a)}$.

*Proof of Claim 3.3.5.* Note that $\mathcal{P}(x) \cap \mathrm{L}[S](a)$ is countable in $V$ for any $x \in \mathcal{H}_{\omega_1} \cap \mathrm{L}[S](a)$. Hence there is a $\mathrm{Coll}(\omega, a)$-generic $g$ over $\mathrm{L}[S](a)$ in $V$. Fix such a $g$. Let $x_g$ be a real coding $a$ from $g$. Then since $S$ is an $\infty$-Borel code for $\Gamma_e$, one can compute whether $e(x_g) \supseteq s$ for each finite binary sequence $s$ or not in $\mathrm{L}[S](a, g)$, hence $e(x_g) \in \mathrm{L}[S](a, g)$. Therefore $f(a)$ is countable in $\mathrm{L}[S](a, g)$. But $\Theta^{\mathrm{L}[S](a)}$ stays an uncountable cardinal in $\mathrm{L}[S](a, g)$. Hence $f(a) < \Theta^{\mathrm{L}[S](a)}$, as desired. $\square$

By the normality of $U$, the following choice principle holds: For any function $F\colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to V$ such that $\emptyset \neq F(a) \in \mathrm{L}[S](a)$ for almost $a$ with respect to $U$, then there is a function $f\colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to V$ such that $f(a) \in F(a)$ for almost all $a$ with respect to $U$. This implies Łoś's Theorem for the ultraproduct $\prod_U \mathrm{L}[S](a)$.

Let $S^* = j(S)$. Then $\left(\prod_U \mathrm{L}[S](a), \in_U\right)$ is isomorphic to $\left(\mathrm{L}[S^*](\mathbb{R}), \in\right)$ by looking at the map $g \mapsto j(g)(\mathbb{R})$. (Note that $\mathrm{Ult}(V, U)$ is wellfounded by DC.) Hence

$$[f]_U < [a \mapsto \Theta^{\mathrm{L}[S](a)}]_U = \Theta^{\mathrm{L}[S^*](\mathbb{R})} \leq \Theta^V,$$

as desired. $\square$

We now introduce strong $\infty$-Borel codes. An $\infty$-Borel code $S$ is *strong* if the tree of $S$ is a tree on $\gamma$ for some $\gamma < \Theta$ and for any $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}$ such that $a$ is closed under $f$, $S{\restriction}\pi[a]$ is an $\infty$-Borel code, and $B_{S{\restriction}\pi[a]} \subseteq B_S$. Note that the choice of $\gamma$ does not depend on the definition of strong $\infty$-Borel codes. A set of reals $A$ is *strong $\infty$-Borel* if $A = B_S$ for some strong $\infty$-Borel code $S$. There is a finer version of Fact 1.13.2 as follows:

**Fact 3.3.6.**

1. Let $S$ be a strong $\infty$-Borel code and $\gamma < \Theta$ be such that $S$ is a tree on $\beta$ for some $\beta < \gamma$ and $\mathrm{L}_\gamma[S, x] \models$ "$\mathrm{KP} + \Sigma_1$-Separation" for any real $x$. Let $\phi(S, x)$ be a $\Sigma_1$-formula expressing "$x \in B_S$". Then for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $a$ is closed under $f$ and for any real $x$, if $\mathrm{L}_{\bar{\gamma}}[\bar{S}, x] \models \phi(\bar{S}, x)$, then $\mathrm{L}_\gamma[S, x] \models \phi(S, x)$, where $\mathrm{L}_{\bar{\gamma}}[\bar{S}]$ is the transitive collapse of the Skolem hull of $\pi[a] \cup \{S\}$ in $\mathrm{L}_\gamma[S]$.

2. Let $\gamma$ be an ordinal with $\gamma < \Theta$, $\phi$ be a $\Sigma_1$-formula, and $S$ be a bounded subset of $\gamma$ such that $\mathrm{L}_\gamma[S, x] \models$ "$\mathrm{KP} + \Sigma_1$-Separation" for any real $x$. Set $A = \{x \in \mathbb{R} \mid \mathrm{L}_\gamma[S, x] \models \phi(S, x)\}$. Assume that for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $a$ is closed under $f$ and for any real $x$, if $\mathrm{L}_{\bar{\gamma}}[\bar{S}, x] \models \phi(\bar{S}, x)$, then $\mathrm{L}_\gamma[S, x] \models \phi(S, x)$, where $\mathrm{L}_{\bar{\gamma}}[\bar{S}]$ is the

transitive collapse of the Skolem hull of $\pi[a] \cup \{S\}$ in $\mathrm{L}_\gamma[S]$. Then $A$ is strong $\infty$-Borel.

*Proof.* This can be done by closely looking at the argument for Fact 1.13.2 in [80]. □

**Theorem 3.3.7.** Assume Bl-$\mathrm{AD}_{\mathbb{R}}$ and DC. Then every set of reals is strong $\infty$-Borel.

*Proof.* Fix a set of reals $A$. We show that $A$ is strong $\infty$-Borel. Let $\big((M_a, \mathbb{Q}^*_a, b_a) \mid a \in \mathcal{P}_{\omega_1}(\mathbb{R})\big)$ and $(M_\infty, \mathbb{Q}^*_\infty, b_\infty)$ be as in the proof of Theorem 3.2.9, but we construct them in $V$, not in $M$. Since we have DC now, we can prove the following equivalences in exactly the same way as in Theorem 3.2.9: For all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and all real $x$ inducing the filter $G_x$ which is $\mathbb{Q}^*_a$-generic over $M_a$,

$$x \in A \iff b_a \in G_x \text{ (in } \mathbb{Q}^*_a).$$

Also,

$$(\forall x \in \mathbb{R})\ x \in A \iff b_\infty \in G_x \text{ (in } \mathbb{Q}^*_\infty).$$

For any $a$, let $D_a$ be the set of all dense subsets of $\mathbb{Q}^*_a$ in $M_a$ and let $D_\infty = \prod_U D_a$. Let $\phi$ be a $\Sigma_1$-formula such that for all $a$,

$$\begin{aligned}
\phi(\mathbb{Q}^*_a, b_a, D_a, x) \iff\ & x \text{ determines the filter } G_x \subseteq \mathbb{Q}^*_a \text{ such that} \\
& (\forall D \in D_a)\ G_x \cap D \neq \emptyset \text{ and } b_a \in G_x, \\
\phi(\mathbb{Q}^*_\infty, b_\infty, D_\infty, x) \iff\ & x \text{ determines the filter } G_x \subseteq \mathbb{Q}^*_\infty \text{such that} \\
& (\forall D \in D_\infty)\ G_x \cap D \neq \emptyset \text{ and } b_\infty \in G_x.
\end{aligned}$$

Let $S_a$ and $S_\infty$ be sets of ordinals coding the two triples $(\mathbb{Q}^*_a, b_a, D_a)$ and $(\mathbb{Q}^*_\infty, b_\infty, D_\infty)$ respectively. For an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let $\alpha_a$ be the least ordinal $\alpha$ such that $S_a$ is a bounded subset of $\alpha$ and for all $x \in a$, $\mathrm{L}_\alpha[S_a, x]$ is a model of KP+$\Sigma_1$-Separation and let $\alpha_\infty$ be the least ordinal $\alpha$ such that $S_\infty$ is a bounded subset of $\alpha$ and for all $x \in \mathbb{R}$, $\mathrm{L}_\alpha[S_\infty, x]$ is a model of KP+$\Sigma_1$-Separation. Note that by Łoś's Theorem, $(\prod_U \mathrm{L}_{\alpha_a}[S_a, x], \in_U)$ is isomorphic to $(\mathrm{L}_{\alpha_\infty}[S_\infty, x], \in)$ for every real $x$. Since each $\alpha_a$ is countable, by Lemma 3.3.3, $\alpha_\infty < \Theta$. Also, by the above equivalences, for all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and all reals $x$,

$$\begin{aligned}
x \in A &\iff \mathrm{L}_{\alpha_a}[S_a, x] \vDash \phi\ (S_a, x) \\
x \in A &\iff \mathrm{L}_{\alpha_\infty}[S_\infty, x] \vDash \phi\ (S_\infty, x).
\end{aligned}$$

By the second item of Fact 3.3.6, it suffices to show the following: For any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \alpha_\infty$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $a$ is closed under $f$ and for any real $x$, if $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}, x] \vDash \phi(\bar{S_\infty}, x)$, then $\mathrm{L}_{\alpha_\infty}[S_\infty, x] \vDash \phi(S_\infty, x)$, where $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}]$ is the transitive collapse of the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Let us fix $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and $\pi\colon \mathbb{R} \to \alpha_\infty$. Since $x \in A \iff \mathrm{L}_{\alpha_b}[S_b, x] \vDash \phi\ (S_b, x)$ for each real $x$ and $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$, the following claim completes the proof:

**Claim 3.3.8.** There are $a$ and $b$ in $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $a$ is closed under $f$ and $(X_a, \in)$ is isomorphic to $(\mathrm{L}_{\alpha_b}[S_b], \in)$, where $X_a$ is the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

*Proof of Claim 3.3.8.* Let $\Gamma_f = \{(x,s) \in \mathbb{R} \times {}^{<\omega}2 \mid f(x) \supseteq s\}$. For each $b$, consider the following game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$: In $\omega$ rounds,

1. Player I and II produce a countable elementary substructure $X$ of $\mathrm{L}_{\alpha_b}[S_b]$,
2. Player II produces an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ which is closed under $f$, and
3. Player II tries to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$, where $X_a$ is the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Player II wins if she succeeds to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$. This is an open game on some set of the form $T_b \times \mathbb{R}$ where $T_b$ is wellorderable. Hence by $\mathrm{DC}_{\mathbb{R}}$, it is determined.

**Subclaim 3.3.9.** There is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy in the game $\hat{G}_b$.

*Proof of Subclaim 3.3.9.* To derive a contradiction, suppose there is no $b$ such that player II has a winning strategy in the game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$. By the determinacy of the game $\hat{G}_b$, player I has a winning strategy in the game $\hat{G}_b$. Let $j \colon V \to \mathrm{Ult}(V, U)$ be the ultrapower map. Then by Łoś's Theorem, $\prod_U \big(\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi], \in_U, \Gamma_f, \pi\big)$ is isomorphic to $\big(\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)], \in, \Gamma_f, j(\pi)\big)$. Then the game $\hat{G}_\infty = \prod_U \hat{G}_b$ is an open game on some set of the form $T_\infty \times \mathbb{R}$ where $T_\infty$ is wellorderable in $\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)]$ such that in $\omega$ rounds,

1. Players I and II produce a countable elementary substructure $Y$ of $\mathrm{L}_{\alpha_\infty}[S_\infty]$,
2. Player II produces an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ which is closed under $f$, and
3. Player II tries to construct an isomorphism between $(Y, \in)$ and $(Y_a, \in)$, where $Y_a$ is the Skolem hull of $j(\pi)[a] \cup \{j(S_\infty)\}$ in $\mathrm{L}_{j(\alpha_\infty)}[j(S_\infty)]$.

Player II wins if she succeeds to construct an isomorphism between $Y$ and $Y_a$. By Łoś's Theorem, player I has a winning strategy $\sigma$ in $\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)]$. By Theorem 1.12.6, $\sigma$ is also winning in $V$. In $V$, let player II move in such a way that she can arrange that $a$ is closed under $f$, $j[Y] = Y_a$, and $j \restriction Y$ is the candidate for the isomorphism. This is possible by a bookkeeping argument. But then player II wins because $j \restriction Y$ is an isomorphism between $Y$ and $j[Y]$ and defeats the strategy $\sigma$, contradiction! □ (Subclaim 3.3.9)

Hence there is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy $\tau$ in the game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$. By Theorem 1.12.6, $\tau$ is also winning in $V$. Since $\mathrm{L}_{\alpha_b}[S_b]$ is countable in $V$, we can let player I move in such a way that $X = \mathrm{L}_{\alpha_b}[S_b]$ and let player II follow $\tau$. Since $\tau$ is winning in $V$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $a$ is closed under $f$ and $\mathrm{L}_{\alpha_b}[S_b] = X$ is isomorphic to $X_a$, as desired. □ (Claim 3.3.8)

□

We are now ready to prove the key statement toward Conjecture 3.3.1: Recall that for a natural number $n$ with $n \geq 1$ and a subset $A$ of $\mathbb{R}^{n+1}$, $\exists^{\mathbb{R}} A = \{x \in \mathbb{R}^n \mid (\exists y \in \mathbb{R})\ (x, y) \in A\}$.

**Theorem 3.3.10.** Assume Bl-AD$_{\mathbb{R}}$ and DC. Let $A$ be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a strict well-founded relation on a set of reals. Suppose $A$ has a strong $\infty$-Borel code $S$ and let $\gamma$ be an ordinal less than $\Theta$ such that the tree of $S$ is on $\gamma$. Then the length of $\exists^{\mathbb{R}} A$ is less than $\gamma^+$.

*Proof.* Let $A, S$, and $\gamma$ be as in the assumptions. We show that the length of $\exists^{\mathbb{R}} A$ is less than $\gamma^+$. Fix a surjection $\pi \colon \mathbb{R} \to \gamma$. Let us start with the following lemma:

**Lemma 3.3.11.** There is a function $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if $a$ is closed under $f$, then $S \restriction \pi[a]$ is an $\infty$-Borel code and $B_{S \restriction \pi[a]} \subseteq B_S$.

Note that the assertion of the above lemma is the strengthening of the definition of strong $\infty$-Borel codes.

*Proof of Lemma 3.3.11.* Let us consider the following game: Player I and II choose reals one by one and produce an $\omega$-sequence $x$ of reals. Setting $a = \mathrm{ran}(f)$, player I wins if $S \restriction \pi[a]$ is an $\infty$-Borel code and $B_{S \restriction \pi[a]} \subseteq B_S$. Since $S$ is a strong $\infty$-Borel code, player I can defeat any strategy for player II because strategies can be seen as functions from ${}^{<\omega}\mathbb{R}$ to $\mathbb{R}$ by Claim 3.1.5. Since the payoff set of this game is range-invariant, by Lemma 3.1.4, this game is determined. Hence player I has a winning strategy and by Claim 3.1.5, there is a function $f$ as desired. □ (Lemma 3.3.11)

We fix an $f_0$ satisfying the conclusion of Lemma 3.3.11 for the rest of this proof. Recall that $U$ is the fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ we fixed at the beginning of this section. Using $\pi$, we can transfer this measure to a fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$ as follows: Let $\pi_* \colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\gamma)$ be such that $\pi_*(a) = \pi[a]$ for each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$. For $A \subseteq \mathcal{P}_{\omega_1}(\gamma)$, $A \in U_\pi$ if $\pi_*^{-1}(A) \in U$. It is easy to check that $U_\pi$ is a fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$.

We now prove the key lemma for this theorem:

**Lemma 3.3.12.** Let $G$ be $\mathrm{Coll}(\omega, \gamma)$-generic over $V$. Then in $V[G]$, there is an elementary embedding $j \colon \mathrm{L}(\mathbb{R}, S, f_0, \pi) \to \mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$ such that all the reals in $V[G]$ are contained in $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$.

*Proof of Lemma 3.3.12.* The argument is based on the result of Kechris and Woodin [47, Theorem 6.2]. We first introduce the notion of weakly meager sets. A subset $B$ of ${}^{\omega}\gamma$ is *weakly meager* if there is an $X \in U_\pi$ such that $(\forall b \in X)\, {}^{\omega}b \cap B$ is meager in the space ${}^{\omega}b$. Since $b$ is countable, the space ${}^{\omega}b$ is homeomorphic to the Baire space in most cases. Note that if $B$ is a meager set in the space ${}^{\omega}\gamma$, then it is weakly meager. A subset $B$ of ${}^{\omega}\gamma$ is *weakly comeager* if its complement is weakly meager. Let $I$ be the set of weakly meager sets.

**Sublemma 3.3.13.**

1. The ideal $I$ is a $\sigma$-ideal on ${}^{\omega}\gamma$.

2. For any $s \in {}^{<\omega}\gamma$, $[s]$ is not weakly meager.

3. If a subset $B$ of ${}^{\omega}\gamma$ is not weakly meager, then there is an $s \in {}^{<\omega}\gamma$ such that $[s] \setminus B$ is weakly meager.

4. Let $g$ be a function from ${}^{\omega}\gamma$ to On. Then for any $B$ which is not weakly meager, there is a $B' \subseteq B$ which is not weakly meager such that for all $x$ and $y$ in $B'$, if $\mathrm{ran}(x) = \mathrm{ran}(y)$, then $g(x) = g(y)$.

*Proof.* The first statement follows from the $\sigma$-completeness of $U_\pi$. The second statement follows from the fineness of $U_\pi$.

For the third statement, suppose $B$ is not weakly meager. Then since $U_\pi$ is an ultrafilter, there is an $X \in U_\pi$ such that $(\forall b \in X)\ {}^{\omega}b \cap B$ is not meager in ${}^{\omega}b$. We may assume that each $b$ in $X$ is infinite because the set of finite subsets of $\gamma$ is measure zero with respect to $U_\pi$ by the fineness of $U_\pi$. Take any $b$ in $X$. Since the space ${}^{\omega}b$ is homeomorphic to the Baire space, the set ${}^{\omega}b \cap B$ has the Baire property in ${}^{\omega}b$. Hence there is an $s_b \in {}^{<\omega}b$ such that $[s_b] \setminus B$ is meager in ${}^{\omega}b$. By normality of $U_\pi$, there is a $Y \in U_\pi$ such that $Y \subseteq X$ and there is an $s \in {}^{<\omega}\gamma$ such that $s_b = s$ for any $b \in Y$. Hence $[s] \setminus B$ is weakly meager.

For the last statement, let $g$ be such a function and $B$ be not weakly meager. Then there is an $X \in U_\pi$ such that $\forall b \in X$, ${}^{\omega}b \cap B$ is not meager in ${}^{\omega}b$. Since ${}^{\omega}b \cap B$ has the Baire property in ${}^{\omega}b$, there is an $s_b \in {}^{<\omega}b$ such that $[s_b] \setminus B$ is meager in ${}^{\omega}b$. By normality of $U_\pi$, there are a $Y \subseteq X$ and $s_0 \in {}^{<\omega}\gamma$ such that $Y \in U_\pi$ and $s_b = s_0$ for every $b \in Y$. We use the following fact:

**Fact 3.3.14** (Folklore)**.** Assume every set of reals has the Baire property. Then the meager ideal in the Baire space is closed under any wellordered union.

Take any $b \in Y$. Since $[s_0] \cap {}^{\omega}b$ is homeomorphic to the Baire space, we can apply Fact 3.3.14 to the space $[s_0] \cap {}^{\omega}b$ and hence there is an $\alpha_b$ such that $[s_0] \cap {}^{\omega}b \cap g^{-1}(\alpha_b)$ is not meager in $[s_0] \cap {}^{\omega}b$. Since the set $[s_0] \cap {}^{\omega}b \cap g^{-1}(\alpha_b)$ has the Baire property in $[s_0] \cap {}^{\omega}b$, there is an $s_b \in {}^{<\omega}b$ such that $s_b \supseteq s_0$ and $[s_b] \setminus g^{-1}(\alpha_b)$ is meager in ${}^{\omega}b$. By normality of $U_\pi$, there are a $Z \in U_\pi$ with $Z \subseteq Y$ and an $s_1 \supseteq s_0$ such that $[s_1] \setminus g^{-1}(\alpha_b)$ is meager in ${}^{\omega}b$ for each $b \in Z$. Then $B' = B \cap [s_1] \cap \{x \mid g(x) = \alpha_{\mathrm{ran}(x)}\}$ is as desired. □ (Sublemma 3.3.13)

Now we prove Lemma 3.3.12. Let $G$ be $\mathrm{Coll}(\omega, \gamma)$-generic over $V$. Consider the Boolean algebra $\mathcal{P}({}^{\omega}\gamma)/I$. Then it is naturally forcing equivalent to $\mathrm{Coll}(\omega, \gamma)$: In fact, for $s \in {}^{<\omega}\gamma$, let $i(s) = [s]/I$. Then by the third item of Sublemma 3.3.13, $i$ is a dense embedding from $\mathrm{Coll}(\omega, \gamma)$ to $\mathcal{P}({}^{\omega}\gamma)/I \setminus \{\mathbf{0}\}$. Define $U'$ as follows: For a subset $B$ of ${}^{\omega}\gamma$ in $V$, $B$ is in $U'$ if there is a $p \in G$ such that $[p] \setminus B$ is weakly meager. By the genericity of $G$ and the third item of Sublemma 3.3.13, $U'$ is an ultrafilter on $({}^{\omega}\gamma)^V$ and $U'$ contains all the weakly comeager sets. Take an ultrapower $\mathrm{Ult}\big(\mathrm{L}(\mathbb{R}, S, f_0, \pi), U'\big) = \big({}^{({}^{\omega}\gamma)^V}\mathrm{L}(\mathbb{R}, S, f_0, \pi) \cap V\big)/U'$ and let $j$ be the ultrapower map. (Note that we consider $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$-valued functions in $V$ which are not necessarily in $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$.)

We show that $j$ is the desired map. We first check Łoś's Theorem for this ultrapower. It is enough to show that for any $B \in U'$ and a function $F$ from $B$ to $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$ such that all the values of $F$ are nonempty, then there is a function $f$ on $B$ in $V$ such that $f(x) \in F(x)$ for all $x$ in $B'$. Since there is a surjection from $\mathbb{R} \times \mathrm{On}$ to $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$, we may assume that the values of $F$ are sets of reals. But then by uniformization for every relation on the reals by Theorem 1.14.9, we get the desired $f$.

Next, we check the well-foundedness of $\mathrm{Ult}\big(\mathrm{L}(\mathbb{R}, S, f_0, \pi), U'\big)$. By DC, we know that the ultrapower $\mathrm{Ult}(V, U_\pi)$ is wellfounded. Hence it suffices to show the following: For a function $f \colon \mathcal{P}_{\omega_1}(\gamma) \to \mathrm{On}$, let $g_f \colon {}^{\omega}\gamma \to \mathrm{On}$ be as follows: $g_f(x) = f\big(\mathrm{ran}(x)\big)$.

**Sublemma 3.3.15.** The map $[f]_{U_\pi} \mapsto [g_f]_{U'}$ is an isomorphism from $\big(({}^{\mathcal{P}_{\omega_1}(\gamma)}\mathrm{On} \cap V)/U_\pi, \in_{U_\pi}\big)$ to $\big(({}^{{}^{\omega}\gamma}\mathrm{On} \cap V)/U', \in_{U'}\big)$.

*Proof of Sublemma 3.3.15.* We first show that if $f_1 \in_{U_\pi} f_2$, then $g_{f_1} \in_{U'} g_{f_2}$. Since $f_1 \in_{U_\pi} f_2$, there is an $X \in U_\pi$ such that for any $b$ in $X$, $f_1(b) \in f_2(b)$. Fix a $b$ in $X$. Since the set $\{x \in {}^{\omega}b \mid \mathrm{ran}(x) = b\} \cap {}^{\omega}b$ is comeager in ${}^{\omega}b$, the set $\{x \in {}^{\omega}b \mid f_1\big(\mathrm{ran}(x)\big) \in f_2\big(\mathrm{ran}(x)\big)\}$ is comeager in ${}^{\omega}b$. Hence for every $b \in X$, the set $\{x \in {}^{\omega}b \mid g_{f_1}(x) \in g_{f_2}(x))\}$ is comeager in ${}^{\omega}b$ and the set $\{x \in {}^{\omega}\gamma \mid g_{f_1}(x) \in g_{f_2}(x)\}$ is weakly comeager and hence is in $U'$. Therefore, $g_{f_1} \in_{U'} g_{f_2}$. In the same way, one can prove that if $f_1 =_{U_\pi} f_2$, then $g_{f_1} =_{U'} g_{f_2}$.

Next, we show that the map is surjective. Take any function $g \colon {}^{\omega}\gamma \to \mathrm{On}$ in $V$. We show that there is an $f \colon \mathcal{P}_{\omega_1}(\gamma) \to \mathrm{On}$ in $V$ such that $g_f =_{U'} g$. By the last item of Sublemma 3.3.13 and the genericity of $G$, there is an $Y$ in $U'$ such that if $x$ and $y$ are in $Y$ with the same range, then $g(x) = g(y)$. Since $Y$ is in $U'$, there is a $p \in G$ such that $[p] \setminus Y$ is weakly meager, hence there is an $X$ in $U_\pi$ such that for all $b$ in $X$, $([p] \setminus Y) \cap {}^\omega b$ is meager in ${}^\omega b$. This means that $g$ is constant on a comeager set in $[p] \cap {}^\omega b$ for each $b \in X$. Let $\alpha_b$ be the constant value for each $b \in X$ and $f$ be such that $f(b) = \alpha_b$ if $b$ is in $Y$ and $f(b) = 0$ otherwise. Then it is easy to check that $g_f =_{U'} g$, as desired. □ (Sublemma 3.3.15)

We have shown that $j$ is elementary and we may assume that the target model of $j$ is transitive. Then $j$ is an elementary embedding from $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$ to $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. Let $M = \mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. We finally check that all the reals in $V[G]$ are in $M$. Let $x$ be a real in $V[G]$ and $\tau$ be a $\mathbb{P}$-name for a real in $V$ such that $\tau^G = x$. We claim that $[f_\tau]_{U'} = x$, where $f_\tau$ is the Baire measurable function from $\mathrm{St}\big(\mathrm{Coll}(\omega, \gamma)\big)$ to the reals induced by $\tau$ from Lemma 2.1.2, which completes the proof.

Take any natural number $n$ and set $m = x(n)$. We show that $[f_\tau]_{U'}(n) = m$. Since $x(n) = m$, there is a $p \in G$ such that $p \Vdash \tau(\check{n}) = \check{m}$. By the definition of $f_\tau$, for any $x \in [p]$, $f_\tau(x)(n) = m\}$. Since $p$ is in $G$, by the definition of $U'$, the set $\{x \mid f_\tau(x)(n) = m$ is in $U'$, as desired. □ (Lemma 3.3.12)

We now finish the proof of Theorem 3.3.10. Let us keep using $M$ to denote $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. We first claim that $S$ and $j[S]$ are in $M$. Since $\gamma$ is countable in $V[G]$, there is a real $x$ coding $S$ in $V[G]$. But by Lemma 3.3.12, such an $x$ is in $M$. Hence $S$ is also in $M$. Since $\gamma$ is countable in $V[G]$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\pi[a] = S$ and hence $j(\pi)[a] = j[S]$ in $V[G]$. But since $j(\pi) \in M$ and $a \in M$ by Lemma 3.3.12, $j[S] = j(\pi)[a]$ is also in $M$, as desired. By Lemma 3.3.11 and elementarity of $j$, the following is true in $M$: For any $a$ closed under $j(f)$, $j(S) \restriction a$ is an $\infty$-Borel code and $B_{j(S)\restriction a} \subseteq B_{j(S)}$. Also, by elementarity of $j$, $\exists^{\mathbb{R}} B_{j(S)}$ is a well-founded relation on a set of reals in $M$. Set $a = j[S]$. Since $a$ is closed under $j(f)$, in $M$, $j(S) \restriction a$ is an $\infty$-Borel code, $B_{j(S)\restriction a} \subseteq B_{j(S)}$, and $\exists^{\mathbb{R}} B_{j[S]}$ is also a wellfounded relation on a set of reals in $M$. Since $j[S]$ is countable in $M$, the relation $\exists^{\mathbb{R}} B_{j[S]}$ is $\mathbf{\Sigma}^1_1$ and hence by Kunen-Martin Theorem (see [66, 2G.2]), its rank is less than $\omega_1$ in $M$ which is the same as $\gamma^+$ in $V$. Finally, since $S$ and $j[S]$ are equivalent as Borel codes, $\exists^{\mathbb{R}} B_S$ has length less than $\omega_1$ in $M$ and since $M$ has more reals than $V$, $\big(\exists^{\mathbb{R}} B_S\big)^V \subseteq \big(\exists^{\mathbb{R}} B_S\big)^M$. Therefore, the length of $\big(\exists^{\mathbb{R}} B_S\big)^V$ is less than $\omega_1^M = (\gamma^+)^V$, as desired. □

Becker proved the following:

**Theorem 3.3.16** (Becker)**.** Assume AD, DC, and the uniformization for every relation on the reals. Suppose that the conclusion of Theorem 3.3.10 holds, i.e., let $A$ be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a well-founded relation on a set of reals. Suppose $A$ has a strong $\infty$-Borel code $S$ and let $\gamma$ be an ordinal less than $\Theta$ such that the tree of $S$ is on $\gamma$. Then the length of $\exists^{\mathbb{R}} A$ is less than $\gamma^+$. Then every set of reals is Suslin.

*Proof.* See [9]. □

We try to simulate Becker's argument, make a small conjecture, and reduce Conjecture 3.3.1 to the small conjecture.

As preparation, we prove a weak version of Moschovakis' Coding Lemma. Let us introduce some notions for that. Let $A$ be a set of reals. Let $\mathrm{IND}(A)$ be the set of all $\mathrm{pos}\Sigma^1_n(A)$-inductive sets of reals for some natural number $n \geq 1$. For the definition of $\mathrm{pos}\Sigma^1_n(A)$-inductive sets, see [66, 7C]. All we need is as follows:

**Fact 3.3.17.** For any set of reals $A$, $\mathrm{IND}(A)$ is the smallest Spector pointclass containing $A$ and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$.

*Proof.* The argument is the same as [66, 7C.3]. □

**Theorem 3.3.18** (Weak version of Moschovakis' Coding Lemma)**.** Assume Bl-AD. Let $<$ be a strict wellfounded relation on a set $A$ of reals with rank function $\rho\colon A \to \gamma$ onto and let $\Gamma$ be a Spector pointclass containing $<$ and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$. Then for any subset $S$ of $\gamma$, there is a set of reals $C \in \boldsymbol{\Gamma}$ such that $\rho[C] = S$.

By Fact 3.3.17, $\mathrm{IND}(<)$ satisfies the conditions for $\Gamma$.

*Proof.* The argument is based on Moschovakis' original argument [66, 7D.5].

Let $S$ be a subset of $\gamma$. We show that for any $\alpha \leq \gamma$, there is a set of reals $C_\alpha \in \boldsymbol{\Gamma}$ with $\rho[C_\alpha] = S \cap \alpha$ by induction on $\alpha$.

It is trivial when $\alpha = 0$ and it is also easy when $\alpha$ is a successor ordinal because $\boldsymbol{\Gamma}$ is a boldface pointclass. So assume $\alpha$ is a limit ordinal and the above claim holds for each $\xi < \alpha$. We show that there is a $C \in \boldsymbol{\Gamma}$ with $\rho[C] = S \cap \alpha$.

Since $\Gamma$ is $\omega$-parametrized and closed under recursive substitutions, we have $\{G^n \subseteq \mathbb{R} \times \mathbb{R}^n \mid n \geq 1\}$ given in Lemma 1.7.1. Let $G^2_a = \{x \in \mathbb{R} \mid (a,x) \in G^2\}$ for each real $a$. For a real $a$, we say $G^2_a$ *codes a subset $S'$ of $S$* if $G^2_a \subseteq A$ and $\rho[G^2_a] = S'$.

Let us consider the following game $\mathcal{G}_\alpha$: Player I and II choose 0 or 1 one by one and they produce reals $a$ and $b$ separately and respectively. Player II wins if either ($G^2_a$ does not code $S \cap \xi$ for any $\xi < \alpha$) or ($G^2_a$ codes $S \cap \xi$ for some $\xi < \alpha$ and $G^2_b$ codes $S \cap \eta$ for some $\eta < \alpha$ with $\eta > \xi$). By Bl-AD, one of the players has an optimal strategy in this game.

**Case 1**: Player I has an optimal strategy $\sigma$ in $\mathcal{G}_\alpha$.

For a real $b$, let $\tau_b$ be the mixed strategy for player II such that player II produces $b$ with probability 1 no matter how player I plays. Since $\sigma$ is optimal for player I, for each real $b$, for $\mu_{\sigma,\tau_b}$-measure one many reals $a$, $G^2_a$ codes $S \cap \xi$ for some $\xi < \alpha$. Fix a real $b$. We use the following fact analogous to Fact 3.3.14:

**Fact 3.3.19** (Folklore)**.** Let $\mu$ be a Borel probability measure on the Baire space and assume every set of reals is $\mu$-measurable. Then the set of $\mu$-null sets is closed under wellordered unions.

Since every set of reals is Lebesgue measurable by Theorem 1.14.8, every set of reals is $\mu_{\sigma,\tau_b}$-measurable. By Fact 3.3.19, there is a unique $\xi_b < \alpha$ such that for $\mu_{\sigma,\tau_b}$-positive measure many reals $a$, $G^2_a$ codes $S \cap \xi_b$ and the set of reals $a$ such that $G^2_a$ codes $S \cap \xi$ for some $\xi < \xi_b$ is $\mu_{\sigma,\tau_b}$-measure zero. Let $C$ be the following: A real $x$ is in $C$ if there is a real $b$ such that for $\mu_{\sigma,\tau_b}$-positive measure many reals $a$, they code the same subset $S'$ of $\gamma$, and no proper subsets of $S'$ can be coded by $\mu_{\sigma,\tau_b}$-positive measure many reals, and $x \in G^2_a$ for some real $a$ such that $G^2_a$ codes $S'$. Since $\Gamma$ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, $C$ is in $\Gamma(\sigma)$. By induction hypothesis, for any $\xi < \alpha$, there is a real $b$ such that $G^2_b$ codes $S \cap \xi$. Since $\sigma$ is optimal, $C$ codes $S \cap \alpha$, as desired.

**Case 2**: Player II has an optimal strategy $\tau$ in $\mathcal{G}_\alpha$.

Let $(a, x) \mapsto \{a\}(x)$ be the partial function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ which is universal for all the partial functions from $\mathbb{R}$ to itself that are $\boldsymbol{\Gamma}$-recursive on their domain. For reals $a$ and $w$, define a set of reals $A_{a,w}$ as follows: a real $x$ is in $A_{a,w}$ if there exists $z < w$ such that $\{a\}(z)$ is defined and $\big(\{a\}(z), x\big) \in G^2$. It is easy to see that $A_{a,w}$ is in $\boldsymbol{\Gamma}$. By Lemma 1.7.1, there is a $\Gamma$-recursive function $\pi \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $A_{a,w} = G^2_{\pi(a,w)}$ for each $a$ and $w$.

For each real $a$ and $w$, define a set of reals $C_{a,w}$ as follows: A real $x$ is in $C_{a,w}$ if for $\mu_{\sigma_{\pi(a,w)},\tau}$-positive measure many $b$, they code the same subset $S'$ of $\gamma$, no proper subsets of $S'$ can be coded by $\mu_{\sigma,\tau_b}$-positive measure many reals, and $x$ is in $G^2_b$ for some real $b$ such that $G^2_b$ codes $S'$. It is easy to see that $C_{a,w}$ is in $\boldsymbol{\Gamma}$. Hence by Lemma 1.7.1, there is a $\Gamma$-recursive function $\pi' \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $C_{a,w} = G^2_{\pi'(a,w)}$ for each $a$ and $w$.

Since the function $(a, w) \mapsto \pi'(a, w)$ is $\Gamma$-recursive in $\tau$ and total, by Recursion Theorem 1.7.3, we can find a fixed $a^*$ such that for all $w$, $\{a^*\}(w) = \pi'(a^*, w)$. Let $g(w) = \{a^*\}(w)$.

**Claim 3.3.20.** For each $w \in A$ with $\rho(w) < \alpha$, there is some $\eta(w) < \alpha$ with $\rho(w) < \eta(w)$ such that $G^2_{g(w)}$ codes $S \cap \eta(w)$.

*Proof of Claim 3.3.20.* We show the claim by induction on $w$. Suppose it is done for all $x < w$. Then $A_{a^*,w}$ codes $S \cap \xi$ where $\xi = \sup\{\eta(x) \mid x < w\} \geq \rho(w)$. Since $\tau$ is optimal for II, $C_{a^*,w}$ codes $S \cap \eta$ for some $\eta > \xi$. Since $G^2_{g(w)} = C_{a^*,w}$, setting $\eta(w) = \eta$, $\eta(w) > \rho(w)$ and $G^2_{g(w)}$ codes $S \cap \eta(w)$. □ (Claim 3.3.20)

Let $C = \bigcup_{w\in A,\rho(w)<\alpha} G^2_{g(w)}$. Then by Claim 3.3.20, $C$ codes $S \cap \alpha$ and $C$ is in $\boldsymbol{\Gamma}$, as desired. □

We also need a weak version of Wadge's Lemma: Let $A$ be a set of reals. For a natural number $n \geq 1$, a set of reals $B$ is $\boldsymbol{\Sigma}^1_n$ *in* $A$ if $B$ is definable by a $\Sigma^1_n$ formula in the structure $\mathcal{A}^2_A$ that is the second order structure with $A$ as an unary predicate with a parameter $x$ for some real $x$. A set of reals $B$ is *projective in* $A$ if $B$ is $\boldsymbol{\Sigma}^1_n(A)$ for some $n \geq 1$.

**Lemma 3.3.21** (Weak version of Wadge's Lemma)**.** Assume Bl-AD. Then for any two sets of reals $A$ and $B$, either $A$ is $\mathbf{\Sigma}^1_2$ in $B$ or $B$ is $\mathbf{\Sigma}^1_2$ in $A$.

*Proof.* Recall the Wadge game $G_{\mathrm{W}}(A,B)$ from § 1.15. By Bl-AD, one of the players has an optimal strategy in $G_{\mathrm{W}}(A,B)$. Assume player II has an optimal strategy $\tau$ in $G_{\mathrm{W}}(A,B)$. Then for any real $x$,

$$x \in A \iff \mu_{\sigma_x,\tau}\big(\{(x',y) \mid x' = x \text{ and } y \in B\}\big) = 1.$$

It is easy to see that the right hand side of the equivalence is $\mathbf{\Sigma}^1_2$ in $B$. If player I has an optimal strategy in $G_{\mathrm{W}}(A,B)$, then one can prove that $B$ is $\mathbf{\Sigma}^1_2$ in $A^{\mathrm{c}}$ in the same way and hence $B$ is $\mathbf{\Sigma}^1_2$ in $A$. $\square$

For the rest of this section, we assume Bl-AD$_{\mathbb{R}}$ and DC. We fix a set of reals $A$ and give a scenario to prove that $A$ is Suslin. We fix a simple surjection $\rho$ from the reals to $\{0,1\}$, e.g., $x \mapsto x(0)$.

**Claim 3.3.22.** There is a sequence $\big((\Gamma_n,<_n,\gamma_n,) \mid n<\omega\big)$ such that for all $n$,

1. $\Gamma_n$ is a Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, $\Gamma_n \subseteq \Gamma_{n+1}$, and $A \in \Gamma_0$,
2. every relation on the reals which is projective in a set in $\mathbf{\Gamma}_n$ can be uniformized by a function in $\mathbf{\Gamma}_{n+1}$,
3. $<_n$ is in $\mathbf{\Gamma}_n$ and a strict wellfounded relation on the reals with length $\gamma_n$ and every set of reals which is projective in a set in $\mathbf{\Gamma}_n$ has a strong $\infty$-Borel code whose tree is on $\gamma_{n+1}$.

*Proof of Claim 3.3.22.* We construct them by induction on $n$. For $n=0$, let $\Gamma_0$ be any Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ containing $A$ which exists by Fact 3.3.17, and $<_0$ be any strict wellfounded relation on the reals in $\Gamma_0$. Then they satisfy all the items above.

Suppose we have constructed $(\Gamma_n,<_n,\gamma_n)$ with the above properties. We construct $\Gamma_{n+1},<_{n+1}$, and $\gamma_{n+1}$ . First note that there is a set $B_n$ of reals which is not projective in any set in $\mathbf{\Gamma}_n$ by uniformization for every relation on the reals. Then by Lemma 3.3.21, every set projective in a set in $\mathbf{\Gamma}_n$ is $\mathbf{\Sigma}^1_2$ in $B_n$. Let $H_n$ and $H'_n$ be universal sets for $\mathbf{\Sigma}^1_2(B_n)$ sets of reals and $\mathbf{\Sigma}^1_2(B_n)$ subsets of $\mathbb{R}^2$, respectively. By uniformization, there is a function $f_n$ uniformizing $H'_n$. By Theorem 3.3.7, there is a $\gamma<\Theta$ such that $H_n$ has a strong $\infty$-code whose tree is on $\gamma$. Let $\gamma_{n+1}=\gamma$, $<_{n+1}$ be a strict wellfounded relation on the reals with length $\gamma_{n+1}$, and let $\Gamma_{n+1}$ be a Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ containing $\Gamma_n \cup \{H_n,H'_n,f_n,<_{n+1}\}$. We show that they satisfy all the items above for $n+1$. The first item is trivial. The second item is easy by noting that if $f_n$ uniformizes $H'_n$ then $(f_n)_a$ uniformizes $(H'_n)_a$ for any real $a$. The third item follows from that if $H_n$ has a strong $\infty$-code whose tree is on $\gamma_{n+1}$, then $(H_n)_a$ has a strong $\infty$-code whose tree is on $\gamma_{n+1}$for every real $a$. $\square$ (Claim 3.3.22)

Note that in the proof of Claim 3.3.22, we have essentially used DC.

We fix $\big((\Gamma_n, <_n, \gamma_n) \mid n < \omega\big)$ as above and let $\Gamma^{\mathrm{I}}_n = \Gamma_{2n}, \Gamma^{\mathrm{II}}_n = \Gamma_{2n+1}, <^{\mathrm{I}}_n$ be induced by $\rho$, $<^{\mathrm{II}}_n = <_{2n+1}$, $\gamma^{\mathrm{I}}_n = \omega$ and $\gamma^{\mathrm{II}}_n = \gamma_{2n+1}$, Let $\rho^{\mathrm{I}}_n = \rho$ and $\rho^{\mathrm{II}}_n$ be the surjection between the reals onto ${}^n\gamma_{2n+1}$ induced by $<_{2n+1}$. Let $\pi^{\mathrm{II}}_n$ be the function $a \mapsto \rho^{\mathrm{II}}_n[G^n_a]$ where $G^n$ is a universal set for $\boldsymbol{\Gamma}^{\mathrm{II}}_n$ sets of reals (we do not use $\pi^{\mathrm{I}}_n$). Then by Theorem 3.3.18, $\pi^{\mathrm{II}}_n$ is a surjection from the reals onto ${}^n\gamma^{\mathrm{II}}_n$. Consider the following game $\hat{G}_A$: Player I plays 0 or 1 and player II plays reals one by one in turn and they produce a real $z$ and a sequence $t \in {}^\omega\mathbb{R}$, respectively. Setting $T_n = \pi^{\mathrm{II}}_n\big(t(n)\big)$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded, where $T_m\restriction n = \{s\restriction n \mid s \in T_m\}$. This is an integer-real game in the sense player I chooses integers and player II chooses reals.

We introduce an integer-integer game $\tilde{G}_A$ simulating the game $\hat{G}_A$. In the game $\tilde{G}_A$, players choose pairs of 0 or 1 one by one and produce a pair of reals $(x_0, y_0)$ and $(a_0, b_0)$ in $\omega$ rounds respectively. From $(x_0, y_0)$ and $(a_0, b_0)$, we "decode" a real $z$ and an $\omega$-sequence of reals $t$ respectively as follows: For each pointclass $\Gamma$ above, we fix a set $U^\Gamma$ universal for relations in $\boldsymbol{\Gamma}$. Setting $F_0 = U^{\Gamma^{\mathrm{I}}_0}_{x_0}$, $F_0$ is a function from the reals to perfect sets of reals (or codes of them) (otherwise player I loses). Let $P_{x_0} = F(x_0)$. Then $y_0$ is an element of $P_{x_0}$ (otherwise player I loses) and is identified with a triple $(u_0, x_1, y_1)$ of reals by looking at a canonical homeomorphism between $P_{x_0}$ and $\mathbb{R}^3$. Then setting $F_1 = U^{\Gamma^{\mathrm{I}}_1}_{x_1}$, $F_1$ is a function from the reals to perfect trees on 2 (or codes of trees) (otherwise player I loses). Let $P_{x_1} = F(x_1)$. Then $y_1$ is an element of $P_{x_1}$ (otherwise player I loses) and is identified with a triple $(u_1, x_2, y_2)$ of reals by looking at a canonical homeomorphism between $P_{x_1}$ and $\mathbb{R}^3$. Continuing this process, one can unwrap $(x_n, y_n)$ and obtain $(u_n, x_{n+1}, y_{n+1})$ for each $n$ and get an $\omega$-sequence $(u_n \mid n < \omega)$. Let $z(n) = \rho(u_n)$. In the same way, one can obtain an $\omega$-sequence $(t_n \mid n < \omega)$ of reals from $(a_0, b_0)$. Setting $T_n = \pi^{\mathrm{II}}_n\big(t(n)\big)$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded.

Becker proved the following:

**Lemma 3.3.23.**

1. If player I has a winning strategy in the game $\tilde{G}_A$, then player I has a winning strategy $\sigma$ in the game $\hat{G}_A$ such that $\sigma$ is a countable union of sets in $\boldsymbol{\Gamma}^{\mathrm{II}}_n$ for some $n$ as a set of reals.

2. If player II has a winning strategy in the game $\tilde{G}_A$, then player II has a winning strategy in the game $\hat{G}_A$.

*Proof.* See [9, Lemma A & B]. □

We show and conjecture the following: Let $B \subseteq {}^\omega\mathbb{R}$. A mixed strategy $\sigma$ for player I is *weakly optimal in $B$* if for any $s \in \mathbb{R}^{\mathrm{Even}}$, the set $\{x \mid \sigma(s)(x) \neq 0\}$ is finite and for any $\omega$-sequence $y$ of reals, $\mu_{\sigma,\tau_y}(B) > 1/2$. One can introduce the weak optimality for mixed strategies for player II in the same way. Note that if player I has an optimal strategy in some payoff set, then player I has a weakly optimal strategy in the same payoff set. The same holds for player II.

**Lemma 3.3.24.** If player I has an optimal strategy in the game $\tilde{G}_A$, then player I has a weakly optimal strategy $\sigma$ in the game $\hat{G}_A$ such that $\sigma$ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some $n$ as a set of reals.

**Conjecture 3.3.25.** If player II has an optimal strategy in the game $\tilde{G}_A$, then player II has a weakly optimal strategy in the game $\hat{G}_A$.

*Proof of Lemma 3.3.24.* We first topologize the set $\mathrm{Prob}(\mathbb{R})$ of all Borel probabilities on the reals. Consider the following map $\iota\colon \mathrm{Prob}(\mathbb{R}) \to {}^{<\omega}2[0,1]$: Given a Borel probability $\mu$ on the reals, for any finite binary sequence $s$, $\iota(\mu)(s) = \mu([s])$. We topologize ${}^{<\omega}2[0,1]$ by the product topology where each coordinate $[0,1]$ is equipped with the relative topology of the real line and we identify $\mathrm{Prob}(\mathbb{R})$ with its image via $\iota$ and topologize it with the relative topology of ${}^{<\omega}2[0,1]$. Then the space $\mathrm{Prob}(\mathbb{R})$ is compact.

**Claim 3.3.26.** For any set $B$ of reals, the map $\mu \mapsto \mu(B)$ is a continuous map from $\mathrm{Prob}(\mathbb{R})$ to $[0,1]$.

*Proof of Claim 3.3.26.* This is easy when $B$ is closed or open. In general, it follows from the following equations: For any $\mu \in \mathrm{Prob}(\mathbb{R})$,

$$\begin{aligned}\mu(B) &= \sup\{\mu(C) \mid C \subseteq B \text{ and } C \text{ is closed}\}\\ &= \inf\{\mu(O) \mid O \supseteq B \text{ and } O \text{ is open}\}.\end{aligned}$$

$\square$ (Claim 3.3.26)

Next, we introduce a complete metric $d$ on $\mathrm{Prob}(\mathbb{R})$ compatible with the topology we consider. Let $(s_n \mid n \in \omega)$ be an injective enumeration of finite binary sequences. For $\mu$ and $\mu'$ in $\mathrm{Prob}(\mathbb{R})$, $d(\mu,\mu') = \sum_{n\in\omega} |\mu([s_n]) - \mu'([s_n])|/2^{n+1}$. Then $d$ is a complete metric compatible with our topology. Since $\mathrm{Prob}(\mathbb{R})$ is compact, the map $\mu \mapsto \mu(A)$ is uniformly continuous with the metric $d$. Hence there is an $\epsilon > 0$ such that if $d(\mu,\mu') < \epsilon$, then $|\mu(A) - \mu'(A)| < 1/2$. Let us fix a sequence $(\epsilon_n \mid n \in \omega)$ of positive real numbers such that $\sum_{n\in\omega} \epsilon_n/2^{n+1} < \epsilon$. For any finite binary sequence $s'$, let $n_{s'}$ be the natural number such that $s_{n'_s} = s'$.

Let $\sigma$ be an optimal strategy for player I in the game $\tilde{G}_A$. We show that there is a weakly optimal strategy $\tilde{\sigma}$ for player I in the game $\hat{G}_A$. Given a real $a$. Consider the function $F^0_a\colon \mathbb{R} \to {}^2[0,1]$ as follows: Given a real $b$, $F^0_a(b)(i) = \mu_{\sigma,\tau_{(a,b)}}\big(\{(x_0,y_0) \mid \rho(u_0) = i\}\big)$ for $i = 0,1$, where $y_0$ is identified with $(u_0,x_1,y_1)$ as discussed. Since every set of reals has the Baire property, $F^0_a$ is continuous on a comeager set. Then there is a perfect set $P$ of reals such that for any $b$ and $b'$

in $P$, $|F^0_a(b)(i) - F^0_a(b')(i)| < \epsilon_{n_{(i)}}$. Since the set $X_0 = \{(a, P) \mid (\forall b, b' \in P)\ (\forall i < 2)\ |F^0_a(b)(i) - F^0_a(b')(i)| < \epsilon_{n_{(i)}}\}$ is projective in $\Gamma^{\mathrm{I}}_0$, there is a real $a_0$ such that the function $f_0 = U^{\Gamma^{\mathrm{II}_0}}_{a_0}$ uniformizes $X_0$. Let $\tilde{\sigma}(\emptyset)(0) = \max\{F^0_{a_0}(b)(0) \mid b \in f_0(a_0)\}$ and $\tilde{\sigma}(\emptyset)(1) = 1 - \tilde{\sigma}(\emptyset)(0)$. We have specified $\tilde{\sigma}$ for the first round.

Next, suppose player II played a real $t_0$ for her first round. We decide the probability $\tilde{\sigma}(t_0)$ on 2. Let $a$ be a real. Consider the function $F^1_a\colon \mathbb{R} \to {}^2[0, 1]$ as follows: For a real $b$, $F^1_a(b)(i) = \mu_{\sigma,\tau_{(a_0,(t_0,a,b))}}\big(\{(x_0, y_0) \mid \rho(u_1) = i\}\big)$ for $i = 0, 1$, where $y_1 = (t_1, x_2, y_2)$ as discussed. Then the function $F^1_a$ is continuous on a comeager set. Then there is a perfect set $P$ of reals such that for any $b$ and $b'$ in $P$, $|F^1_a(b)(i) - F^1_a(b')(i)| < \min\{\epsilon_{n_{s^\frown\langle i\rangle}} \mid s \in {}^12\}$ for $i = 0, 1$. Since the set $X_1 = \{(a, P) \mid (\forall b, b' \in P)\ (\forall i < 2)\ |F^1_a(b)(i) - F^1_a(b')(i)| < \min\{\epsilon_{n_{s^\frown\langle i\rangle}} \mid s \in {}^12\}\}$ is projective in $\mathbf{\Gamma}^{\mathrm{I}_1}$, there is a real $a_1$ such that the function $f_1 = U^{\Gamma^{\mathrm{II}_1}}_{a_1}$ uniformizes $X_1$. Let $\tilde{\sigma}(t_0)(0) = \max\ \{F^1_{a_1}(b)(i) \mid b \in f_1(a_1)\}$ and $\tilde{\sigma}(t_0)(1) = 1 - \tilde{\sigma}(t_0)(0)$.

Continuing this process, we can specify $\tilde{\sigma}$ with the following property: For any natural number $m$ and $m$-tuple reals $(t_0, \ldots, t_{m-1})$, $|\tilde{\sigma}(t_0, \ldots, t_{m-1})(i) - F^m_{a_m}(b)(i)| < \min\{\epsilon_{n_{s^\frown\langle i\rangle}} \mid s \in {}^m2\}$ for each $b \in f_m(a_m)$. Also we have specified the reals $a_m$ and $b_m$ for all $m < \omega$.

We show that $\tilde{\sigma}$ is weakly optimal in the game $\hat{G}_A$. Let $(t_n \mid n < \omega)$ be an $\omega$-sequence of reals such that the tree $\bigcup_{n<\omega} T_{n+1}\restriction n$ is illfounded. We show that the probability of the payoff set via $\mu_{\tilde{\sigma},\tau_{(t_n|n<\omega)}}$ is greater than 1/2. (The case when the tree is wellfounded is dealt with in the same way.)

First note that together with $(t_n \mid n < \omega)$, $\tilde{\sigma}$ produces a Borel probability $\mu$ on the reals such that for any finite binary sequence $s$, $\mu([s]) = \prod_{i<m} \tilde{\sigma}(t_j \mid j < i)\big(s(j)\big)$, where $m$ is the length of $s$. Since the tree from $(t_n \mid n < \omega)$ is illfounded, it suffices to show that $\mu(A) > 1/2$. On the other hand, the measure $\mu_{\sigma,\tau_{(a_0,b_0)}}$ induces a Borel probability measure $\nu$ on the reals as follows: For a finite binary sequence $s$, $\nu([s]) = \mu_{\sigma,\tau_{(a_0,b_0)}}\big(\{(x_0, y_0) \mid (\forall i < m)\ \rho(t_i) = s(i)\}\big)$, where $m$ is the length of $s$. By the property of $\tilde{\sigma}$, $d(\mu, \nu) < \epsilon$. Hence $|\mu(A) - \nu(A)| < 1/2$. Since $\sigma$ is optimal for player I in the game $\tilde{G}_A$ and the tree from $(t_n \mid n < \omega)$ is illfounded, $\nu(A) = 1$. Therefore, $\mu(A) > 1/2$, as desired. $\square$

From Lemma 3.3.24 together with Theorem 3.3.10, one can conclude the following:

**Lemma 3.3.27.** There is no optimal strategy for player I in the game $\tilde{G}_A$.

*Proof.* To derive a contradiction, suppose player I has an optimal strategy in the game $\tilde{G}_A$. Then by Lemma 3.3.24, player I has a weakly optimal strategy $\sigma$ in the game $\hat{G}_A$ such that $\sigma$ is in a countable union of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some $n$ as a set of reals.

Consider the following set:

$$X = \{(t, s) \in {}^\omega\mathbb{R} \times {}^{<\omega}\mathbb{R} \mid \mu_{\sigma,\tau_t}\big(\{(z, t') \mid t' = t \text{ and } z \in A\}\big) > 1/2 \text{ and } (\forall i < \mathrm{s})\ \big(|s(0)|_{<^{\mathrm{II}}_0}, \ldots, |s(i)|_{<^{\mathrm{II}}_i}\big) \in T_{i+1}\restriction i\},$$

where $|s(i)|_{<^{\mathrm{II}}_i}$ is the rank of $s(i)$ with respect to the wellfounded relation $<^{\mathrm{II}}_i$ and $T_i = \rho^{\mathrm{II}}_i\big(t(i)\big)$. For $(t,s)$ and $(t',s')$ in $X$, $(t,s) < (t',s')$ if $t$ and $t'$ code the same tree $T$ and $s$ codes a node in $T$ extending a node coded by $s'$. Note that for any $(t,s)$ in $X$, if $T$ is the tree coded by $t$, $T$ is wellfounded because $\sigma$ is weakly optimal in the game $\hat{G}_A$. Hence $(X,<)$ is a strict wellfounded relation on $X$. Let $\gamma_\omega = \sup\{\gamma^{\mathrm{II}}_n \mid n \in \omega\}$. By DC, the cofinality of $\Theta$ is greater than $\omega$. Hence $\gamma_\omega < \Theta$. Note that for any ordinal $\alpha < \gamma^+_\omega$, there is a wellfounded tree $T$ coded by some real $t$ as in the definition of $X$ such that the length of $T$ is $\alpha$. Hence the length of $(X,<)$ is $\gamma^+_\omega$.

Since $\sigma$ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some $n$ as a set of reals, the set $<$ on $X$ is in $\exists^{\mathbb{R}} \bigwedge^\omega \bigvee^\omega \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{I}}_n$, i.e., it is a projection of a countable intersection of countable unions of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some $n$. Since every set in $\mathbf{\Gamma}^{\mathrm{I}}_n$ has a strong $\infty$-Borel code whose tree is on $\gamma^{\mathrm{II}}_n$ for every $n$, every set in $\bigwedge^\omega \bigvee^\omega \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{I}}_n$ has a strong $\infty$-Borel code whose tree is on $\gamma^+_\omega$. By Theorem 3.3.10, the length of $<$ must be less than $\gamma^+_\omega$, which is not possible because it was equal to $\gamma^+_\omega$. Contradiction! $\square$

We close this section by proving that Conjecture 3.3.25 implies Conjecture 3.3.1.

*Proof of Conjecture 3.3.1 from Conjecture 3.3.25.* By Lemma 3.3.27, player I does not have an optimal strategy in the game $\tilde{G}_A$. Hence by Bl-AD, player II has an optimal strategy in the game $\tilde{G}_A$. By Conjecture 3.3.25, player II has a weakly optimal strategy $\tau$ in the game $\hat{G}_A$. Note that $\tau$ can be seen as a real because each measure on the reals given by $\tau$ is with finite support by the weak optimality of $\tau$. For each finite binary sequence $s$ with length $n$, let $t_s = \{u \in {}^n\mathbb{R} \mid (\forall i < n)\ \tau\Big(\big(s\restriction i\big) * \big(u\restriction(i-1)\big)\Big)\big(s(i)\big) \neq 0\}$, where $(s\restriction i) * \big(u\restriction(i-1)\big)$ is the concatenation of $s\restriction i$ and $u\restriction(i-1)$ bit by bit. For each finite binary sequence $s$, we identify $t_s$ with a set of $n$-tuples of natural numbers via a map $\pi_s$ by using the isomorphisms between $(a, <_{\mathbb{R}})$ and $(n, \in)$ for a finite set of reals $a$ and a natural number, where $<_{\mathbb{R}}$ is a standard total order on the reals. For any real $x$, $t_x = \bigcup_{n\in\omega} t_{x\restriction n}$ is a tree on natural numbers and $(\pi_s \mid s \in {}^{<\omega}\omega)$ induces a homeomorphism $\pi_x$ between $[t_x]$ and $[\{t' \in {}^{<\omega}\mathbb{R} \mid \mu_{\sigma_x,\tau}([t']) \neq 0\}]$. Consider the following tree:

$$T = \{(s,t,u) \in \bigcup_{n\in\omega} ({}^n2 \times {}^n\omega \times {}^n\gamma_\omega) \mid\ t \in \pi_s(t_s) \text{ and } \big(\forall i < \mathrm{lh}(s)\big)\ u(i) = |x_i|_{<^{\mathrm{II}}_i}\},$$

where $x_i$ is the $t(i)$th real of the set of successors of $(x_j \mid j < i)$ in $t_s\restriction i$. Then by the weak optimality of $\tau$, the following holds: Setting $B = \{(x,y) \in \mathbb{R} \times {}^\omega\omega \mid (\exists f \in {}^\omega\gamma_\omega)\ (x,y,f) \in [T]\}$, for any real $x$,

$$\begin{aligned} x \in A &\iff \mu_{\sigma_x,\tau}\big(\pi_x[B_x]\big) > 1/2 \\ &\iff (\exists T' :\ \text{a tree on } 2)\ [T'] \subseteq B_x \text{ and } \mu_{\sigma_x,\tau}\big(\pi_x\big[[T']\big]\big) > 1/2. \end{aligned}$$

Since $B$ is Suslin, the set $\{(x, T') \mid [T'] \subseteq B_x\}$ is also Suslin. Hence $A$ is Suslin, as desired.

We have shown that every set of reals is Suslin. Then by Theorem 1.14.5, AD holds. Now by Theorem 3.3.2 and Theorem 1.14.9, $\mathrm{AD}_{\mathbb{R}}$ holds. □

## 3.4 Toward the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$

In the last section, we have discussed the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ under AD+DC. Solovay proved the following:

**Theorem 3.4.1** (Solovay)**.** If we have $\mathrm{AD}_{\mathbb{R}}$ and DC, then we can prove the consistency of $\mathrm{AD}_{\mathbb{R}}$. Hence the consistency of $\mathrm{AD}_{\mathbb{R}}$+DC is strictly stronger than that of $\mathrm{AD}_{\mathbb{R}}$.

*Proof.* See [78]. □

Hence assuming DC to see the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ is not optimal. One can ask whether they are equivalent without DC. So far we do not have any scenario to answer this question. Instead, one could ask the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In this section, we discuss the following conjecture:

**Conjecture 3.4.2.** $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equiconsistent.

Woodin conjectured the following:

**Conjecture 3.4.3** (Woodin)**.** Assume the following:

1. The principle $\mathrm{DC}_{\mathbb{R}}$ holds,
2. Every Suslin & co-Suslin set of reals is determined, and
3. There is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Then either there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model $M$ of $\mathrm{AD}^+$ such that $M$ contains all the reals and $\Theta^M = \Theta^V$.

We show that Conjecture 3.4.3 implies Conjecture 3.4.2.

*Proof of Conjecture 3.4.2 from Conjecture 3.4.3.* First note that the assumptions in Conjecture 3.4.3 hold if we assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Hence by Conjecture 3.4.3, there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model $M$ of $\mathrm{AD}^+$ such that $M$ contains all the reals and $\Theta^M = \Theta^V$. If there is an inner model of $\mathrm{AD}_{\mathbb{R}}$, then we are done. Hence we assume that there is an inner model $M$ of $\mathrm{AD}^+$ such that $M$ contains all the reals and $\Theta^M = \Theta^V$.

We show that $\mathrm{AD}_{\mathbb{R}}$ holds in $V$. First we claim that $M$ contains all the sets of reals. Suppose not. Then there is a set of reals $A$ which is not in $M$. Then by Lemma 3.3.21, every set of reals in $M$ is $\mathbf{\Sigma}^1_2(A)$. Then $\Theta^M$ must be less than $\Theta^V$ because one can code all the prewellorderings by reals using $A$ in $V$, which contradicts the condition of $M$. Hence every set of reals is in $M$. Since we have uniformization for every relation on the reals in $V$, it is also true in $M$. We use the following fact:

**Fact 3.4.4.** Assume $\mathrm{AD}^+$. Then the following are equivalent:

1. The axiom $\mathrm{AD}_{\mathbb{R}}$ holds, and
2. Every relation on the reals can be uniformized.

By Fact 3.4.4, since every relation on the reals can be uniformized in $M$, $M$ satisfies $\mathrm{AD}_{\mathbb{R}}$. Since $\mathcal{P}(\mathbb{R}) \cap M = \mathcal{P}(\mathbb{R})$, $\mathrm{AD}_{\mathbb{R}}$ holds in $V$, as desired. $\square$

## 3.5 Questions

We close this chapter by raising questions.

**The equivalence between $\mathbf{AD}_{\mathbb{R}}$ and Bl-$\mathbf{AD}_{\mathbb{R}}$ under ZF+DC** As discussed in § 3.3, it is enough to show Conjecture 3.3.25 to prove the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In the proof of Lemma 3.3.24, in each round, we shrank the reals into a perfect set sufficiently enough so that the strategy we constructed gives us a measure on the reals which is close enough to the measure derived from a given optimal strategy and the opponent's moves, which yields the weak optimality of the strategy. But the same argument does not work for Conjecture 3.3.25 because one cannot shrink the reals into a perfect set to get the continuity of a given function from $\mathbb{R}$ to ${}^{\mathbb{R}}[0,1]$. Nonetheless, we can proceed the similar argument to the coded space $\prod_{n\in\omega} \mathcal{P}({}^n\gamma^{\mathrm{II}}_n)$ from the space ${}^{\omega}\mathbb{R}$ by using the fact that the meager ideal on the reals is closed under any wellordered union and deciding the probability on the space $\prod_{n\in\omega} \mathcal{P}({}^n\gamma^{\mathrm{II}}_n)$ is enough to determine the probability of the payoff set. Although the details of the argument seem complicated and it is not yet done, we believe it is possible and it is not so difficult.

**The equiconsistency between $\mathbf{AD}_{\mathbb{R}}$ and Bl-$\mathbf{AD}_{\mathbb{R}}$** By the argument in § 3.4, it is enough to show Conjecture 3.4.3 to prove the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. It seems possible because Bl-$\mathrm{AD}_{\mathbb{R}}$ gives us a generic embedding similar to the one obtained by an $\omega_1$-dense ideal on $\omega_1$, CH and "The restriction of the generic embedding given by the ideal to On is definable in $V$". Let us see more details. If one takes a generic filter $G$ of the partial order ${}^{<\omega}\mathbb{R}$ ordered by reverse inclusion, then this filter generates an ultrafilter $U'$ extending the dual filter of the meager ideal in ${}^{\omega}\mathbb{R}$ in the same way as we have seen in Lemma 3.3.12. If one takes the generic ultrapower of $V$ via $U'$ and lets $M$ be the target model of the ultrapower embedding $j$, then Łoś's Theorem holds for $M$ if the cofinality of $\Theta$ is $\omega$, the reals in $V$ belongs to $M$ as an element (as a real), $M$ contains all the reals in $V[G]$ and $j \restriction \mathrm{On}$ is definable in $V$ (the last statement is ensured by the existence of a fine normal measure $U$ in Theorem 3.1.2, in fact, the ultrapower embedding via $U'$ agrees with $j$ on ordinals as we have seen). In general, $M$ is not well-founded (in the case $\mathrm{cof}(\Theta) = \omega$). But $\Theta$ is always in the well-founded part of $M$. Together with the determinacy of Suslin & co-Suslin sets of reals, this seems enough to proceed the Core Model Induction up to $\Theta = \Theta_\omega$, i.e., a minimal model of $\mathrm{AD}_{\mathbb{R}}$.

**A stronger weak Moschovakis' Lemma** As we have seen in § 3.3, a weak version of Moschovakis's Lemma 3.3.18 holds assuming Bl-AD. One can ask whether one can prove a stronger version of Moschovakis's Lemma formulated in [66, 7D.5] from Bl-AD. If this is possible, it would be plausible to show that the set of strong partition cardinals is unbounded in $\Theta$ and that every Suslin set of reals is determined from Bl-AD.

# Chapter 4

# Games and Large Cardinals

In this chapter, we investigate the upper bound of the consistency strength of the existence of alternating chains with length $\omega$, which are essential objects proving projective determinacy from Woodin cardinals.

## 4.1 The consistency strength of the existence of alternating chains

In late 1980s, Martin and Steel [60] proved that if there are $n$ Woodin cardinals and a measurable above them, then every $\boldsymbol{\Pi}^1_{n+1}$ set of reals is determined for each natural number $n$, where they introduced the notion of *iterations trees* which originally comes from the development of the inner model theory for strong cardinals. To build the inner model theory above one strong cardinal, one would have to iterate premice not only linearly but in more complicated way which would give us tree structures labeled with extenders that they call iteration trees. This generalization gives us another difficulty when we iterate premice more than $\omega$ times: In a limit stage, there could be many cofinal branches in the tree we have constructed and we have to choose one of them so that the direct limit through that branch will be wellfounded. This problem occurs when we reach the region of Woodin cardinals and Martin and Steel used this obstacle to prove projective determinacy by coding one second-order existential quantifier by the existence of cofinal wellfounded branch of suitable iteration trees (in their case, they arranged the iteration trees in such a way that the wellfounded branch is always unique). *Alternating chains* are the simplest iteration trees with this obstacle: They are iteration trees with length $\omega$ such that their tree structure is given as follows: For all natural numbers $n, m$,

$$mTn \iff m = 0 \text{ or } n - m \text{ is a positive even number.}$$

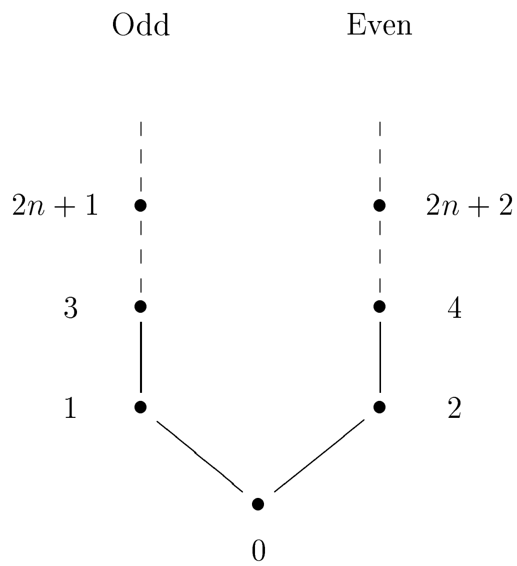


Figure 4.1: An alternating chain with length $\omega$

This is the simplest tree structure with two cofinal branches. Let us call these two branches Even ($= \{2n \mid n \in \omega\}$) and Odd ($= \{2n+1 \mid n \in \omega\} \cup \{0\}$). Since these two branches are completely symmetric with respect to the tree structure, there is no canonical way to choose one of them so that the chosen one is wellfounded. This gives us the basic idea of how to code certain information via iteration trees. Actually, in the proof of projective determinacy, Martin and Steel replaced the odd part by ${}^{<\omega}\omega$ and ensured that the branch Even is ill-founded and that exactly one of the cofinal branches is wellfounded. This is how they code a real via a wellfounded cofinal branch.

But the above argument works only when there is only one wellfounded cofinal branch in the iteration tree. So the question is: Is there any iteration tree with length $\omega$ with more than one wellfounded branches? Martin and Steel [61] (independently by Woodin) proved that if there is a Woodin cardinal, then there are a countable transitive model $M$ of (a large enough fragment of) ZFC and an alternating chain on $M$ such that both branches are wellfounded. Conversely, they proved that if there is an iteration tree with limit length and two cofinal wellfounded branches, then there is a transitive model of ZF which satisfies "There is a Woodin cardinal". Hence there is a tight connection between Woodin cardinals and the existence of iteration trees with more than one cofinal wellfounded branches. In fact, what they proved is stronger:

**Theorem 4.1.1** (Martin and Steel)**.** Suppose there is an iteration tree $T$ with limit length and two cofinal branches $b$ and $c$. Let $\delta$ be the supremum of the length of extenders used in $T$ and $\alpha$ be an ordinal with $\alpha > \delta$ and $\alpha$ is in the wellfounded part of both $M_b$ and $M_c$ where $M_b$ and $M_c$ are the direct limit of models in $T$ through $b$ and $c$ respectively. Then $\mathrm{L}_\alpha(V_\delta^{M_b}) \models$ "$\delta$ is Woodin".

*Proof.* See [62, Corollary 2.3]. $\square$

This theorem gives us more information: Note that $V_\delta^{M_b} = V_\delta^{M_c}$ and it is always a subset of the wellfounded part of both models. Since every wellfounded part of a model of KP is also a model of KP, we have the following: If one of $M_b$ and $M_c$ is wellfounded and $\theta$ is the least ordinal that is not in the wellfounded part of one of $M_b$ and $M_c$ and $\theta > \delta$, then $\mathrm{L}_\theta(V_\delta^{M_b}) \models$ "KP + $\delta$ is Woodin". Hence we get the Woodin-in-the-next-admissibleness from the assumption, here we say $\delta$ is *Woodin-in-the-next-admissible* if there is an ordinal $\theta > \delta$ such that $\mathrm{L}_\theta(V_\delta) \models$ "KP+$\delta$ is Woodin". Andretta [2] proved the following stronger converse:

**Theorem 4.1.2** (Andretta)**.** Suppose $\delta$ is Woodin-in-the-next-admissible. Then for any tree order on $\omega$ with an infinite branch, there is an iteration tree such that for any infinite branch $b$ of the tree, $\delta_\omega$ is in the wellfounded part of $M_b$, where $\delta_\omega$ is the supremum of the length of extenders in the iteration tree.

*Proof.* See [2, Theorem 1.3]. $\square$

Hence Woodin-in-the-next-admissible cardinals are intimately correlated to iteration trees with more than one cofinal branches. The natural question would be: What if we do not demand that $\delta_\omega$ is in the wellfounded part of $M_b$? In this section, we partially answer this question in the case of alternating chains. In fact, we do not need Woodin-in-the-next-admissible cardinals to construct alternating chains:

**Theorem 4.1.3.** Suppose $\delta$ is an ordinal such that $\delta$ is $\Sigma_2$-Woodin and $V_\delta \prec_{\Sigma_2} V$. Then there is an alternating chain with length $\omega$.

The assumption of the above theorem (which we will explain later) is much weaker than Woodin-in-the-next-admissibleness. Hence we do not need Woodin-in-the-next-admissibleness just to construct alternating chains.

Let us prepare for introducing the notions in the above theorem. For a transitive model $M$ of ZFC and an ordinal $\alpha$ in $M$, we write $M|\alpha$ for abbreviating $V_\alpha^M$. Furthermore, for a subset $A$ of $M$, $\mathrm{Thy}_\Gamma(M;\in,A)$ denotes the $\Gamma$-theory of $M$ with parameters in $A$ where $\Gamma$ is $\Sigma_n$ for some natural number $n \geq 1$. Also, for a set $A$ and an ordinal $\alpha$, $A \upharpoonright \alpha$ denotes $A \cap V_\alpha$.

Let $\kappa < \delta$ be ordinals and $\Gamma$ be $\Sigma_n$ for some natural number $n \geq 1$. We say $\kappa$ is *$<\delta$-$\Gamma$-strong* if it is $<\delta$-$A$-strong where $A = \mathrm{Thy}_\Gamma(V|\delta;\in,V|\delta)$, i.e., for any ordinal $\alpha < \delta$ there is a non-trivial elementary embedding $j\colon V \to M$ with critical point $\kappa$ where $M$ is transitive such that $V_\alpha \subseteq M$, $j(\kappa) > \alpha$ and $A \upharpoonright \alpha = j(A) \upharpoonright \alpha$. If $\delta$ is a limit of inaccessible cardinals, such an embedding can be easily coded by an extender in $V_\delta$. An ordinal $\delta$ is *$\Gamma$-Woodin* if it is a limit of $<\delta$-$\Gamma$-strongs.

Note that if $\delta$ is a limit of $<\delta$-strong cardinals, then $\delta$ is $\Sigma_1$-Woodin and $V_\delta$ is a $\Sigma_1$ elementary substructure of $V$. Hence we cannot replace $\Sigma_2$ with $\Sigma_1$ in

Theorem 4.1.3 because if we could, then we could run the argument in a mouse below $0^{\P}$ with a cardinal $\delta$ which is a limit of $<\delta$-strong cardinals, which is impossible by [73, Lemma 2.4].

Also note that $\Sigma_n$-Woodinness for a natural number $n$ is much weaker than Woodin-in-the-next-admissibleness. In fact, if $\delta$ is Woodin-in-the-next-admissible, then for any natural number $n \geq 1$, $\delta$ is a limit of $<\delta$-strong cardinals $\kappa$ such that the set of $<\kappa$-$A_n$-strong cardinals is stationary in $\kappa$ where $A_n = \mathrm{Thy}_{\Sigma_n}(V|\delta; \in, V|\delta)$, which immediately gives us that the set of $\Sigma_n$-Woodin cardinals $\delta'$ with $V_{\delta'} \prec_{\Sigma_n} V_\kappa$ is stationary in $\kappa$. Hence the assumption of Theorem 4.1.3 is much weaker than Woodin-in-the-next-admissibleness.

*Proof of Theorem 4.1.3.* We will construct $\big((\kappa_n, E_n, \beta_n) \mid n < \omega\big)$ with the following properties:

$(1)_n$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n}\Big) = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n\dot{-}1}|\beta_n; \in, M_{2n\dot{-}1}|\kappa_{2n}\Big)$,

$(2)_n$ $\kappa_{2n}$ is $<\delta$-$\Sigma_2$-strong in $M_{2n}$,

$(3)_n$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1; \in, M_{2n+1}|\kappa_{2n+1}+1\Big) = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1; \in, M_{2n}|\kappa_{2n+1}+1\Big)$, and

$(4)_n$ $\kappa_{2n+1}$ is $<\beta_{n+1}$-$\Sigma_2$-strong in $M_{2n+1}$,

where $n \dot{-} 1 = \max\{n-1, 0\}$, $M_0 = V$ and $M_{n+1} = \mathrm{Ult}(M_{n\dot{-}1}, E_n)$ for each $n \in \omega$. At the same time, we will arrange that $\kappa_{n+1}$ is less than the strength and the length of $E_n$ for each $n \in \omega$, which will ensure that each $M_n$ is well-founded by the result of Martin and Steel [61, Theorem 3.7].

Also note that all the extenders we will use belong to $V_\delta$. Since $\delta$ is a limit of inaccessible cardinals, $\delta$ will not move under any embedding we will consider.

Let $\beta_0 = \delta$. Then $(1)_0$ is true. Since $\delta$ is $\Sigma_2$-Woodin in $V$, we can pick $\kappa_0 < \delta$ such that $\kappa_0$ is $<\delta$-$\Sigma_2$-strong in $V$, hence $(2)_0$ is also true.

Suppose we have constructed $(\kappa_i \mid i \leq 2n), (E_i \mid i < 2n), (\beta_i \mid i \leq n)$ with the properties $(1)_n$ and $(2)_n$. We will find $\kappa_{2n+1}, E_{2n}, \beta_{n+1}, \kappa_{2n+2}$ and $E_{2n+1}$ with the properties $(3)_n, (4)_n, (1)_{n+1}$ and $(2)_{n+1}$.

Since $\delta = \pi_{0,2n}(\delta)$ is $\Sigma_2$-Woodin in $M_{2n}$, we can pick $\kappa_{2n+1} > \kappa_{2n}$ such that $\kappa_{2n+1}$ is $<\delta$-$\Sigma_2$-strong in $M_{2n}$. By $(2)_n$, $\kappa_{2n}$ is $<\delta$-$\Sigma_2$-strong in $M_{2n}$. Hence we can pick $E_{2n} \in M_{2n}$ such that $E_{2n}$ is an extender with critical point $\kappa_{2n}$ and length and strength greater than $\kappa_{2n+1}+3$ in $M_{2n}$, such that $\pi_{E_{2n}}(A) \restriction (\kappa_{2n+1} + 3) =$

$A \restriction (\kappa_{2n+1}+3)$ in $M_{2n}$, where $A = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\delta\Big)$. Then

$$\begin{aligned}
&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n); \in, M_{2n+1}|\kappa_{2n+1}+3\Big)\\
=&\pi_{2n\dot{-}1,2n+1}\Big(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n\dot{-}1}|\beta_n; \in, M_{2n\dot{-}1}|\kappa_{2n}\Big)\Big) \restriction \kappa_{2n+1}+3\\
=&\pi_{E_{2n}}\Big(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n}\Big)\Big) \restriction \kappa_{2n+1}+3\\
=&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n+1}+3\Big).
\end{aligned}$$

Now the following is true in $M_{2n}$ witnessed by $\beta = \delta$:

$(*)$ There is an ordinal $\beta$ such that $B = \mathrm{Thy}_{\Sigma_2}\Big(V|\beta+1; \in, V|\kappa_{2n+1}+1\Big)$ and $\kappa_{2n+1}$ is $<\beta$-$\Sigma_2$-strong and $\beta$ is $\Sigma_2$-Woodin,

where $B = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1; \in, M_{2n}|\kappa_{2n+1}+1\Big)$. Note that this statement is $\Sigma_2$ in $M_{2n}$ with parameters $B$ and $\kappa_{2n+1}$ because the statement "$\kappa_{2n+1}$ is $<\beta$-$\Sigma_2$-strong and $\beta$ is $\Sigma_2$-Woodin" is definable in $V|\beta$ if $\beta$ is a limit of inaccessibles, which is also $\Sigma_2$ definable.

Since $V_\delta$ is a $\Sigma_2$-elementary substructure of $V$, $M_{2n}|\delta = M_{2n}|\pi_{0,2n}(\delta)$ is a $\Sigma_2$-elementary structure of $M_{2n}$. Hence $(*)$ is also true in $M_{2n}|\delta$. But by the previous calculation, $(*)$ is also true in $M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n)$.

Let $\beta_{n+1}$ be a witness for $(*)$ in $M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n)$. Then it follows that

$$\begin{aligned}
&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1; \in, M_{2n+1}|\kappa_{2n+1}+1\Big)\\
=\ &\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1; \in, M_{2n}|\kappa_{2n+1}+1\Big),
\end{aligned}$$

that is $(3)_n$. Also we have that $\beta_{n+1}$ is $\Sigma_2$-Woodin and $\kappa_{2n+1}$ is $<\beta_{n+1}$-$\Sigma_2$-strong in $M_{2n+1}$, that is $(4)_n$. Since $\beta_{n+1}$ is $\Sigma_2$-Woodin in $M_{2n+1}$ and $\beta_{n+1} > \kappa_{2n+1}$, we can pick $\kappa_{2n+2} < \beta_{n+1}$ large enough and such that $\kappa_{2n+2}$ is $<\beta_{2n+1}$-$\Sigma_2$-strong in $M_{2n+1}$.

By $(4)_n$, we can take $E_{2n+1} \in M_{2n+1}$ such that $E_{2n+1}$ is an extender with critical point $\kappa_{2n+1}$ and length and strength greater than $\kappa_{2n+2}+3$ in $M_{2n+1}$ such that $\pi_{E_{2n+1}}(A') \restriction \kappa_{2n+2}+3 = A' \restriction \kappa_{2n+2}+3$, where $A' = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}; \in,$

$M_{2n+1}|\beta_{n+1}\Big)$. Then

$$\begin{aligned}&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+2}|\delta+1;\in M_{2n+2}|\kappa_{2n+2}+1\Big)\\ =&\pi_{2n,2n+2}\bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1;\in, M_{2n}|\kappa_{2n+1}+1\Big)\bigg)\restriction \kappa_{2n+2}+1\\ =&\pi_{E_{2n+1}}\bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1;\in, M_{2n+1}|\kappa_{2n+1}+1\Big)\bigg)\restriction \kappa_{2n+2}+1\\ =&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1;\in, M_{2n+1}|\kappa_{2n+2}+1\Big),\end{aligned}$$

and by this calculation, we obtain $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+2}|\delta;\in, M_{2n+2}|\kappa_{2n+2}\Big) =$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1};\in, M_{2n+1}|\kappa_{2n+2}\Big)$ and $\kappa_{2n+2}$ is $<\delta$-$\Sigma_2$-strong in $M_{2n+2}$, which are $(1)_{n+1}$ and $(2)_{n+1}$ respectively, as desired. $\square$

Note that in the above construction, we have arranged that $\beta_{n+1} < \pi_{2n\dot{-}1,2n+1}(\beta_n)$ for each $n \in \omega$. Hence $M_{\mathrm{Odd}}$ is always ill-founded.

## 4.2 Questions

We close this chapter with asking one question.

**Question 4.2.1.** What is the consistency strength of the existence of alternating chains with length $\omega$?

# Chapter 5

# Wadge reducibility for the real line

In this chapter, we study the Wadge reducibility for the *real line* and show that the Wadge's Lemma fails and that the Wadge order for the real line is ill-founded. This situation is completely different from the case of the Baire space as given in § 1.15 and it is not possible to get the same kind of game characterization of continuous functions from the real line to itself as in the case of the Baire space.

Throughout this chapter, we work in ZF+DC. In case we need more assumptions, we explicitly mention them. In this chapter, $\mathbb{R}$ denotes the real line, not the Baire space or the Cantor space.

## 5.1 Wadge reducibility for the real line

It was probably known to the Polish school of mathematicians before the Wadge reducibility was introduced that Wadge's Lemma for the Wadge order $\leq_{\mathrm{W}}^{\mathbb{R}}$ fails: Let $A$ be a subset of the real line and assume $A$ and $A^{\mathrm{c}}$ are dense. Then $A$ cannot be a continuous preimage of any nowhere dense subset of the real line. In particular, there are subsets $A$ and $B$ of the real line such that neither $A \leq_{\mathrm{W}}^{\mathbb{R}} B$ nor $B \leq_{\mathrm{W}}^{\mathbb{R}} A^{\mathrm{c}}$ holds (e.g., $A = \mathbb{Q}$, $B =$ any nowhere dense, non-$\boldsymbol{\Pi}_2^0$ set).

We say that a subset $A$ of the real line is *non-trivial* if it is neither the empty set $\emptyset$ nor the whole space $\mathbb{R}$. We remark that the condition for $A^{\mathrm{c}}$ in the above remark is not necessary:

**Proposition 5.1.1.** Let $A$, $B$ be subsets of the real line and assume $A$ is non-trivial and dense and $B$ is nowhere dense. Then $A$ cannot be a continuous preimage of $B$.

*Proof.* Toward a contradiction, suppose there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $A = f^{-1}(B)$. Since $A$ is non-trivial, $f$ is not a constant function. Hence the range of $f$ contains an interval. But since $A$ is dense and $f$ is continuous, the range of $f$ is included in the closure of $B$, which contradicts the fact that $B$ is nowhere dense. $\square$

Note that the failure of Wadge's Lemma for the real line occurs for subsets of the real line which are the difference of the two open sets (see Corollary 5.1.7).

Next, we discuss the failure of the wellfoundedness of the Wadge order $\leq_{\mathrm{W}}^{\mathbb{R}}$, which was proved by Peter Hertling in his Ph.D. thesis [32]. We prove the following stronger result:

**Theorem 5.1.2.** There is an embedding $i$ from $(\mathcal{P}(\mathbb{N}), \subseteq_{\mathrm{fin}})$ to $(\mathcal{P}(\mathbb{R}), \leq_{\mathrm{W}}^{\mathbb{R}})$ such that the range of $i$ consists of subsets of real numbers which are the difference of open sets, where $a \subseteq_{\mathrm{fin}} b$ if $a \setminus b$ is finite for subsets $a, b$ of $\mathbb{N}$ and $\mathbb{N} = \omega \setminus \{0\}$.

*Proof.* Let us start with an easy observation:

**Observation 5.1.3.** Let $a, b, c, d, e, f, g$ be real numbers with $a < b < c < d$ and $e < f < g$ and $h$ be a continuous function from the real line to itself with $h(b) \in [e, f)$, $h\big([a, b)\big) \cap [e, f) = \emptyset$, $h\big([b, c)\big) \cap [f, g) = \emptyset$, $h\big([c, d)\big) \cap [e, f) = \emptyset$ and $h\big([b, c)\big) \not\supseteq (e - \epsilon, e)$ for any $\epsilon > 0$. Then $h(b) = e$, $h(c) = f$ and $h\big([b, c)\big) = [e, f)$.

This observation allows us to encode subsets of $\mathbb{N}$ into sets formed from a sequence of half-open intervals by suitably inserting points between them.

Let us discuss this idea in detail. We first construct increasing sequences of real numbers $\langle a_\alpha, b_\alpha \mid \alpha < \omega^\omega \rangle$ and $\langle c_n \mid n \in \mathbb{N} \rangle$ with the following properties: For $\alpha < \omega^\omega$, a limit $\gamma$, and a natural number $n \geq 1$,

$$\begin{aligned}
&a_\alpha < b_\alpha < a_{\alpha+1}, \\
&\sup\{a_\alpha \mid \alpha < \gamma\} < a_\gamma, \text{ and} \\
&\sup\{a_\alpha \mid \alpha < \omega^n\} < c_n < a_{\omega^n},
\end{aligned}$$

where $\omega^\omega$ and $\omega^n$ are ordinals given by ordinal exponentiation. Hence the point $c_n$ is inserted after the first $\omega^n$ many intervals. Now define $i \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{R})$ as follows: For a subset $x$ of $\mathbb{N}$,

$$i(x) = \bigcup_{\alpha < \omega^\omega} [a_\alpha, b_\alpha) \cup \{c_n \mid n \in \mathbb{N} \setminus x\}.$$

It is easy to see that each $i(x)$ is the difference of two open sets. For simplicity, let $a_\gamma^- = \sup\{a_\alpha \mid \alpha < \gamma\}$. The sets are constructed in such a way that $i(x) \leq_{\mathrm{W}}^{\mathbb{R}} i(y)$ for all $x \subseteq y \subseteq \mathbb{N}$. To see this, we construct a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1}\big(i(y)\big) = i(x)$. For each $n \in \mathbb{N}$, we pick a real number $d_n$ between $c_n$ and $a_{\omega^n}$. Note that $i(x) \supseteq i(y)$ and $i(x) \setminus i(y) = \{c_n \mid n \in y \setminus x\}$. Now define $f(t) = t$ unless $t \in [a_{\omega^n}^-, a_{\omega^n}]$ and $n \in y \setminus x$. If $t \in [a_{\omega^n}^-, a_{\omega^n}]$ and $n \in y \setminus x$, then we map the interval $[a_{\omega^n}^-, c_n]$ to $[a_{\omega^n}^-, a_{\omega^n}]$ by preserving the order, mapping the end points to the end points. We further map $[c_n, d_n]$ to $[d_n, a_{\omega^n}]$ with $f(c_n) = a_{\omega^n}$ and $f(d_n) = d_n$ by switching the order around and then map $[d_n, a_{\omega^n}]$ to itself by the identity function. This completes our construction of $f$ and it is easy to check that $f$ is as desired.

By modifying the above argument, we get the following:

**Claim 5.1.4.** If $x \subseteq_{\text{fin}} y$ (i.e., $x \setminus y$ is finite), then $i(x) \leq^{\mathbb{R}}_{\mathrm{W}} i(y)$.

*Proof of Claim 5.1.4.* Let $n = \max(x \setminus y) + 2$. Define $g \colon \mathbb{R} \to \mathbb{R}$ as follows: $g$ is equal to $f$ above on $[a^-_{\omega^n}, \infty)$ and since the order type of the set $\{a_\alpha \mid \omega^{n-1} \leq \alpha < \omega^n\}$ is $\omega^n$, we can define $g$ on $(-\infty, a^-_{\omega^n}]$ to $(d_{n-1}, a^-_{\omega^n}]$ in the same way as we did before so that $g^{-1}(i(y)) = i(x)$ (the point is that there is no point $c_m$ in $i(y)$ inserted between $a_{\omega^{n-1}}$ and $a^-_{\omega^n}$). This $g$ is the witness for $i(x) \leq^{\mathbb{R}}_{\mathrm{W}} i(y)$. □

**Claim 5.1.5.** If $i(x) \leq^{\mathbb{R}}_{\mathrm{W}} i(y)$, then $x \subseteq_{\text{fin}} y$.

*Proof of Claim 5.1.5.* Suppose $i(x) \leq^{\mathbb{R}}_{\mathrm{W}} i(y)$ via $h \colon \mathbb{R} \to \mathbb{R}$. If $h(a_\alpha) = c_n$ for some $\alpha$ and $n$, then $h(b_\alpha) = c_n$ by continuity, which is absurd. Hence for each $\alpha < \omega^\omega$ there is some $\beta < \omega^\omega$ such that $h(a_\alpha) \in [a_\beta, b_\beta)$. Let $\alpha_0$ and $n_0$ be such that $h(a_0) \in [a_{\alpha_0}, b_{\alpha_0})$ and $\alpha_0 < \omega^{n_0}$.

We prove that $h(a_\alpha) = a_{\alpha_0+\alpha}$, $h(b_\alpha) = b_{\alpha_0+\alpha}$ and $h\big([a_\alpha, b_\alpha)\big) = [a_{\alpha_0+\alpha}, b_{\alpha_0+\alpha})$ for every $\alpha < \omega^\omega$ by induction on $\alpha$.

The case $\alpha = 0$ is done by Observation 5.1.3 for $a = a_0 - 1, b = a_0, c = b_0, d = a_1, e = a_{\alpha_0}, f = b_{\alpha_0}$ and $g = a_{\alpha_0+1}$.

If $\alpha$ is a successor ordinal, let $\alpha = \beta + 1$. By induction hypothesis, $h(b_\beta) = b_{\alpha_0+\beta}$. By Observation 5.1.3 for $a = a_\beta, b = b_\beta, c = a_{\beta+1}, d = b_{\beta+1}, e = b_{\alpha_0+\beta}, f = a_{\alpha_0+\beta+1}$ and $g = b_{\alpha_0+\beta+1}$, $h(a_{\beta+1}) = a_{\alpha_0+\beta+1}$ and $h\big([b_\beta, a_{\beta+1})\big) = [b_{\alpha_0+\beta}, a_{\alpha_0+\beta+1})$. Again by Observation 5.1.3 for $a = b_\beta, b = a_{\beta+1}, c = b_{\beta+1}, d = a_{\beta+2}, e = a_{\alpha_0+\beta+1}, f = b_{\alpha_0+\beta+1}$ and $g = a_{\alpha_0+\beta+2}$, $h(b_{\beta+1}) = b_{\alpha_0+\beta+1}$ and $h\big([a_{\beta+1}, b_{\beta+1})\big) = [a_{\alpha_0+\beta+1}, b_{\alpha_0+\beta+1})$.

If $\alpha$ is a limit ordinal, then by the continuity of $h$, we have $h(a^-_\alpha) = a^-_{\alpha_0+\alpha}$. If $\alpha$ is not of the form $\omega^n$ for some $n$, by the same argument as when $\alpha$ is a successor ordinal, we can conclude that $h(a_\alpha) = a_{\alpha_0+\alpha}$, $h(b_\alpha) = b_{\alpha_0+\alpha}$, and $h\big([a_\alpha, b_\alpha)\big) = [a_{\alpha_0+\alpha}, b_{\alpha_0+\alpha})$. If $\alpha$ is of the form $\omega^n$ for some $n$, then there are two cases: When $n \in x$, there is no inserted point in $i(x)$ between $a^-_\alpha$ and $a_\alpha$. Hence there is no inserted point in $i(y)$ between $a^-_{\alpha_0+\alpha}$ and $a_{\alpha_0+\alpha}$, otherwise $h$ would not reduce $i(x)$ to $i(y)$. By the same argument as before, $h(a_\alpha) = a_{\alpha_0+\alpha}$, $h(b_\alpha) = b_{\alpha_0+\alpha}$ and $h\big([a_\alpha, b_\alpha)\big) = [a_{\alpha_0+\alpha}, b_{\alpha_0+\alpha})$. When $n \notin x$, there is an inserted point $c_n$ in $i(x)$. But no matter whether there is an inserted point in $i(y)$ between $a^-_{\alpha_0+\alpha}$ and $a_{\alpha_0+\alpha}$, $h$ will map $[a^-_\alpha, a_\alpha]$ to $[a^-_{\alpha_0+\alpha}, a_{\alpha_0+\alpha}]$ and $h(a_\alpha) = a_{\alpha_0+\alpha}$ by the similar argument as before. Hence $h(a_\alpha) = a_{\alpha_0+\alpha}$, $h(b_\alpha) = b_{\alpha_0+\alpha}$, and $h\big([a_\alpha, b_\alpha)\big) = [a_{\alpha_0+\alpha}, b_{\alpha_0+\alpha})$.

Therefore, $h(a_\alpha) = a_{\alpha_0+\alpha}$, $h(b_\alpha) = b_{\alpha_0+\alpha}$ and $h$ maps $[a_\alpha, b_\alpha)$ to $[a_{\alpha_0+\alpha}, b_{\alpha_0+\alpha})$ for each $\alpha$. The above argument (for the limit case) also shows that if there is no inserted point in $i(x)$ between $a^-_{\omega^n}$ and $a_{\omega^n}$, then there is no inserted point in $i(y)$ between $a^-_{\alpha_0+\alpha}$ and $a_{\alpha_0+\alpha}$, which implies that $x \setminus y \subseteq n_0$ by the definition of $i$. Hence $x \subseteq_{\text{fin}} y$ as desired. □

The above two claims complete our proof. □

It is easy to construct a descending sequence of subsets of $\mathbb{N}$ with length $\omega$ with respect to $\subseteq_{\text{fin}}$. Hence,

**Corollary 5.1.6** (Hertling)**.** The Wadge order $\leq^{\mathbb{R}}_{\mathrm{W}}$ is ill-founded.

**Corollary 5.1.7.** There are two sets $A$ and $B$ which are the difference of two open sets such that neither $A \leq^{\mathbb{R}}_{\mathrm{W}} B$ nor $B \leq^{\mathbb{R}}_{\mathrm{W}} A^{\mathrm{c}}$ holds.

*Proof.* Let Even and Odd be the set of even natural positive numbers and the set of odd natural numbers respectively and set $A = i(\mathrm{Even})$ and $B = i(\mathrm{Odd})$ where $i$ is from the proof of Theorem 5.1.2. By Theorem 5.1.2, $A \not\leq^{\mathbb{R}}_{\mathrm{W}} B$. Hence it suffices to show that $B \not\leq^{\mathbb{R}}_{\mathrm{W}} A^{\mathrm{c}}$.

Suppose $B \leq^{\mathbb{R}}_{\mathrm{W}} A^{\mathrm{c}}$. Then there is a continuous function $f$ from $\mathbb{R}$ to itself such that $f^{-1}(A^{\mathrm{c}}) = B$. We show that this is impossible. Note that

$$A^{\mathrm{c}} = (-\infty, a_0) \cup \bigcup_{\alpha<\omega^\omega} [b_\alpha, a_{\alpha+1}) \cup \left( \bigcup_{\gamma \mathrm{limit}} [a^-_\gamma, a_\gamma) \setminus \{c_n \mid n \notin \mathrm{Even}\} \right).$$

Since $a_0$ is in $B$, $f(a_0) = b_\alpha$ for some $\alpha < \omega^\omega$ or $f(a_0) = a^-_\gamma$ for some limit $\gamma$. In the former case, by the continuity of $f$, $f(a^-_\omega) = a^-_{\alpha+\omega}$ while $a^-_\omega \notin B$ and $a^-_{\alpha+\omega} \in A^{\mathrm{c}}$, a contradiction. In the latter case, if $\gamma \neq \omega^n$ for any odd $n$, then by the same argument as the former case, $f(a^-_\omega) = a^-_{\gamma+\omega}$ and we can derive a contradiction. If $\gamma = \omega^n$ for some odd $n$, then we cannot reduce $[a_0, b_0)$ to a half interval inside $A^{\mathrm{c}}$ with $a^-_\gamma$ being the left endpoint because $c_n$ is not in $A^{\mathrm{c}}$ in this case. □

We now investigate the lower levels of the Wadge order on the real line and compare it with the ones of the Wadge order on the Baire space. The first obvious observation is as follows: The empty set $\emptyset$ and the whole space $\mathbb{R}$ are the only minimal elements with respect to $\leq^{\mathbb{R}}_{\mathrm{W}}$, i.e., for any subset $A$ of the real line, either $A = \emptyset$, $A = \mathbb{R}$, or $\emptyset, \mathbb{R} <^{\mathbb{R}}_{\mathrm{W}} A$. This statement holds for any topological space. Recall that a subset $A$ of the real line is *non-trivial* if $A$ is neither the empty set nor the whole space. Non-trivial subsets are non-trivial in the sense of the Wadge order on the real line.

The next observation is that closed sets and open sets on the real line behave in the same way as those in the Baire space with respect to Wadge reducibility:

**Proposition 5.1.8** (Folklore)**.** Any two non-trivial open sets are Wadge equivalent. The same holds for non-trivial closed sets.

*Proof.* It is enough to see that $(0,1) \equiv^{\mathbb{R}}_{\mathrm{W}} U$ for any non-trivial open set $U$.

The fact $(0,1) \leq_{\mathrm{W}} U$ is easy to see: $U$ consists of disjoint open intervals and we let $(a,b)$ be one of them, then we can easily map $(0,1)$ into a subset of $(a,b)$ and the complement of $(0,1)$ to the point $a$ continuously (when $a = -\infty$, we map $(0,1)$ to a subset of $(a,b)$ and the complement of $(0,1)$ to the point $b$). This continuous function witnesses $(0,1) \leq^{\mathbb{R}}_{\mathrm{W}} U$.

For $U \leq_{\mathrm{W}} (0,1)$, if $\{(a_n, b_n) \mid n \in \omega\}$ is a set of pairwise-disjoint open intervals with $U = \bigcup_{n\in\omega}(a_n, b_n)$, then we can continuously map $(a_n, b_n)$ into a subset of

$(0,1)$ for each $n$ and the complement of $U$ into the point 0 in the same way as above. This continuous function witnesses $U \leq^{\mathbb{R}}_{\mathrm{W}} (0,1)$.

The assertion for closed sets follows from the observation that if $A \leq^{\mathbb{R}}_{\mathrm{W}} B$, then $A^{\mathrm{c}} \leq^{\mathbb{R}}_{\mathrm{W}} B^{\mathrm{c}}$. □

As we have seen in Theorem 5.1.2, once we go up to the sets obtained by the difference of two open sets, then there are a lot of subsets of the real line which are not Wadge comparable each other while sets of reals in the Baire space are almost Wadge comparable in the sense of Theorem 1.15.1. Hence the agreement of the real line and the Baire space with respect to Wadge reducibility is limited to closed sets and open sets.

Since $\mathbb{R}$ is connected, there is no clopen subset of the real line except $\emptyset$ and the whole space $\mathbb{R}$. Hence non-trivial open sets cannot be reduced to non-trivial closed sets and vice versa, i.e., non-trivial closed sets are not comparable to non-trivial open sets with respect to $\leq^{\mathbb{R}}_{\mathrm{W}}$. Also they are minimal in the sense that there is no subset of the real line between the empty set (or the whole space) and closed sets (or open sets) with respect to the Wadge order. We say that a subset $A$ of a topological space $X$ is *$<^X_W$-minimal* if $\emptyset <^X_{\mathrm{W}} A$ and there is no $B$ with $\emptyset <^X_{\mathrm{W}} B <^X_{\mathrm{W}} A$. Non-trivial open sets and non-trivial closed sets are $<^{\mathbb{R}}_{\mathrm{W}}$-minimal and in the case of the Baire space, the $<^{\omega\omega}_{\mathrm{W}}$-minimal sets are exactly the non-trivial clopen sets by Wadge's Lemma, in particular every set of reals is Wadge comparable to a clopen set in the Baire space. But as we have seen in the paragraph after Proposition 5.1.1: The rationals $\mathbb{Q}$ are not comparable to any non-trivial closed set and to any non-trivial open set. We now consider which subsets of the real line are not comparable to non-trivial open sets or non-trivial closed sets.

**Definition 5.1.9.** For $A \subseteq \mathbb{R}$, we consider the following two conditions for $A$:

- $(\mathrm{I}_1)$: Every point in $A$ is an accumulation point in $A$ from both sides, i.e., for any point $x$ in $A$ any open set $U$ with $x \in U$, there are points $y, z$ in $A$ such that $y < x < z$.
- $(\mathrm{I}_2)$: If $A$ contains a bounded interval $(a, b)$, then $a, b$ belong to $A$.

We say $A$ satisfies (I) if $A$ satisfies the conditions $(\mathrm{I}_1)$ and $(\mathrm{I}_2)$.

Any countable dense subset and its complement satisfy the condition (I).

**Proposition 5.1.10.** For any non-trivial subset $A$ of $\mathbb{R}$, the following are equivalent:

1. The set $A$ satisfies the condition $(\mathrm{I}_1)$,
2. The complement of $A$ satisfies the condition $(\mathrm{I}_2)$, and
3. Any non-trivial closed set is not Wadge reducible to $A$.

Hence $A$ is not comparable to any non-trivial open set and any non-trivial closed set if and only if $A$ satisfies the condition (I). In particular, if $A$ satisfies (I), so do the complement of $A$ and any continuous preimage of $A$.

*Proof.* We show the implication 1 to 2 by contraposition. Suppose that the complement of $A$ does not satisfy the condition $(\mathrm{I}_2)$. Then there is an interval $(a, b)$ which is included in $A^{\mathrm{c}}$ but either $a$ or $b$ does not belong to $A^{\mathrm{c}}$, i.e., belongs to $A$. We may assume $a$ is in $A$. Then the point $a$ is a counter-example of the condition $(\mathrm{I}_1)$ for $A$.

We show the implication 2 to 3 by contraposition. Suppose $F \leq_{\mathrm{W}}^{\mathbb{R}} A$ for some non-trivial closed set $F$ via a continuous function $f$. By Proposition 5.1.8, we may assume $F^{\mathrm{c}} = (0, 1)$. Then $f[(0, 1)]$ is a subset of $A^{\mathrm{c}}$. Since $f$ is continuous and $0, 1$ do not belong to $A^{\mathrm{c}}$, $f[(0, 1)]$ is an interval contained in $A^{\mathrm{c}}$ such that at least one of the end-points of it does not belong to $A^{\mathrm{c}}$. This shows the negation of 2.

We show the implication 3 to 1 by contraposition. Suppose there is a point $x$ in $A$ such that $x$ is not an accumulation point of $A$ from the right side, i.e., there is a $b$ in $A^{\mathrm{c}}$ such that $(x, b)$ is contained in $A^{\mathrm{c}}$. By the same argument as in Proposition 5.1.8, we can reduce $(0, 1)$ to $A^{\mathrm{c}}$. Hence the complement of $(0, 1)$ is Wadge reducible to $A$ as desired. $\square$

The subsets of the real line which are not Wadge comparable to any non-trivial open set and to any non-trivial closed set cannot be very simple:

**Proposition 5.1.11.** Let $A$ be a non-trivial subset of $\mathbb{R}$ satisfying (I). Then $A$ is not $\mathbf{\Delta}^0_2$.

*Proof.* Let $A$ be as above and $F$ be the boundary of $A$, i.e., $\overline{A} \cap \overline{A^{\mathrm{c}}}$. We use the following fact:

**Fact 5.1.12.** If $A$ is $\mathbf{\Delta}^0_2$, then either $A \cap F$ or $A^{\mathrm{c}} \cap F$ is not dense in $F$.

*Proof.* See [53, pp. 98, 99, 258, 417]. $\square$

Hence it suffices to show that $A \cap F$ and $A^{\mathrm{c}} \cap F$ are dense in $F$. By Proposition 5.1.10, it suffices to see that $A \cap F$ is dense in $F$. We show that for any open interval $U$ with $U \cap F \neq \emptyset$, $U \cap F \cap A \neq \emptyset$.

Take any such $U$. Since $U \cap F \neq \emptyset$, there is a point $x$ which is in $U$ and $F$. If $x$ is in $A$, then $x \in U \cap F \cap A$ and we are done.

So suppose $x$ is not in $A$. Since $x \in F \subseteq \overline{A}$, there is a point $y$ in $A$ such that $y \in U$. Consider the connected component $C_y$ containing $y$ in $A$. It will remain connected in $\mathbb{R}$. Hence $C_y$ is a singleton or an interval. If $C_y$ is a singleton namely $\{y\}$, then we are done because $y \in U \cap F \cap A$.

So suppose $C_y$ is an interval with endpoints $a$ and $b$ ($a$ or $b$ might be $-\infty$ or $\infty$). Since $x$ is not in $A$ and $x$ is in $U$, $C_y \not\supseteq U$. Therefore either $a$ or $b$ belongs to $U$. Assume $a$ is in $U$. Then since $(a, b) \subseteq A$, by the condition $(\mathrm{I}_2)$ for $A$, $a$ belongs to $A$ and also to $\overline{A^{\mathrm{c}}}$. Hence $a$ is in $U \cap F \cap A$ and $U \cap F \cap A \neq \emptyset$. $\square$

Since $\mathbb{Q}$ and the complement of it satisfy the condition (I), the above proposition is optimal with respect to the complexity.

We now investigate the Wadge structure below the rationals $\mathbb{Q}$. The first observation is a trivial application of a back-and-forth argument:

**Proposition 5.1.13** (Folklore)**.** Any countable dense subset of the real line is Wadge equivalent to the rationals.

*Proof.* Let $A$ be any countable dense subset of the real line. By a standard back-and-forth argument, there is an order isomorphism $i$ between $(A, <)$ and $(\mathbb{Q}, <)$. Let $\bar{\imath}$ be the canonical order isomorphism from $\mathbb{R}$ to itself extending $i$, i.e., for a real number $r$,

$$\bar{\imath}(r) = \sup\{i(a) \mid a \in A \text{ and } a < r\}.$$

This is well-defined and $\bar{\imath}$ is homeomorphism because the topology of the real line is the order topology with its natural order. It is easy to check that $\bar{\imath}^{-1}(\mathbb{Q}) = A$ and $\bar{\imath}(A) = \mathbb{Q}$. Hence $A \equiv_{\mathrm{W}} \mathbb{Q}$. □

It is natural to ask whether $\mathbb{Q}$ is $<^{\mathbb{R}}_{\mathrm{W}}$-minimal. The answer is “No”:

**Proposition 5.1.14.** The rationals $\mathbb{Q}$ is not $<_{\mathrm{W}}$-minimal.

*Proof.* We will show that there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(\mathbb{Q})$ is nowhere dense. By Proposition 5.1.1, $\mathbb{Q}$ is not Wadge-reducible to $f^{-1}(\mathbb{Q})$. Hence $f^{-1}(\mathbb{Q}) <_{\mathrm{W}} \mathbb{Q}$. Therefore, it suffices to construct such a continuous function $f$.

Let $g\colon [0, 1] \to [0, 1]$ be the Cantor function, i.e.,

$$g\left(\sum_{n\in\omega} \frac{2a_n}{3^{n+1}}\right) = \sum_{n\in\omega} \frac{a_n}{2^{n+1}}$$

on the Cantor set and $g$ is constant on each open interval disjoint from the Cantor set in such a way that $g$ is continuous. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the continuous extension of $g$ obtained by translation, i.e.,

$$h(x) = g(x - n) + n \quad \text{if } n \le x < n + 1 \text{ for some integer } n.$$

Let $f = h + \sqrt{2}$. Then $f$ is continuous and surjective. Since the preimage of the irrationals of $g$ is a subset of the Cantor set, it is nowhere dense. Hence the preimage of the irrationals of $h$ is nowhere dense, which implies that the preimage of the rationals of $f$ (i.e., $f^{-1}(\mathbb{Q})$) is nowhere dense. □

In the above proof, the set $f^{-1}(\mathbb{Q})$ is countable and satisfies the condition (I). Hence there are two countable sets with the condition (I) such that they are not Wadge equivalent.

We do not know whether there is a $<^{\mathbb{R}}_{\mathrm{W}}$-minimal set below $\mathbb{Q}$ with respect to the Wadge order.

We now discuss long ascending and descending sequences of subsets of the real line with respect to the Wadge order. In the case of the Baire space, by Borel determinacy proved by Martin, all the Borel sets are almost prewellordered and the supremum of the rank of them is an ordinal between $\omega_1$ and $\omega_2$ by the work of Wadge. Assuming AD, all the sets of reals in the Baire space are prewellordered and the supremum of the rank of them is equal to $\Theta$, where $\Theta$ is the supremum of the ordinals which are the surjective images from the reals. Under AD we can prove that $\Theta$ is quite huge, e.g., it is a limit of measurable cardinals. Hence one can construct a very long ascending sequence of sets of reals in the Baire space with respect to the Wadge order while there is no infinite descending sequence by Theorem 1.15.2.

By Theorem 5.1.2, it is natural (and easier) to consider long ascending and descending sequences of sets of natural numbers with respect to $\subseteq_{\mathrm{fin}}$ when we discuss long ascending and descending sequences of subsets of the real line with respect to the Wadge order. Since $(\mathcal{P}(\mathbb{N}), \subseteq_{\mathrm{fin}})$ and $(\mathcal{P}(\mathbb{N}), \supseteq_{\mathrm{fin}})$ are isomorphic, it suffices to consider only ascending sequences.

**Proposition 5.1.15.** For any countable ordinal $\alpha$, there is an ascending sequence of sets of natural numbers with length $\alpha$ with respect to $\subseteq_{\mathrm{fin}}$.

*Proof.* Let $\alpha$ be any countable ordinal. Fix a bijection $\pi$ between $\alpha \times \mathbb{N}$ and $\mathbb{N}$ and for each $\xi < \alpha$, let $a_\xi = \{\pi(\xi, n) \mid n \in \mathbb{N}\}$. Then $\{a_\xi \mid \xi < \alpha\}$ is a pairwise disjoint family of infinite subsets of $\mathbb{N}$. For $\xi < \alpha$, set $b_\xi = \bigcup_{\eta<\xi} a_\eta$. Then the sequence $\langle b_\xi \mid \xi < \alpha \rangle$ is the desired sequence. $\square$

**Corollary 5.1.16.** For any countable ordinal $\alpha$, there are ascending and descending sequences of subsets of the real line with length $\alpha$ with respect to the Wadge order.

Note that by Theorem 5.1.2, the above sequences consist of sets that are the difference of two open sets. Given a countable ordinal $\xi \geq 1$, by replacing half-open intervals with proper $\boldsymbol{\Sigma}^0_\xi$ sets which are dense and co-dense in a half open interval in the construction of $i$ in Theorem 5.1.2, one could embed $(\mathcal{P}(\mathbb{N}), \subseteq_{\mathrm{fin}})$ into proper $\boldsymbol{\Sigma}^0_\xi$ sets of the real line with respect to the Wadge order, where *proper* $\boldsymbol{\Sigma}^0_\xi$ *sets* are $\boldsymbol{\Sigma}^0_\xi$ sets which are not $\boldsymbol{\Pi}^0_\xi$ sets and sets are *co-dense* if their complements are dense. Hence

**Corollary 5.1.17.** For any countable ordinals $\xi \geq 2$ and $\alpha$, there are ascending and descending sequences of proper $\boldsymbol{\Sigma}^0_\xi$ subsets of the real line with length $\alpha$ with respect to the Wadge order.

We do not know whether one could construct an ascending (or descending) sequence of subsets of the real line with length $\omega_1$ with respect to the Wadge order without using the Axiom of Choice. In the presence of the Axiom of Choice, it is possible by the following well-known result:

**Proposition 5.1.18** (AC, folklore)**.** There is an ascending sequence of sets of natural numbers with length $\omega_1$ with respect to $\subseteq_{\text{fin}}$. Moreover, if Martin's Axiom (MA) holds, then there is an ascending sequence of sets of natural numbers with length continuum.

*Proof.* We first show the former statement. Given a $\subseteq_{\text{fin}}$-increasing sequence $\langle a_n \mid n \in \omega \rangle$ of infinite and co-infinite sets of natural numbers, it is easy to find an infinite and co-infinite set of natural numbers $a$ such that $a_n \subseteq_{\text{fin}} a$ for each $n$. Using this, we can recursively construct a $\subseteq_{\text{fin}}$-increasing sequence of natural numbers with length $\omega_1$.

For the second statement, by [65, Theorem 4.23], MA implies that there is a $\subseteq_{\text{fin}}$-increasing sequence of sets of natural numbers with length continuum. □

**Corollary 5.1.19** (AC)**.** Let $\xi$ be any countable ordinal with $\xi \geq 1$. Then there are ascending and decreasing sequences of proper $\mathbf{\Sigma}^0_\xi$ subsets of the real line with length $\omega_1$ with respect to the Wadge order. Moreover if MA holds, then there are ascending and decreasing sequences of proper $\mathbf{\Sigma}^0_\xi$ subsets of the real line with length continuum with respect to the Wadge order.

Before closing this section, we come to the question whether there is a maximal set in $\mathbf{\Sigma}^0_\xi$ sets for a countable ordinal $\xi \geq 1$ with respect the Wadge order. In the case of the Baire space, any proper $\mathbf{\Sigma}^0_\xi$ set is maximal in $\mathbf{\Sigma}^0_\xi$ sets by Wadge's Lemma. In the case of the real line, this fails dramatically:

**Proposition 5.1.20** (AC)**.** There is a family $\{A_\alpha \mid \alpha < \omega_1\}$ of sets, each being the difference of two open sets in the real line such that there is no subset $B$ of the real line such that $A_\alpha \leq^{\mathbb{R}}_{\mathrm{W}} B$ for every $\alpha < \omega_1$.

*Proof.* For each countable ordinal $\alpha$, let $A_\alpha$ be the union of a sequence of half-open intervals with order type $\alpha$ (we need AC to pick up such an $\omega_1$-sequence of sequences of half-open intervals). The following is the key point:

**Claim 5.1.21.** If $A_\alpha \leq^{\mathbb{R}}_{\mathrm{W}} B$ via $f$, then $f(A_\alpha)$ is the disjoint union of $\alpha$-many half open intervals inside $B$. Hence there is a sequence of disjoint half open intervals inside $B$ with length at least $\alpha$.

*Proof of Claim 5.1.21.* We show the statement by induction on $\alpha < \omega_1$. The case $\alpha = 0$ is trivial. If $\alpha$ is a successor ordinal and $\alpha = \beta + 1$ for some $\beta$, then by induction hypothesis, $f(A_\alpha \setminus I_\beta)$ is the disjoint union of $\beta$-many half open intervals inside $B$, where $I_\beta$ is the last half-open interval in $A_\alpha$. By arguments like in Observation 5.1.3, $f(I_\beta)$ is a half open interval disjoint from $f(A_\alpha \setminus I_\beta)$. Hence $f(A_\alpha)$ is the disjoint union of $\alpha$-many half open intervals as desired. The $\alpha$ is a limit ordinal is also trivial. □

Hence if $A_\alpha \leq^{\mathbb{R}}_{\mathrm{W}} B$ for every $\alpha < \omega_1$, then $B$ must contain $\alpha$-many half open intervals for every $\alpha < \omega_1$. But any subset of the real line cannot contain $\omega_1$-many half open intervals. Hence there is no $B$ such that $A_\alpha \leq^{\mathbb{R}}_{\mathrm{W}} B$ for every $\alpha < \omega_1$. $\square$

## 5.2 Conclusion and Questions

Although we often identify the real line with the Baire space in set theory, continuous functions are sensitive objects and give us two completely different aspects of Wadge reducibility (i.e., continuous reduction) in the Baire space and the real line. It is known that Wadge's Lemma for the real line dramatically fails while it holds for the Baire space. We showed that the Wadge order for the real line is ill-founded while it is known that the Wadge order for the Baire space is well-founded. We also investigated several properties of the Wadge order for the real line and compare it with the one for the Baire space.

Let us finish this chapter by raising questions:

**The Wadge order below the rationals $\mathbb{Q}$.** As we have seen, the rationals $\mathbb{Q}$ is Wadge incomparable to non-trivial closed sets and open sets and $\mathbb{Q}$ is not $<^{\mathbb{R}}_{\mathrm{W}}$-minimal by Proposition 5.1.14. But we do not know how the structure of the Wadge order below $\mathbb{Q}$ looks like.

**Question 5.2.1.** Is there a $<^{\mathbb{R}}_{\mathrm{W}}$-minimal set below $\mathbb{Q}$?

**Long ascending and descending sequences of the Wadge order without AC.** As mentioned, we can produce ascending and descending sequences of the Wadge order with an uncountable length assuming the Axiom of Choice. How about without AC?

**Question 5.2.2.** Can we prove the existence of ascending and descending sequences of the Wadge order for the real line with length $\omega_1$ without using AC?

**The Wadge order for Polish spaces.** We have investigated the Wadge order for the real line. For this analysis, the connectedness of the space was essential. The question is how far can we generalize the above results for connected Polish spaces. Some work by Philipp Schlicht [74] deals with related issues.

# Chapter 6

# Fixed-Point Logic and Product Closure

Standard first-order logic has some simple but important closure properties. First, it is closed under *relativization*: Given a formula $\psi$ with one free variable, for every formula $\phi$, there is a formula $(\phi)^{\psi}$ which says that $\phi$ holds in the submodel consisting of all objects satisfying $\psi$. Also useful is closure under *predicate substitutions*: Given unary predicate letter $P$ and a formula $\psi$ with one free variable, for every formula $\phi$, there is a formula $[\psi/P]\phi$ which says that $\phi$ holds in the model that interprets $P$ as the set of all objects satisfying $\psi$ in the original model and the rest of the interpretation is the same as the original one. Moreover, it is closed under some kind of *product construction* which allows us to interpret the rationals $\mathbb{Q}$ by the integers $\mathbb{Z}$ as a definable subset of the Cartesian product $(\mathbb{Z} \times \mathbb{Z})$.

The three mentioned properties also hold in many languages extending first-order logic, for example LFP(FO), first-order logic with added fixed-point operators. In this chapter, we define a precise sense of 'product closure' in terms of modal languages which originally comes as an extension of public announcement in epistemic logic where we formulate logic of knowledge and information flow. Then we investigate the product closure of modal fixed-point logics including PDL and the modal $\mu$-calculus.

There are certain infinite games, called *parity games*, which serve as the game semantics for modal fixed-point logics, and the history-free determinacy of parity games is important for the semantics of modal fixed-point logics. The proofs of this chapter could be reformulated in terms of parity games, but this would not be to the benefit of the clarity of the argument, so we decided not to do it.

## 6.1 Basic notions and background

### Basic setting

We assume that readers are familiar with basics of modal logic (e.g., given in [14]). We first fix our setting throughout this chapter. In the modal logics we are going to work with, we have Boolean connectives (negation, disjunction, conjunction, and implication) and modal operators ($[i]$ and $\langle i\rangle$) for $i \in I$ where $I$ is a fixed finite set throughout this chapter (we do not use first-order quantifiers in our modal languages). Hence Kripke models are of the form $(M, \{R_i\}_{i\in I}, V)$ where $M$ is the universe of the structure, $R_i$ is an accessibility relation (i.e., a binary relation on $M$) for each $i \in I$, and $V$ is the valuation for the structure (i.e., $V\colon \mathrm{Prop} \to \mathcal{P}(M)$ and Prop is a fixed countable infinite set of all propositional letters). The semantics of the propositional letters, Boolean connectives, and modal operators for Kripke models are standard. Let us review it only for modal operators: For $i \in I$, a formula $\phi$, a Kripke model $\mathbf{M} = (M, \{R_i\}_{i\in I}, V)$ and a world $s \in M$,

$$\begin{aligned}\mathbf{M}, s \vDash [i]\phi &\iff \text{for all } t,\ \text{if } sR_it, \text{then } \mathbf{M}, t \vDash \phi,\\ \mathbf{M}, s \vDash \langle i\rangle\phi &\iff \text{for some } t, sR_it \text{ and } \mathbf{M}, t \vDash \phi.\end{aligned}$$

### Relativization and public announcement

Next, we introduce the *relativization* of a given Kripke model via a formula. For a Kripke model $\mathbf{M} = (M, \{R_i\}_{i\in I}, V)$ and a formula $P$, consider the following Kripke model $\mathbf{M}|P$: The universe is the set of all worlds $s$ in $M$ with $\mathbf{M}, s \vDash P$ (denoted by $M|P$), and all the relations and the valuation are the restriction of the original ones to the new universe, i.e., for each $i \in I$, $R'_i = R_i \cap (M|P \times M|P)$ and $V'(p) = V(p) \cap M|P$. For each formula $P$, we add the new modal operator $[!P]$ with the following semantics: Given a Kripke model $\mathbf{M}$ with a world $s$ and a formula $\phi$,

$$\mathbf{M}, s \vDash [!P]\phi \iff \text{if } \mathbf{M}, s \vDash P, \text{then } \mathbf{M}\,|\,P, s \vDash \phi.$$

The dual modal operator $\langle !P\rangle$ can be introduced in the standard way.

For a modal logic $\mathcal{L}$, let $\mathcal{L}^{\mathrm{rel}}$ be the least modal logic containing $\mathcal{L}$ and the operators $[!P]$ for each formula $P$ in $\mathcal{L}^{\mathrm{rel}}$ (i.e., closed under the operation mapping pairs $(P, \phi)$ to formulas $[!P]\phi$). A modal logic $\mathcal{L}$ is *closed under relativization* if any formula in $\mathcal{L}^{\mathrm{rel}}$ is semantically equivalent to some formula in $\mathcal{L}$.

Philosophically speaking, we regard $I$ as the set of agents and modalities $[i]$ as what agent $i$ knows or what is true to the best of $i$'s information via the accessibility relation $R_i$, i.e., given a formula $\phi$, the formula $[i]\phi$ means "The agent $i$ knows $\phi$". From this point of view, the formula $[!P]\phi$ means "After the 'event' $P$ happens, $\phi$ holds" because the new accessibility relation $R'_i$ is restricted to the worlds in $M|P$ where the formula $P$ is true. Hence each agent $i$ in $I$ is 'announced' the 'event' $P$ in the new model $\mathbf{M}|P$. In this way, we express the *public announcement* of the event $P$ to each agent and this is why we call the basic modal logic with operators $[!P]$ *public announcement logic*.

Many modal logics are not only closed under relativization but also have simple recursive translations from formulas in the expanded languages to semantically equivalent formulas in the original languages. For example, let the *basic modal logic* be the smallest modal logic in the setting we have fixed at the beginning (i.e., it has Boolean connectives, modal operators $[i], \langle i \rangle$ for $i \in I$, and propositional letters in Prop). Then the following equivalences (so-called *reduction axioms*) give us the translation witnessing its closure under relativization:

$$\begin{array}{rcl}
[!P]p & \leftrightarrow & P \to p \quad \text{for propositional letter } p, \\
[!P]\neg\phi & \leftrightarrow & P \to \neg[!P]\phi, \\
[!P](\phi \wedge \psi) & \leftrightarrow & [!P]\phi \wedge [!P]\psi, \\
[!P][i]\phi & \leftrightarrow & P \to [i](P \to [!P]\phi).
\end{array}$$

Is this always the case? No. For example, let us add the following modal operators $C_G$ for $G \subseteq I$ to the basic modal logic expressing common knowledge (e.g., everyone knows that everyone knows that, and so on...). Formally, for any formula $\phi$ and Kripke model $\mathbf{M} = (M, \{R_i\}_{i \in I}, V)$ with world $s$,

$$\mathbf{M}, s \vDash C_G\phi \iff \begin{array}{l} \text{for all worlds } t \text{ reachable from } s \text{ by some finite} \\ \text{sequence of } \bigcup_{i \in G} R_i \text{ steps, } \mathbf{M}, t \vDash \phi. \end{array}$$

This amounts to adding an operator of *reflexive-transitive closure* over the union of all individual accessibility relations. This infinitary operation takes us from the basic modal language into a fragment of so-called *propositional dynamic logic* (PDL) that we will define later. It can be shown that this fragment does not have the relativization property: Indeed, the formula $[!p]C_G q$ is not definable without modalities $[!p]$. Van Benthem, van Eijck and Kooi [11] proved this undefinability and go on to propose richer epistemic languages, using richer fragments of PDL which do have relativization closure, using so-called 'conditional common knowledge' $C_G(\phi, \psi)$ which says that $\phi$ is true in every world reachable with steps staying inside the $\psi$-worlds.

## Event models and product update

In public announcement $[!P]$, all the agents obtain the same amount of information, namely $P$. In real-life scenarios, different agents often have different powers of observation. *Product update* was introduced to model these situations. We work with *event models*

$$\mathbf{E} = (E, \{R_i\}_{i\in I}, \mathrm{PRE}),$$

where $E$ is a finite set of "events", $R_i$ is an accessibility relation on $E$ for the agent $i$ (hence $R_i \subseteq E \times E$), and PRE is a precondition function that maps events $e \in E$ to precondition formulas $\mathrm{PRE}_e$ (i.e., formulas in a given modal logic) which must hold in order for the event to occur. For a formula $P$, the basic event model $\mathbf{E}_P$ is as follows: It has only one event $e_0$ and $R_i = \{(e_0, e_0)\}$ and $\mathrm{PRE}_{e_0} = P$. This event model will play the same role as the operator $[!P]$ does.

Given an event model $\mathbf{E} = (E, \{R_i\}_{i\in I}, \mathrm{PRE})$, "product update" turns a Kripke model $\mathbf{M}$ into another Kripke model $\mathbf{M} \times \mathbf{E}$ as follows: The universe of $\mathbf{M} \times \mathbf{E}$ (we write $|\mathbf{M} \times \mathbf{E}|$) is the set of all pairs $(s, e)$ in $M \times E$ such that $(\mathbf{M}, s) \vDash \mathrm{PRE}_e$, the new accessibility relation satisfies $(s, e)R_i(t, f)$ if both $sR_it$ and $eR_if$ for each $i \in I$, and the new valuation is the same as $\mathbf{M}$, i.e., $V(p) = \{(s, e) \in |\mathbf{M} \times \mathbf{E}| \mid s \in V(p)\}$ for each $p$ in PROP. Note that if $\mathbf{E}$ is $\mathbf{E}_P$ for some formula $P$, then $\mathbf{M} \times \mathbf{E}_P$ is naturally isomorphic to $\mathbf{M}|P$.

The product model $\mathbf{M}\times\mathbf{E}$ with a world $(s, e)$ records the information of different agents after some event $e$ has taken place in the epistemic setting represented by $\mathbf{E}$. The uncertainty among new worlds $(s, e), (t, f)$ can only come from old uncertainty among $s, t$ via indistinguishable events $a, b$.

Given an event model $\mathbf{E}$ with an event $e$, we introduce the modal operator $[\mathbf{E}, e]$ as follows: For a formula $\phi$ and Kripke model $\mathbf{M}$ with world $s$,

$$\mathbf{M}, s \vDash [\mathbf{E}, e]\phi \iff \text{if } \mathbf{M}, s \vDash \mathrm{PRE}_e, \text{ then } \mathbf{M} \times \mathbf{E}, (s, e) \vDash \phi.$$

The dual modal operator $\langle \mathbf{E}, e\rangle$ can be introduced in the standard way. It is easy to see that if $\mathbf{E}$ is $\mathbf{E}_P$ for some formula $P$, then the modal operator $[\mathbf{E}_P, e_0]$ is really the same as $[!P]$.

We now introduce the product update closure of modal logics. For a modal logic $\mathcal{L}$, let $\mathcal{L}^{\mathrm{p}}$ be the least modal logic containing $\mathcal{L}$ and the operators $[\mathbf{E}, e]$ for each event model $\mathbf{E}$ with an event $e$ whose precondition function maps events to formulas in $\mathcal{L}^{\mathrm{p}}$. A modal logic $\mathcal{L}$ is *closed under product update* if any formula in $\mathcal{L}^{\mathrm{p}}$ is semantically equivalent to a formula in $\mathcal{L}$.

As is expected, the basic modal logic is closed under product update by the following equivalences:

$$\begin{aligned}
[\mathbf{E}, e]p &\quad\leftrightarrow\quad \mathrm{PRE}_e \to p \quad \text{for propositional letter } p,\\
[\mathbf{E}, e]\neg\phi &\quad\leftrightarrow\quad \mathrm{PRE}_e \to \neg[\mathbf{E}, e]\phi,\\
[\mathbf{E}, e](\phi \wedge \psi) &\quad\leftrightarrow\quad [\mathbf{E}, e]\phi \wedge [\mathbf{E}, e]\psi,\\
[\mathbf{E}, e][i]\phi &\quad\leftrightarrow\quad \mathrm{PRE}_e \to \bigwedge_{eR_if \text{ in } \mathbf{E}} [i][\mathbf{E}, f]\phi.
\end{aligned}$$

This is due to Baltag, Moss and Solecki [8].

But again, the situation gets more complicated when we add common knowledge operators $C_G$ for $G \subseteq I$. In this case, no reduction to the language without

$[\mathbf{E}, e]$ modalities is possible. This problem can be solved by moving to *propositional dynamic logic* (PDL) which allows more modalities $[\pi]$ than just the basic modalities $[i]$ for $i \in I$. The set of such $\pi$s (called programs in the context of PDL) is the smallest set satisfying the following: It contains $i$ for all $i \in I$ and the tests $?\phi$ for each formula $\phi$ in the language, and is closed under the operations "unions" $\pi_1 \cup \pi_2$, "compositions" $\pi_1; \pi_2$ and "Kleene iterations" $\pi^*$. More precisely, in the language of PDL, the set of formulas $\phi$ and the set of programs $\pi$ are simultaneously and recursively defined in the following way:

$$\begin{aligned}
\phi &::= p \ (p \in \mathrm{PROP}) \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid [\pi]\phi \mid \langle\pi\rangle\phi, \\
\pi &::= i \ (i \in I) \mid ?\phi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*.
\end{aligned}$$

Semantics of formulas in PDL are given by assigning the relations $R_\pi$ on the universe of a given Kripke model to each program $\pi$ given the relations $R_i$ for each $i \in I$ and interpreting $[\pi]\phi$ and $\langle\pi\rangle\phi$ in exactly the same way as for formulas $[i]\phi$ and $\langle i\rangle\phi$ with using $R_\pi$ instead of $R_i$. Given a Kripke model $\mathbf{M} = (M, \{R_i\}_{i\in I}, V)$, the relations $R_\pi$ are recursively defined as follows:

$$\begin{aligned}
R_{?\phi} &= \{(s,s) \mid (\mathbf{M}, s) \vDash \phi\}, \\
R_{\pi_1 \cup \pi_2} &= R_{\pi_1} \cup R_{\pi_2}, \\
R_{\pi_1;\pi_2} &= \{(s,t) \mid (\exists u \in M)\ (s,u) \in R_{\pi_1} \text{ and } (u,t) \in R_{\pi_2}\}, \\
R_{\pi^*} &= R_\pi^*,
\end{aligned}$$

where $R_\pi^*$ is the reflexive and transitive closure of $R_\pi$.

**Theorem 6.1.1** (Van Benthem and Kooi [13])**.** The modal logic PDL is closed under product update.

The product update closure of PDL was first proved by van Benthem and Kooi [13] using finite automata to serve as "controllers" restricting state sequences in product models $\mathbf{M}\times\mathbf{E}$. The second proof of this fact was given by van Benthem, van Eijck and Kooi [11] where they use Kleene's Theorem for regular languages connecting the theory of finite automata with PDL. The third proof, which we present here, is given by van Benthem and the author [12] where they regard PDL as a weak fragment of the modal $\mu$-calculus (which we define in the next section). In this chapter, we strengthen the last point of view: We give a uniform proof of the product update closure for three fixed-point logics: The modal $\mu$-calculus, PDL and the continuous fragment of the modal $\mu$-calculus (CF). We first give the proof for the modal $\mu$-calculus as a proto-type and then apply the same argument for the other two logics using Venema's characterization of PDL as a fragment of the modal $\mu$-calculus.

## 6.2 The case for the modal $\mu$-calculus

In this section, we introduce the modal $\mu$-calculus and prove its product update closure. In the syntax of the modal $\mu$-calculus, we add two fixed-point operators $\mu$ and $\nu$ to the basic modal logic which denote the "least fixed-point" and the "largest fixed-point" respectively. More precisely, the set of formulas in the modal $\mu$-calculus is recursively defined as follows:

$$\phi ::= p \ (p \in \mathrm{PROP}) \mid \neg\phi \mid \phi \wedge \phi \mid \langle i \rangle \phi \mid \mu x.\phi(x),$$

where any occurrence of the variable $x$ (which is formally an element of PROP) is *positive* in $\phi(x)$, if the number of negation symbols binding the occurrence is even. (We say it is *negative* if the number is odd.) As is usual, one can define $\phi \vee \psi$ to be $\neg(\neg\phi \wedge \neg\psi)$, $[i]\phi$ to be $\neg\langle i \rangle \neg\phi$, and $\nu x.\phi(x)$ to be $\neg\mu x.\neg\phi(\neg x)$ respectively.

Formulas $\phi(x)$ with only positive occurrences of the proposition letter $x$ define a monotonic set transformation from $\mathcal{P}(M)$ to itself in any Kripke model $\mathbf{M}$:

$$F_\phi^{\mathbf{M}}(X) = \{s \in M \mid (\mathbf{M}[x := X], s) \vDash \phi\},$$

where the model $\mathbf{M}[x := X]$ is obtained by replacing $V(x)$ with $X$ and giving the same structure for the rest as $\mathbf{M}$, i.e., the universe of $\mathbf{M}[x := X]$ is $M$, $R_i^{\mathbf{M}]x:=X]} = R_i^{\mathbf{M}}$, $V^{\mathbf{M}[x:=X]}(p) = X$ if $p = x$, and otherwise $V^{\mathbf{M}[x:=X]}(p) = V^{\mathbf{M}}(p)$. Note that the map $F_\phi^{\mathbf{M}}$ is monotone in the sense that $X \subseteq Y$ implies that $F_\phi^{\mathbf{M}}(X) \subseteq F_\phi^{\mathbf{M}}(Y)$.

The formula $\mu x.\phi(x)$ defines the smallest fixed-point of this transformation and $\nu x.\phi(x)$ defines the greatest fixed-point of $F_\phi^{\mathbf{M}}$, i.e., the subsets $X$, $Y$ of $M$ such that $F_\phi^{\mathbf{M}}(X) = X, F_\phi^{\mathbf{M}}(Y) = Y$ and $X$ is the smallest set with this property and $Y$ is the largest set with this property respectively. Both exist for monotone maps by the Tarski-Knaster theorem (for the proof, see, e.g., the *Handbook* article by Bradfield and Stirling [18]). This means that the semantics of $\mu x.\phi(x)$ is given as follows: $(\mathbf{M}, s) \vDash \mu x.\phi(x)$ if $s$ is in the least fixed-point of the operator $F_\phi^{\mathbf{M}}$. The semantics of $\nu x.\phi(x)$ is defined in the same way with the greatest fixed-point of the operator $F_\phi^{\mathbf{M}}$. For convenience, we assume that each occurrence of a fixed-point operator binds a unique proposition letter.

Now we are ready to prove the product update closure for the modal $\mu$-calculus.

**Theorem 6.2.1.** The modal $\mu$-calculus is closed under product update.

*Proof.* We prove the statement by induction on the complexity of formulas. We only consider the least fixed-point case $\mu x.\phi(x)$ because the greatest fixed-point case can be reduced to the ones for the negation and for the least fixed-point and other cases have been dealt for the basic modal logic in the last section.

Our main task is to analyze fixed-point computations in product models $\mathbf{M} \times \mathbf{E}$ in terms of similar computations in the original model $\mathbf{M}$. The following idea turns out to work here. Let $X$ be a subset of $\mathbf{M} \times \mathbf{E}$. Modulo the event preconditions possibly ruling out some pairs, we can describe $X$, without loss of information, in terms of the sequence of its projections to the events in $\mathbf{E}$, viewed as a finite set of indices. Thus, we can describe the computation in $\mathbf{M} \times \mathbf{E}$ by means of a finite set of computations in $M$. The following set of definitions and observations makes this precise.

Take any Kripke model $\mathbf{M}$ and any event model $\mathbf{E}$. Let $n$ be the number of elements of $\mathbf{E}$ and let $\mathbf{E} = \{e_j\}_{1 \leq j \leq n}$. There are canonical mappings $\pi \colon \mathcal{P}(M)^n \to \mathcal{P}(|\mathbf{M} \times \mathbf{E}|)$ and $\iota \colon \mathcal{P}(|\mathbf{M} \times \mathbf{E}|) \to \mathcal{P}(M)^n$ with $\pi \circ \iota = \mathrm{id}$:

$$\begin{aligned} \pi(\vec{X}) &= \bigcup_{1 \leq j \leq n} (X_j \times \{e_j\}) \cap (|\mathbf{M} \times \mathbf{E}|), \\ \iota(Y) &= \{Y_j\}_{1 \leq j \leq n}, \end{aligned}$$

where $Y_j = \{x \in M \mid (x, e_j) \in Y\}$.

Given a positive formula $\phi(x)$ in the modal $\mu$-calculus, let $F_\phi^{\mathbf{M} \times \mathbf{E}} \colon \mathcal{P}(|\mathbf{M} \times \mathbf{E}|) \to \mathcal{P}(|\mathbf{M} \times \mathbf{E}|)$ be the monotone function induced by $\phi(x)$. Define $F^{\phi(x)} \colon \mathcal{P}(M)^n \to \mathcal{P}(M)^n$ as follows:

$$F^{\phi(x)} = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}} \circ \pi.$$

We claim that $F_\phi^{\mathbf{M} \times \mathbf{E}}$ is monotone if and only if $F^{\phi(x)}$ is monotone. Suppose $F_\phi^{\mathbf{M} \times \mathbf{E}}$ is monotone. Since $\pi, \iota$ are monotone and compositions of monotone functions are monotone, $F^{\phi(x)}$ is also monotone. To prove the converse, suppose $F^{\phi(x)}$ is monotone. Pick any $X, Y \in \mathcal{P}(|\mathbf{M} \times \mathbf{E}|)$ with $X \subseteq Y$. First note that $F_\phi^{\mathbf{M} \times \mathbf{E}}(X) \subseteq F_\phi^{\mathbf{M} \times \mathbf{E}}(Y)$ holds if and only if $\iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(X) \subseteq \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(Y)$ holds. Hence all we have to check is $\iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(X) \subseteq \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(Y)$. But

$$\begin{aligned} \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(X) &= \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}\big(\pi \circ \iota(X)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}} \circ \pi\big(\iota(X)\big) \\ &= F^{\phi(q)}\big(\iota(X)\big) \subseteq F^{\phi(q)}\big(\iota(Y)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}} \circ \pi\big(\iota(Y)\big) \\ &= \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}\big(\pi \circ \iota(Y)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{E}}(Y), \end{aligned}$$

where the above inclusion follows from the monotonicity of $F^{\phi(x)}$ and $\iota$.

Moreover, there is a further canonical correspondence: if $\vec{X}$ is an $F^{\phi(x)}$-fixed-point, then $\pi(\vec{X})$ is an $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point, and if $Y$ is an $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point, then $\iota(Y)$ is an $F^{\phi(x)}$-fixed-point. Since $\pi$ and $\iota$ preserve inclusions, the least $F^{\phi(x)}$-fixed-point corresponds to the least $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point in the following way: If $\vec{X}$ is the least $F^{\phi(x)}$-fixed-point, then $\pi(\vec{X})$ is the least $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point. Also if $\vec{Y}$ is the least $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point, then $\iota(\vec{Y})$ is the least $F_\phi^{\mathbf{M} \times \mathbf{E}}$-fixed-point.

**Remark 6.2.2** (Relating fixed-point computations in different models)**.** The argument above may be seen as a special case of the following "Transfer Lemma": Given two complete lattices $E$ and $F$, a function $f \colon E \to F$ and an ordinal $\beta$, $f$ is called *$\beta$-sup-continuous* (resp., *$\beta$-inf-continuous*) if for any nondecreasing (resp., nonincreasing) sequence $\langle x_\alpha \mid \alpha < \beta \rangle$ of elements of $E$,

$$\bigvee_F f(\{x_\alpha \mid \alpha < \beta\}) = f\left(\bigvee_E \{x_\alpha \mid \alpha < \beta\}\right)$$

(resp., $\bigwedge_F f(\{x_\alpha \mid \alpha < \beta\}) = f\left(\bigwedge_E \{x_\alpha \mid \alpha < \beta\}\right)$).

**Lemma 6.2.3** (Transfer Lemma)**.** Let $E$ and $F$ be two complete lattices. Let $f \colon E \to F$ be a mapping that is $\beta$-inf-continuous and $\beta$-sup-continuous for any ordinal $\beta$ and such that $f(\bot_E) = \bot_F$ and $f(\top_E) = \top_F$.

Let $g \colon E \to E$ and $h \colon F \to F$ be two monotonic mappings such that $f \circ g = h \circ f$. Let $a$ and $b$ be the least and the greatest fixed points of $g$ and let $a'$ and $b'$ be the least and the greatest fixed points of $h$. Then $a' = f(a)$ and $b' = f(b)$.

*Proof.* See [3, Lemma 1.2.15]. □

This lemma only uses our $\iota$ function, while we added the function $\pi$ for clarity, to restrict an input to the inverse image of $\iota$ — which is why the equation $\pi \circ \iota = \mathrm{id}$ holds. For further background on this kind of argument, cf. [17].

So far, we have seen that the least $F^{\mathbf{M}\times\mathbf{E}}_{\phi}$-fixed-point can be correlated with the least $F^{\phi(x)}$-fixed-point via $\pi$ and $\iota$. Our next task is to show that $[\mathbf{E}, e]\ \mu x.\phi(x)$ is actually definable in the modal $\mu$-calculus. For that purpose, first note that $[\mathbf{E}, e_j]\ \mu x.\phi(x)$ defines the $j$th coordinate of the least $F^{\mathbf{M}\times\mathbf{E}}_{\phi}$-fixed-point. By the definition of $\iota$, it is also the $j$th coordinate of the least $F^{\phi(x)}$-fixed-point. It is easy to see that the modal $\mu$-calculus is closed under simultaneous fixed-point operators by using the following lemma repeatedly:

**Lemma 6.2.4** (Gauss elimination principle)**.** Let $E, F$ be complete lattices and $f_1, f_2$ be monotone operators from $E \times F$ to itself. Let $\mu$ denote the least fixed point, $\nu$ denote the greatest fixed point and $\theta$ be $\mu$ or $\nu$. Let $g_1 \colon F \to F$ be such that $g_1(y) = \theta x.f_1(x, y)$ and let $(a, b) = \theta(x, y).\big(f_1(x, y), f_2(x, y)\big)$. Then $b = \theta y.f_2(g_1(y), y)$ and $a = g_1(b)$.

*Proof.* See [3, Proposition 1.4.7]. □

Hence if we can express $F^{\phi(x)}$ by a formula of the modal $\mu$-calculus with positive variables, then $[\mathbf{E}, e_j]\ \mu x.\phi(x)$ is definable in the modal $\mu$-calculus and we are done.

To prove this, we generalize the syntactic analysis to formulas with many variables $\vec{x} = x_1, \ldots, x_m$. For any formula $\phi(\vec{x})$ in the modal $\mu$-calculus, define $F^{\mathbf{M}\times\mathbf{E}}_{\phi(\vec{x})} \colon \mathcal{P}(|\mathbf{M} \times \mathbf{E}|)^m \to \mathcal{P}(|\mathbf{M} \times \mathbf{E}|)$ as follows:

$$F^{\mathbf{M}\times\mathbf{E}}_{\phi(\vec{x})}(\vec{Y}) = \{(s, a) \mid \big((\mathbf{M} \times \mathbf{E})[x_k := Y_k], (s, a)\big) \vDash \phi(\vec{x})\},$$

where $\vec{Y} \in (|\mathbf{M} \times \mathbf{E}|)^m$.

**Claim 6.2.5.** For any formula $\phi(\vec{x})$ in the modal $\mu$-calculus, there are formulas $\vec{\psi}_\phi(\vec{y})$ such that $F^{\phi(\vec{x})} = F^{\mathbf{M}}_{\vec{\psi}_\phi}$ and

(∗) For any $1 \leq k \leq m$, if all the occurrences of $x_k$ in $\phi$ are positive (resp., negative), then for each $1 \leq j, j' \leq n$, all the occurrences of $y_{k,j}$ in $(\psi_\phi)_{j'}$ are positive (resp., negative),

where $F^{\phi(\vec{q})}\colon \mathcal{P}(M)^{m\cdot n} \to \mathcal{P}(M)^n$ is defined as follows:

$$F^{\phi(\vec{q})}(\vec{X}) = \iota\Big(F^{\mathbf{M}\times\mathbf{E}}_{\phi(\vec{x})}\big((\pi(X_{1,1}, \cdots, X_{1,n}), \cdots, \pi(X_{m,1}, \cdots, X_{m,n})\big)\Big).$$

*Proof of Claim 6.2.5.* In the following definitions, we only display the essential argument variables needed to understand the function values. We prove the statement by induction on the complexity of $\phi$. We identify formulas with their truth sets. Also, if $\vec{\psi}$ is a sequence of formulas, $\psi_j$ is the $j$th coordinate of $\vec{\psi}$.

- Case 1: $\phi = p$ ($p$ is not in $\vec{q}$).

$$F^{\phi(\vec{x})} = \big(p \wedge \mathrm{PRE}_{e_1}, \ldots, p \wedge \mathrm{PRE}_{e_n}\big).$$

Hence $(\psi_{\phi(\vec{x})})_j = p \wedge \mathrm{PRE}_{e_j}$. It is easy to check (∗).

- Case 2: $\phi = x_k$ ($x_k$ is the $k$th coordinate of $\vec{x}$).

$$F^{\phi(\vec{x})}(\vec{X}) = \{X_{k,j} \wedge \mathrm{PRE}_{e_j}\}_{1\leq j\leq n}.$$

Hence $(\psi_{\phi(\vec{x})})_j = y_{k,j} \wedge \mathrm{PRE}_{e_j}$, where $y_{k,j}$ is the $j$th variable in the $k$th block corresponding to $x_k$. It is also easy to check (∗).

- Case 3: $\phi = \phi_1 \wedge \phi_2$.

$$F^{\phi(\vec{x})} = \vec{\psi}_{\phi_1} \wedge \vec{\psi}_{\phi_2}.$$

Hence $\vec{\psi}_{\phi(\vec{x})} = \vec{\psi}_{\phi_1} \wedge \vec{\psi}_{\phi_2}$. It is easy to check (∗).

- Case 4: $\phi = \neg\phi'$.

$$F^{\phi(\vec{x})} = \{\neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{e_j}\}_{1\leq j\leq n}.$$

Hence $(\psi_{\phi(\vec{x})})_j = \neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{e_j}$. It is easy to check (∗) by our inductive hypothesis, and the simultaneous definition for positive and negative occurrences.

- Case 5: $\phi = \langle i\rangle\phi'$.

For any $1 \leq j \leq n$ and $s \in M$,

$$\begin{aligned} &s \in \big(F^{\phi(\vec{x})}(\vec{X})\big)_j \iff \\ &\quad (1 \leq \exists j' \leq n)\ (\exists t \in \mathbf{M})\ \Big(sR_it \wedge e_jR_ie_{j'} \wedge t \in \big(F^{\phi'(\vec{x})}(\vec{X})\big)_{j'}\Big). \end{aligned}$$

To see that this is true, observe that the condition $t \in \left(F^{\phi'(\vec{x})}(\vec{X})\right)_{j'}$ implies $(t, e_{j'}) \in |\mathbf{M} \times \mathbf{E}|$ because $t$ must be the $j'$th coordinate of an image of $\iota$ by the definition of $F^{\phi'(\vec{x})}$. Therefore, we can put

$$\left(\psi_{\phi(\vec{x})}\right)_j = \bigvee_{e_j R_i e_{j'}} \langle i \rangle \left(\psi_{\phi'(\vec{x})}\right)_{j'}.$$

- Case 6: $\phi = \mu x'.\phi'$, where all the occurrences of $x'$ are positive in $\phi'$.

$$\begin{aligned} F^{\phi(\vec{x})}(\vec{X}) =& \left\{ \left( F^{\mathbf{M}\times\mathbf{E}}_{\mu x'.\phi'(x',\vec{x})}\left((\pi(X_{1,1}, \cdots, X_{1,n}), \cdots, \pi(X_{m,1}, \cdots, X_{m,n}))\right) \right)_j \right\}_{1 \le j \le n} \\ =& \left\{ \left( \left( x' \mapsto F^{\mathbf{M}\times\mathbf{E}}_{\phi'(x',\vec{x})}\left(\pi(X_{1,1}, \cdots, X_{1,n}), \cdots, \pi(X_{m,1}, \cdots, X_{m,n})\right) \right)_* \right)_j \right\}_{1 \le j \le n} \\ =& \left( \vec{X'} \mapsto F^{\mathbf{M}}_{\vec{\psi}_{\phi'}}(\vec{X'}, \vec{X}) \right)_*, \end{aligned}$$

where $(F(\cdot))_*$ is the least $F$-fixed-point. In the above equations, the first equality is by the definition of $F^{\phi(\vec{x})}$, the second is by the definition of $F^{\mathbf{M}\times\mathbf{E}}_{\mu x'.\phi'(x',\vec{x})}$, and the third follows from the induction hypothesis and the fact that the simultaneous fixed points are invariant under the order of applications of single fixed points. By the induction hypothesis, all the occurrences of $y'_j$ are positive in $(\psi_{\vec{\phi'}})_{j'}$ for any $1 \le j, j' \le n$, where $\vec{y'}$ corresponds to $x'$. Since the modal $\mu$-calculus is closed under simultaneous fixed-point operators, we can put $\vec{\psi}_{\phi(\vec{x})} = \mu \vec{x'}.\vec{\psi}_{\phi'}(\vec{x})$, which is also in the modal $\mu$-calculus. Since $\mu$-operators do not change the positivity (negativity) of variables not bounded by them, $(*)$ also holds in this case. $\square$

$\square$

The proof of the last case explains why we needed to 'blow-up' in the number of variables in Claim 6.2.5. Also, we proved the claim for arbitrary formulas (not only for positive ones) because otherwise we cannot use the induction hypothesis in Case 4 (if a variable is positive in $\phi$, then it must be negative in $\phi'$).

As in the case for the basic modal logic, we also have a recursive translation for $[\mathbf{E}, e_j]\mu x.\phi(x)$ by taking the $j$th coordinate of the simultaneous fixed-point expression $\mu\vec{y}.\vec{\psi}_{\phi(\vec{y})}$. Since our proof is effective, we can effectively compute the shape of the translation (or the reduction axiom).

## 6.3 The case for PDL

In this section, we prove that PDL is also closed under product update using Venema's characterization of PDL as a fragment of the modal $\mu$-calculus. Let us first see this characterization.

Given a finite subset $P$ of PROP, we define the *completely additive fragment* with respect to $P$ (denoted by $\mathrm{PDL}'(P)$) as follows:

$$\phi ::= p\ (p \in P) \mid \psi \mid \psi \wedge \phi \mid \phi \vee \phi \mid \langle i \rangle \phi \mid \mu x.\phi'(x),$$

where $\psi$ belongs to the $P$-free fragment of the modal $\mu$-calculus (i.e., none of the variables in $P$ has a free occurrence in $\psi$), and $\phi' \in \mathrm{PDL}'(X \cup \{x\})$ and $x$ is not in $P$. (Hence, to be rigorous, the logics $\mathrm{PDL}'(P)$ ($P \subseteq \mathrm{PROP}$ and $P$ is finite) are simultaneously recursively defined with the above rules).

We define $\mathrm{PDL}'$ to be the fragment of the modal $\mu$-calculus where the use of the least fixed-point operator is restricted to the completely additive fragment. Formally,

$$\phi ::= p\ (p \in \mathrm{PROP}) \mid \neg\phi \mid \phi \vee \phi \mid \langle i \rangle \phi \mid \mu x.\psi(x),$$

where $\psi \in \mathrm{PDL}' \cap \mathrm{PDL}'(\{x\})$.

**Theorem 6.3.1** (Venema [85])**.** The modal logic PDL is effectively equivalent to the fragment $\mathrm{PDL}'$, i.e., there is an effective translation from formulas in PDL to ones in $\mathrm{PDL}'$ such that it preserves the truth values of the formulas in any Kripke model and vice versa.

With the help of this theorem, we can apply the same argument for the product update closure of PDL. As mentioned in the last paragraph of § 6.1, Theorem 6.1.1 is due to van Benthem and Kooi [13] and is not new. We will give a new proof of this known result.

*Proof of Theorem 6.1.1.* We will show that the fragment $\mathrm{PDL}'$ is closed under product update instead of PDL itself. The proof is basically the same as the case for the modal $\mu$-calculus. We show the statement by induction on the complexity of formulas. As before, we only consider the fixed-point case. From now on, we fix the event model $\mathbf{E}$.

In the proof for the case of the modal $\mu$-calculus, one of the points was the closure under simultaneous fixed-point operators. Here is the corresponding fact in the case for the fragment $\mathrm{PDL}'$, which is easy to check:

**Remark 6.3.2.** Let $X$, $\{y_1, \ldots, y_n\}$ be disjoint finite subsets of PROP. Then if $\phi_1(y_1, \ldots, y_n), \ldots, \phi_n(y_1, \ldots, y_n)$ are in $\mathrm{PDL}' \cap \mathrm{PDL}'(X \cup \{y_1, \ldots, y_n\})$, then each coordinate of the following formula is in $\mathrm{PDL}' \cap \mathrm{PDL}'(X)$:

$$\mu \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \bullet \begin{pmatrix} \phi_1(y_1, \ldots, y_n) \\ \phi_2(y_2, \ldots, y_n) \\ \vdots \\ \phi_n(y_1, \ldots, y_n) \end{pmatrix}$$

By the same argument as in the case for the modal $\mu$-calculus, if we can express $F^{\phi(q)}$ by formulas in $\mathrm{PDL}' \cap \mathrm{PDL}'(x_1, \ldots, x_n)$ for some fresh variables $x_1, \ldots, x_n$ for any formula $\phi(q)$ in $\mathrm{PDL}' \cap \mathrm{PDL}'(\{q\})$, we are done. The following claim with the above remark is enough for that:

**Claim 6.3.3.** Let $\phi(\vec{q})$ be a formula in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{q})$ where $\vec{q}$ is a sequence of variables (possibly not in $\phi$) with length $m$ and every variable in $\vec{q}$ does not occur in any precondition formula in $\mathbf{E}$. Take fresh variables $x_{k,j}$ $(1 \leq k \leq m, 1 \leq j \leq n)$ which do not appear in any precondition formula in $\mathbf{E}$ or in $\phi$ or in $\vec{q}$. Then there is a sequence $\vec{\psi}_{\phi(\vec{q})}$ of formulas in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$ with length $n$ such that $F^{\phi(\vec{q})} = F^{M}_{\vec{\psi}_{\phi(\vec{q})}}$ for any Kripke model $\mathbf{M}$ and

$(**)$ for a natural number $k$ with $1 \leq k \leq m$, if there is a $j$ such that $x_{k,j}$ is free in the $j$th coordinate of $\vec{\psi}_{\phi(\vec{q})}$, then $q_k$ is also free in $\phi(\vec{q})$.

*Proof of Claim 6.3.3.* In the following definitions, we only display the essential argument variables needed to understand the function values. We prove the statement by induction on the complexity of $\phi$, following the rules in $\mathrm{PDL}'$. We identify formulas with their truth sets. Also, if $\vec{\psi}$ is a sequence of formulas, $\psi_j$ is the $j$th coordinate of $\vec{\psi}$.

- Case 1: $\phi = p$ ($p$ is not in $\vec{q}$).

$$F^{\phi(\vec{q})} = \big(p \wedge \mathrm{PRE}_{e_1}, \ldots, p \wedge \mathrm{PRE}_{e_n}\big).$$

Hence $(\psi_{\phi(\vec{q})})_j = p \wedge \mathrm{PRE}_{e_j}$. Then this is in $\mathrm{PDL}'$. Since each $x_{k,j}$ does not appear in any precondition formula in $\mathbf{E}$, $(\psi_{\phi(\vec{x})})_j$ is also in $\mathrm{PDL}'(\vec{x})$. Since each $x_{k,j}$ does not appear in the formula $p \wedge \mathrm{PRE}_{e_j}$, the condition $(**)$ is immediate.

- Case 2: $\phi = q_k$ ($q_k$ is the $k$th coordinate of $\vec{q}$).

$$F^{\phi(\vec{q})}(\vec{X}) = \{X_{k,j} \wedge \mathrm{PRE}_{e_j}\}_{1 \leq j \leq n}.$$

Hence $(\psi_{\phi(\vec{q})})_j = x_{k,j} \wedge \mathrm{PRE}_{e_j}$, where $x_{k,j}$ is the $j$th variable in the $k$th block corresponding to $q_k$. By the same reasoning as in Case 1, $\mathrm{PRE}_{e_j}$ is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$ and hence $x_{k,j} \wedge \mathrm{PRE}_{e_j}$ is also in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$. It is easy to check $(**)$.

- Case 3: $\phi = \neg\phi'$.

$$F^{\phi(\vec{q})} = \{\neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{e_j}\}_{1 \leq j \leq n}.$$

Hence $(\psi_{\phi(\vec{q})})_j = \neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{e_j}$ and this is in $\mathrm{PDL}'$. Note that in this case, any free variable in $\phi'$ is not in $\vec{q}$ (otherwise $\phi$ would not belong to $\mathrm{PDL}'(\vec{q})$). By the induction hypothesis, the condition $(**)$ is true for $\vec{\psi}_{\phi'}$. Hence there is no free variable in $(\psi_{\phi'})_j$ which is of the form $x_{k,j}$ and the formula $\neg(\psi_{\phi'})_j$ is also in

$\mathrm{PDL}'(\vec{x})$ and so $\neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{e_j}$ is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$ as desired. It is easy to check the condition $(**)$.

• Case 4: $\phi = \phi_1 \vee \phi_2$.

$$F^{\phi(\vec{q})}(\vec{X}) = \vec{\psi}_{\phi_1} \vee \vec{\psi}_{\phi_2}.$$

Hence $\vec{\psi}_{\phi(\vec{q})} = \vec{\psi}_{\phi_1} \vee \vec{\psi}_{\phi_2}$ and this is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{q})$ and $(**)$ is immediately true for this formula.

• Case 5: $\phi = \langle i \rangle \phi'$.

For any $1 \leq j \leq n$ and $s \in M$,

$$\begin{aligned} &s \in \left(F^{\phi(\vec{q})}(\vec{X})\right)_j \iff \\ &\quad (1 \leq \exists j' \leq n)\ (\exists t \in \mathbf{M})\ \left(sR_it \wedge e_jR_ie_{j'} \wedge t \in \left(F^{\phi'(\vec{q})}(\vec{X})\right)_{j'}\right). \end{aligned}$$

To see that this is true, observe that the condition $t \in \left(F^{\phi'(\vec{q})}(\vec{X})\right)_{j'}$ implies $(t, e_{j'}) \in |\mathbf{M} \times \mathbf{E}|$. Therefore, we can put

$$\left(\psi_{\phi(\vec{q})}\right)_j = \bigvee_{e_jR_ie_{j'}} \langle i \rangle \left(\psi_{\phi'(\vec{q})}\right)_{j'},$$

which is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$ and it is easy to check $(**)$.

• Case 6: $\phi = \mu q'.\phi'$.

We may assume that $q'$ is not in any formulas we are concerned except $\phi$ and $\phi'$ and $q'$ is free in $\phi'$. Since $\phi$ is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{q})$, $\phi'$ is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{q} \cup \{q'\})$.

$$\begin{aligned} F^{\phi(\vec{q})}(\vec{X}) =& \left\{ \left( F^{\mathbf{M}\times\mathbf{E}}_{\mu q'.\phi'(q',\vec{q})}\big((\pi(X_{1,1}, \cdots, X_{1,n}), \cdots, \pi(X_{m,1}, \cdots, X_{m,n})\big)\right)_j \right\}_{1 \leq j \leq n} \\ =& \left\{ \left( \left( q' \mapsto F^{\mathbf{M}\times\mathbf{E}}_{\phi'(q',\vec{q})}\big(\pi(X_{1,1}, \cdots, X_{1,n}), \cdots, \pi(X_{m,1}, \cdots, X_{m,n})\big) \right)_* \right)_j \right\}_{1 \leq j \leq n} \\ =& \left(\vec{Y} \mapsto F^{\mathbf{M}}_{\vec{\psi}_{\phi'}}(\vec{Y}, \vec{X})\right)_*, \end{aligned}$$

where $(F(\cdot))_*$ is the least $F$-fixed-point.

By Remark 6.3.2, we can put $\vec{\psi}_{\phi(\vec{q})} = \mu\vec{y}.\vec{\psi}_{\phi'}(\vec{y})$, which is in $\mathrm{PDL}' \cap \mathrm{PDL}'(\vec{x})$, where $\vec{y}$ are variables corresponding to $\vec{Y}$ in the above equations, . It is easy to check $(**)$. $\square$

$\square$

## 6.4 The case for CF

One of the special properties of formulas in PDL (or $\mathrm{PDL}'$) is that when it gives the least-fixed point of a monotone operator (i.e., $\mu x.\phi(x) \in \mathrm{PDL}'$ for some $\phi(x)$), we can compute the least fixed-point of the operator by applying it $\omega$ many times from $\emptyset$ (or $\bot$). This is based on the fact that the only fixed-point operator in PDL is the star operation i.e., $\pi \mapsto \pi^*$ and this corresponds to the complete additivity of the formulas to which we can apply fixed-point operators in PDL$'$ as we have seen in the last section. If we look at the property of the star operation in PDL, we will reach the notion of continuity of the monotone operators: A function $F\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is *continuous* if the value $F(A)$ is covered by $F(C)$s for $C \subseteq A$ which is finite for any $A \in \mathcal{P}(X)$, i.e.,

$$F(A) = \bigcup\{F(C) \mid C \subseteq A \text{ and } A \text{ is finite}\}.$$

This is equivalent to saying that $F$ is *Scott continuous*, i.e., $F$ is continuous if we endow $(\mathcal{P}(X), \subseteq)$ with the Scott topology where open sets are subsets $\mathcal{U}$ of $\mathcal{P}(X)$ which are upward closed (i.e., if $A \in \mathcal{U}, A \subseteq B$, then $B \in \mathcal{U}$) and intersect with every directed subset $\mathcal{D}$ of $\mathcal{P}(X)$ with $\bigcup \mathcal{D} \in \mathcal{U}$ (a subset $\mathcal{D}$ of $\mathcal{P}(X)$ is *directed* if for any two elements $A, B$ of $\mathcal{D}$ there is an element $C$ of $\mathcal{D}$ such that $A, B \subseteq C$). Note that if $F$ is continuous, then $F$ is monotone.

Given a propositional letter $x$, a formula $\phi(x)$ in modal logic is *continuous in $x$* if the operator $F^{\mathbf{M}}_{\phi}\colon \mathcal{P}(M) \to \mathcal{P}(M)$ induced by $\phi(x)$ is continuous for any Kripke model $\mathbf{M}$. It is routine to check that every formula $\phi(x)$ in PDL$'(\{x\})$ is continuous in $x$. Also every monotone operator induced by a continuous formula gives us the least fixed-point within $\omega$ steps.

Fontaine [26] syntactically characterized continuous formulas in the modal $\mu$-calculus with the *continuous fragment* of the modal $\mu$-calculus with respect to a finite subset $P$ of PROP (denoted by CF$(P)$). The formulas in CF$(P)$ are defined as follows:

$$\phi ::= p \ (p \in P) \mid \psi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \langle i \rangle \phi \mid \mu x.\rho(x),$$

where $\psi$ is any formula in the modal $\mu$-calculus without any free variable in $P$ and $\rho(x)$ is a formula in CF$(P \cup \{x\})$ and $x$ is not in $P$. Fontaine proved that a formula in the modal $\mu$-calculus is continuous in $p$ if and only if it is equivalent to a formula in CF$(\{p\})$.

We will define the *continuous fragment* CF of the modal $\mu$-calculus in the same way as PDL$'$ and will prove its product update closure. Formulas in CF are defined as follows:

$$\phi ::= p \ (p \in \text{PROP}) \mid \neg\phi \mid \phi \vee \phi \mid \langle i \rangle \phi \mid \mu x.\psi(x),$$

where $\psi \in \text{CF} \cap \text{CF}(\{x\})$.

It is easy to see that PDL (or PDL$'$) is a fragment of CF and this inclusion is strict: The formula $\phi = \mu x.\big(\langle i \rangle (p \wedge x) \wedge \langle i \rangle (q \wedge x)\big)$ is in CF but not in PDL. This is due to van Benthem [10].

**Theorem 6.4.1.** The modal logic CF is closed under product update.

*Proof.* The argument is exactly the same as the case for PDL (or PDL$'$). ☐

## 6.5 Conclusion and questions

We introduced the product construction of Kripke models with event models generalizing the idea of public announcement in epistemic logic and proved that three modal logics are closed under product update using the fixed-point theory. There could be several ways to extend this work which we will list below:

**Connections with automata theory.** In many fixed-point logics, there is a one-to-one effective translation from formulas to 'equivalent' some kinds of automata (cf. [84]). By using this translation, it is possible to prove the product update closure of the modal $\mu$-calculus in terms of automata. But so far the argument is nothing but the combination of the translation and our argument which is more complicated than the proof in this chapter. We wonder if there is a natural (and elegant) argument for the product update closure starting from an automaton and translating it to another automaton expressing the formula after the update. This would give us more intuitive idea about what is going on when we update a current Kripke model with an event model.

**The product update closure for a general fixed-point logic.** Modal fixed-point logics fit with coalgebras and our work can be coalgebraically expressed with a functor which is essentially the same as the power set functor on the category of sets. There is a general framework of developing modal fixed-point logic via coalgebras so-called "*coalgebraic logic*" (cf. [84]). It would be interesting if we could prove the product update closure for a general fixed-point logic which is coalgebraically defined. The first step would be to formulate the product construction we gave in terms of coalgebras.

**General closure properties of a general fixed-point logic.** If one could formulate the product construction in terms of coalgebras, it would probably be some functor from the category of $F$-coalgebras to itself where $F$ is the functor for the given coalgebraic logic. If this is the case, one could extract the properties of the functor and of $F$ that we need to prove the product closure. This would give us the possibility of exploring the closure of general operations in a general coalgebraic logic.

# Bibliography


[1] John W. Addison, Jr. Some consequences of the axiom of constructibility. *Fund. Math.*, 46:337–357, 1959.

[2] Alessandro Andretta. Large cardinals and iteration trees of height $\omega$. *Ann. Pure Appl. Logic*, 54(1):1–15, 1991.

[3] André Arnold and Damian Niwiński. *Rudiments of $\mu$-calculus*, volume 146. North-Holland, 2001. Studies in Logic and the Foundations of Mathematics.

[4] Joan Bagaria. *Definable forcing and regularity properties of projective sets of reals.* PhD thesis, University of California, Berkeley, 1991.

[5] Joan Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.

[6] Joan Bagaria and Roger Bosch. Proper forcing extensions and Solovay models. *Arch. Math. Logic*, 43(6):739–750, 2004.

[7] Joan Bagaria and Roger Bosch. Solovay models and forcing extensions. *J. Symbolic Logic*, 69(3):742–766, 2004.

[8] Alexandru Baltag, Lawrence S. Moss, and Slawomir Solecki. The logic of public announcements, common knowledge, and private suspicions. In *TARK '98: Proceedings of the 7th conference on Theoretical aspects of rationality and knowledge*, pages 43–56, San Francisco, CA, USA, 1998. Morgan Kaufmann Publishers Inc.

[9] Howard Becker. A Property Equivalent to the Existence of Scales. *Trans. Amer. Math. Soc.*, 287(2):591–612, 1985.

[10] Johan van Benthem. Modal frame correspondences and fixed-points. *Studia Logica*, 83(1-3):133–155, 2006.

[11] Johan van Benthem, Jan van Eijck, and Barteld Kooi. Logics of communication and change. *Inform. and Comput.*, 204(11):1620–1662, 2006.

[12] Johan van Benthem and Daisuke Ikegami. Modal fixed-point logic and changing models. In Arnon Avron, Nachum Dershowitz, and Alexander Rabinovich, editors, *Pillars of Computer Science*, volume 4800 of *Lecture Notes in Computer Science*, pages 146–165. Springer, 2008.

[13] Johan van Benthem and Barteld Kooi. Reduction axioms for epistemic actions. In *AiML-2004: Advances in Modal Logic. International Conference on Advances in Modal Logic, Manchester, UK, September 9-1, 2004, Preliminary Conference Proceedings*, volume UMCS-04-9-1 of *Department of Computer Science. University of Manchester. Technical Report Series*, pages 197–211. King's College Publications, 2004.

[14] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, 2001.

[15] David Blackwell. Infinite $G_\delta$-games with imperfect information. *Zastos. Mat.*, 10:99–101, 1969.

[16] David Blackwell. Games with infinitely many moves and slightly imperfect information. In R. J. Nowakowski, editor, *Games of no chance, Combinatorial games at MSRI, Workshop, July 11-21, 1994 in Berkeley, CA, USA*, Mathematical Sciences Research Institute Publications 29, pages 407–408. Cambridge, 1997.

[17] Stephen L. Bloom and Zoltán Ésik. *Iteration theories: the equational logic of iterative processes.* Springer-Verlag, 1993.

[18] Julian Bradfield and Colin Stirling. Modal mu-calculi. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic (Studies in Logic and Practical Reasoning, vol. 3).* Elsevier Science, 2006.

[19] Jörg Brendle, Lorenz Halbeisen, and Benedikt Löwe. Silver measurability and its relation to other regularity properties. *Math. Proc. Cambridge Philos. Soc.*, 138(1):135–149, 2005.

[20] Jörg Brendle and Benedikt Löwe. Solovay-type characterizations for forcing-algebras. *J. Symbolic Logic*, 64(3):1307–1323, 1999.

[21] Jörg Brendle and Benedikt Löwe. Eventually different functions and inaccessible cardinals. To appear in the *Journal of the Mathematical Society of Japan.*, 2010.

[22] Andrés Eduardo Caicedo. A review of sharps. Preprint, 2004.

[23] Morton Davis. Infinite games of perfect information. In L. S. Shapley M. Dresher and A. W. Tucker, editors, *Advances in game theory*, Annals of Mathematics Studies, no. 52, pages 85–101. Princeton Univ. Press, Princeton, NJ, 1964.

[24] Anthony J. Dodd. *The core model*, volume 61 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1982.

[25] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In H. Judah, W. Just, and H. Woodin, editors, *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[26] Gaëlle Fontaine. Continuous fragment of the mu-calculus. In Michael Kaminski and Simone Martini, editors, *Computer Science Logic. 22nd International Workshop, CSL 2008, 17th Annual Conference of the EACSL, Bertinoro, Italy, September 16-19, 2008, Proceedings*, volume 5213 of *Lecture Notes in Computer Science*, pages 139–153. Springer, 2008.

[27] David Gale and Frank M. Stewart. Infinite games with perfect information. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.

[28] Kurt Gödel. The consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis. *Proc. Natl. Acad. Sci. USA*, 24:556–557, 1938. Reprinted in 26–27 of [29].

[29] Kurt Gödel. *Collected works. Vol. II.* The Clarendon Press Oxford University Press, New York, 1990. Publications 1938–1974, Edited and with a preface by Solomon Feferman.

[30] Lorenz Halbeisen and Haim Judah. Mathias absoluteness and the Ramsey property. *J. Symbolic Logic*, 61(1):177–194, 1996.

[31] Leo Harrington. Analytic determinacy and $0^{\#}$. *J. Symbolic Logic*, 43(4):685–693, 1978.

[32] Peter Hertling. *Unstetigkeitsgrade von Funktionen in der effektiven Analysis.* PhD thesis, Fern Universität, Gesamthochschule in Hagen, 1996.

[33] Greg Hjorth. The size of the ordinal $u_2$. *J. London Math. Soc. (2)*, 52(3):417–433, 1995.

[34] Daisuke Ikegami. Projective absoluteness for Sacks forcing. *Arch. Math. Logic*, 48(7):679–690, 2009.

[35] Daisuke Ikegami. Forcing absoluteness and regularity properties. *Ann. Pure Appl. Logic*, 161(7):879–894, 2010.

[36] Daisuke Ikegami, David de Kloet, and Benedikt Löwe. The Axiom of Real Blackwell Determinacy. Submitted, available at ILLC Publications PP-2009-17, 2009.

[37] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[38] Ronald B. Jensen. Measures of order zero, 1988. Handwritten notes.

[39] Ronald B. Jensen. The core model for non-overlapping extender sequences, 1990. Handwritten notes.

[40] Ronald B. Jensen and John R. Steel. Note on getting almost-true K, 2007. Unpublished manuscript.

[41] Ronald B. Jensen and John R. Steel. Note on weak covering, 2007. Unpublished manuscript.

[42] Haim Judah. Absoluteness for projective sets. In J. Oikkonen and J. Väänänen, editors, *Logic Colloquium '90 (Helsinki, 1990)*, volume 2 of *Lecture Notes Logic*, pages 145–154. Springer, Berlin, 1993.

[43] Haim Judah and Saharon Shelah. $\boldsymbol{\Delta}^1_2$-sets of reals. *Ann. Pure Appl. Logic*, 42(3):207–223, 1989.

[44] Akihiro Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[45] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[46] Alexander S. Kechris and W. Hugh Woodin. Equivalence of partition properties and determinacy. *Proc. Nat. Acad. Sci. U.S.A.*, 80(6):1783–1786, 1983.

[47] Alexander S. Kechris and W. Hugh Woodin. Generic codes for uncountable ordinals, partition properties, and elementary embeddings. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, volume 31 of *Lect. Notes Log.*, pages 379–397. Assoc. Symbol. Logic, Chicago, IL, 2008.

[48] John L. Kelley. *General topology.* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[49] John L. Kelley. *General topology*, volume 27 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1975. Reprint of [48].

[50] Peter Koepke. *The short core model and its application.* PhD thesis, Albert-Ludwigs-Universität Freiburg im Breisgau, 1983.

[51] Motokichi Kondô. Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe. *Japan J. Math.*, 15:197–230, 1939.

[52] Kenneth Kunen. *Set theory. An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1980.

[53] K. Kuratowski. *Topology. Vol. I.* New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.

[54] Grzegorz Labędzki. A topology generated by eventually different functions. *Acta Univ. Carolin. Math. Phys.*, 37(2):37–53, 1996.

[55] Benedikt Löwe. Playing with mixed strategies on infinite sets. *Internat. J. Game Theory*, 31(1):137–150, 2002.

[56] Benedikt Löwe. Set theory of infinite imperfect information games. In Alessandro Andretta, editor, *Set theory: recent trends and applications*, volume 17 of *Quad. Mat.*, pages 137–181. Dept. Math., Seconda Univ. Napoli, Caserta, 2006.

[57] Nicolas Lusin and Wacław Sierpiński. Sur un ensemble non mesurable B. *Journal de Mathématiques Pures et Appliquées*, 2(9):53–72, 1923. Reprinted as [76, p. 504–519].

[58] Donald A. Martin. The determinacy of Blackwell games. *J. Symbolic Logic*, 63(4):1565–1581, 1998.

[59] Donald A. Martin, Itay Neeman, and Marco Vervoort. The strength of Blackwell determinacy. *J. Symbolic Logic*, 68(2):615–636, 2003.

[60] Donald A. Martin and John R. Steel. A proof of projective determinacy. *J. Amer. Math. Soc.*, 2(1):71–125, 1989.

[61] Donald A. Martin and John R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[62] Donald A. Martin and John R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[63] R. Daniel Mauldin, editor. *The Scottish Book.* Birkhäuser Boston, Mass., 1981. Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, TX, May 1979.

[64] William Mitchell. The core model for sequences of measures i. *Math. Prof. Cambridge Philos. Soc.*, 95(2):228–260, 1984.

[65] J. Donald Monk. *Cardinal invariants on Boolean algebras*, volume 142 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 1996.

[66] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2009.

[67] Jan Mycielski. On the axiom of determinateness. *Fund. Math.*, 53:205–224, 1963/1964.

[68] Jan Mycielski and Hugo Steinhaus. A mathematical axiom contradicting the axiom of choice. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 10:1–3, 1962.

[69] John C. Oxtoby. The Banach-Mazur game and Banach category theorem. In A. W. Tucker M. Dresher and P. Wolfe, editors, *Contributions to the theory of games, vol. 3*, Annals of Mathematics Studies, no. 39, pages 159–163. Princeton University Press, Princeton, N. J., 1957.

[70] Jean Raisonnier and Jacques Stern. The strength of measurability hypotheses. *Israel J. Math.*, 50(4):337–349, 1985.

[71] Ralf Schindler. A simple proof of $\mathbf{\Sigma}^1_3$-correctness of K. Preprint.

[72] Ralf Schindler. Iterates of the core model. *J. Symbolic Logic*, 71(1):241–251, 2006.

[73] Ralf-Dieter Schindler. The core model for almost linear iterations. *Ann. Pure Appl. Logic*, 116(1-3):205–272, 2002.

[74] Philipp Schlicht. The Wadge order for Polish spaces. Preprint.

[75] Philipp Schlicht. *Thin equivalence relations in* $\mathrm{L}(\mathbb{R})$ *and inner models.* PhD thesis, Westfälischen Wilhelms-Universität Münster, 2008.

[76] Wacław Sierpiński. *Oeuvres choisies.* PWN–Éditions Scientifiques de Pologne, Warsaw, 1975. Volume II: Théorie des ensembles et ses applications, works of the years 1908–1929, edited by Stanisław Hartman et al.

[77] Robert M. Solovay. The independence of DC from AD. In Alexander S. Kechris and Yiannis N. Moschovakis, editors, *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 171–183. Springer, Berlin, 1978.

[78] Robert M. Solovay. The independence of DC from AD. In Alexander S. Kechris and Yiannis N. Moschovakis, editors, *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 171–183. Springer, Berlin, 1978.

[79] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic.* Springer-Verlag, Berlin, 1996.

[80] John R. Steel. The theory $\mathrm{AD}^+$, 1996. Unpublished lecture notes.

[81] John R. Steel and Philip D. Welch. $\Sigma^1_3$ absoluteness and the second uniform indiscernible. *Israel J. Math.*, 104:157–190, 1998.

[82] Mikhail Suslin. Sur une definition des ensembles mesurables B sans nombres transfinis. *C. R. Acad. Sci. Paris*, 164:88–91, 1917.

[83] Robert Van Wesep. Wadge degrees and descriptive set theory. In Alexander S. Kechris and Yiannis N. Moschovakis, editors, *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 151–170. Springer, Berlin, 1978.

[84] Yde Venema. Automata and fixed point logic: A coalgebraic perspective. *Inf. Comput.*, 204(4):637–678, 2006.

[85] Yde Venema. PDL as a fragment of the modal $\mu$-calculus, 2008. Unpublished manuscript.

[86] Marco R. Vervoort. Blackwell games. In T. S. Ferguson, L. S. Shapley, and J. B. MacQueen, editors, *Statistics, probability and game theory*, volume 30 of *IMS Lecture Notes Monogr. Ser.*, pages 369–390. Inst. Math. Statist., Hayward, CA, 1996.

[87] Petr Vopěnka and Petr Hájek. *The theory of semisets.* Academia (Publishing House of the Czechoslovak Academy of Sciences), Prague, 1972.

[88] William W. Wadge. Degrees of complexity of subsets of the Baire space. *Notices Amer. Math. Soc.*, 19:A714, 1972.

[89] W. Hugh Woodin. On the consistency strength of projective uniformization. In J. Stern, editor, *Proceedings of the Herbrand symposium* (*Marseilles, 1981*), volume 107 of *Stud. Logic Found. Math.*, pages 365–384, Amsterdam, 1982. North-Holland.

[90] Jindřich Zapletal. Descriptive set theory and definable forcing. *Mem. Amer. Math. Soc.*, 167(793), 2004.

[91] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.

[92] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.

[93] Ernst Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. In E.W. Hobson and A.E.H. Love, editors, *Proceedings of the Fifth International Congress of Mathematicians, Cambridge 1912*, volume 2, pages 501–504, Cambridge, 1913. Cambridge.

# Samenvatting

In dit proefschrift bekijken we verschillende soorten oneindige spelen en aanverwante onderwerpen in de verzamelingenleer en de wiskundige logica. Hoofdstuk 1 is gewijd aan de algemene inleiding en technische achtergrondinformatie. Het vervolg is als volgt opgezet:

**Hoofdstuk 2:** Het is bekend dat de Baire-eigenschap een zogeheten *regulariteitseigenschap* is van verzamelingen reële getallen, en dat deze eigenschap gekarakteriseerd kan worden door middel van *Banach-Mazur-spelen*. Wij karakteriseren vrijwel alle bekende *regulariteitseigenschappen* van verzamelingen reële getallen via de Baire-eigenschap van bepaalde topologische ruimtes en we gebruiken *Banach-Mazur-spelen* om de algemene equivalentiestellingen aangaande *regulariteitseigenschappen*, absoluutheid van forcing en transcendentie-eigenschappen over bepaalde canonieke binnenmodellen te bewijzen. Met behulp van deze equivalentieresultaten beantwoorden we een aantal open vragen uit de verzamelingenleer van reële getallen.

**Hoofdstuk 3:** We bespreken het verband tussen *Gale-Stewart-spelen* en *Blackwell-spelen*. De eerste zijn oneindige spelen met volledige informatie en komen uit de verzamelingenleer, de tweede zijn oneindige spelen met onvolledigde informatie en komen uit de speltheorie. Het al dan niet *gedetermineerd* zijn van Gale-Stewart-spelen is een belangrijk onderwerp in de verzamelingenleer en we kunnen ons evengoed over het *gedetermineerd* zijn van Blackwell-spelen buigen. We vergelijken het Gedetermineerdheidsaxioma voor reële getallen ($\mathrm{AD}_{\mathbb{R}}$) met het Blackwell-Gedetermineerdheidsaxioma voor reële getallen ($\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$). We laten zien dat de consistentiekracht van $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ strikt groter is dan die van het Gedetermineerdheidsaxioma (AD) in § 3.1. In § 3.2, laten we zien dat $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ vrijwel alle bekende regulariteitseigenschappen van impliceert voor alle verzamelingen reële getallen . In § 3.3, bespreken we de mogelijkheid dat $\mathrm{AD}_{\mathbb{R}}$ en $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ equivalent zijn onder Zermelo-Fraenkel verzamelingenleer verrijkt met het Axioma van Afhankelijke Keuze (ZF+DC). In § 3.4, bespreken we de mogelijkheid van equiconsistentie van $\mathrm{AD}_{\mathbb{R}}$ en $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$.

**Hoofdstuk 4:** We bestuderen het verband tussen het gedetermineerd zijn van *Gale-Stewart-spelen* en *grote kardinaalgetallen*. *Iteratiebomen* zijn belangrijke objecten bij het bewijzen het gedetermineerd zijn van *Gale-Stewart-spelen* uitgaande van *grote kardinaalgetallen*, en *alternerende ketens* van lengte $\omega$ zijn de belangrijkste *iteratiebomen* die te maken hebben met het gedetermineerd zijn van *Gale-Stewart-spelen*. We onderzoeken de bovengrenzen van de consistentiekracht van het bestaan van *alternerende ketens* met lengte $\omega$.

**Hoofdstuk 5:** *Wadge-reduceerbaarheid* is een manier om de complexiteit van deelverzamelingen van een topologische ruimte te meten via de continue reductie van een deelverzameling van een topologische ruimte naar een andere in de beschrijvende verzamelingenleer. *Wadge-reduceerbaarheid* correspondeert met many-one-reduceerbaarheid in recursietheorie. Met behulp van de karakterisering van *Wadge-reduceerbaarheid* voor de Baire-ruimte door middel van *Wadge-spelen* kan de elegante theorie van de *Wadge-reduceerbaarheid* voor de Baire-ruimte ontwikkeld worden (denk aan bijna-lineariteit, welgefundeerdheid), als we het gedetermineerdheidsaxioma (AD) aannemen. We bestuderen *Wadge-reduceerbaarheid* voor de *reëele rechte*, welke niet op een soortgelijke manier gekarakteriseerd kan worden door middel van oneindige spelen. We laten zien dat het Wadge Lemma niet opgaat voor de reëele rechte en dat de Wadge-ordening voor de reëele rechte niet welgefundeerd is, en we onderzoeken andere eigenschappen van de Wadge-ordening voor de reëele rechte.

**Hoofdstuk 6:** *Modale dekpuntslogica's* zijn modale logica's met dekpuntsoperatoren, welke meerdere wenselijke eigenschappen gemeen hebben met eerste orde-logica. We definiëren een productconstructie van een gebeurtenismodel en een Kripke-model, en we bespreken het gesloten zijn onder het nemen an producten van modale dekpuntslogica's. We laten zien dat PDL, de modale $\mu$-calculus en een fragment van de modale $\mu$-calculus gesloten zijn onder het nemen an producten.

# Abstract

In this dissertation, we discuss several types of infinite games and related topics in set theory and mathematical logic. Chapter 1 is devoted to the general introduction and preliminaries. The rest is organized as follows:

**Chapter 2:** It is known that the Baire property is one of the nice properties for sets of reals called *regularity properties* and that it can be characterized by *Banach-Mazur games*. We characterize almost all the known *regularity properties* for sets of reals via the Baire property for some topological spaces and use *Banach-Mazur games* to prove the general equivalence theorems between *regularity properties*, forcing absoluteness, and the transcendence properties over some canonical inner models. With the help of these equivalence results, we answer some open questions from set theory of the reals.

**Chapter 3:** We discuss the connection between *Gale-Stewart games* and *Blackwell games* where the former are infinite games with perfect information coming from set theory and the latter are infinite games with imperfect information coming from game theory. The *determinacy* of Gale-Stewart games has been one of the main topics in set theory and one could also consider the *determinacy* of Blackwell games. We compare the Axiom of Real Determinacy ($\mathrm{AD}_{\mathbb{R}}$) and the Axiom of Real Blackwell Determinacy ($\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$). We show that the consistency strength of $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ is strictly greater than that of the Axiom of Determinacy (AD) in § 3.1 and that $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ implies almost all the known regularity properties for every set of reals in § 3.2. In § 3.3, we discuss the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ under the Zermelo-Fraenkel set theory with the Axiom of Dependent Choice (ZF+DC). In § 3.4, we discuss the possibility of the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$.

**Chapter 4:** We work on the connection between the determinacy of *Gale-Stewart games* and *large cardinals*. *Iteration trees* are important objects to prove the determinacy of *Gale-Stewart games* from *large cardinals* and *alternating chains* with length $\omega$ are the most fundamental *iteration trees* connected to the determinacy of *Gale-Stewart games*. We investigate the the upper bound of the

consistency strength of the existence of *alternating chains* with length $\omega$.

**Chapter 5:** *Wadge reducibility* measures the complexity of subsets of topological spaces via the continuous reduction of a subset of a topological space to another one in descriptive set theory corresponding to many-one reducibility in recursion theory. With the help of the characterization of the *Wadge reducibility* for the Baire space in terms of *Wadge games*, one can develop the beautiful theory of the *Wadge reducibility* for the Baire space (e.g., almost linearity, wellfoundedness) assuming the Axiom of Determinacy (AD). We study the *Wadge reducibility* for *the real line* which cannot be characterized by infinite games in a similar way. We show that the Wadge Lemma for the real line fails and that the Wadge order for the real line is illfounded and investigate more properties of the Wadge order for the real line.

**Chapter 6:** *Modal fixed point logics* are modal logics with fixed point operators and they enjoy several nice properties as first-order logic has. We define a product construction of an event model and a Kripke model and discuss the product closure of modal fixed point logics. We show that PDL, the modal $\mu$-calculus, and a fragment of the modal $\mu$-calculus are product closed.

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