Lectio Ultima

XVI

Forcing & the Continuum Hypothesis

2 December 2025

RECAP & Summary

$CH : 2^{\aleph_0} = \aleph_1$

$GCH : \forall\alpha\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

$L \models ZFC + GCH$

if $M \models CH$, $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$,

G $\mathbb{P}$-generic $/ M$ $\quad$ $\mathbb{P}$ preserves cardinals

$M[G] \models 2^{\aleph_0} = \kappa$

if $\kappa = \aleph_2, \aleph_3, \aleph_4, \ldots$

NOT $\aleph_\omega$

but $\aleph_{\omega_1}$

(39) Let M be a ctm, $\mathbb{P} := \mathrm{Fn}(\aleph_\omega^M \times \omega, 2) \in M$, and G $\mathbb{P}$-generic over M. Analyse upper and lower bounds for $2^{\aleph_0}$ in $M[G]$ in the style of Lectures XIV & XV. Note that it is not possible that $2^{\aleph_0} = \aleph_\omega$, so not everything can go through exactly as we did it in Lecture XV. Explain what is different.

[The supplementary material on cofinality may be useful.]

(40) Let M be a ctm, $\mathbb{P}_n := \mathrm{Fn}(\omega, \omega_n^M) \in M$, and G_n be $\mathbb{P}_n$-generic over M. Let N be a ctm such that $\{G_n : n \in \omega\} \subseteq N$. Show that $\aleph_\omega^M$ is countable in N.

(41) Presentation Example. Show that if ZFC is consistent, then so is ZFC $+ 2^{\aleph_0} = \aleph_{\omega_1}$.

Q What is the relationship between CH & GCH?
Is is true that CH $\Rightarrow$ GCH?

Concretely: Is $\boxed{2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} > \aleph_2}$ consistent?

How do we increase the size of $2^{\aleph_1}$?

Natural attempt:

$$\mathbb{P} = \mathrm{Fn}(\omega_3^M \times \omega_1^M, 2)$$

Know: $\mathbb{P}$ preserves cardinals (since c.c.c.)

If G is $\mathbb{P}$-generic$/M$, then

$f := \bigcup G$ and $f : \aleph_3 \times \aleph_1 \longrightarrow 2$

and $A_\alpha := \{\gamma < \omega_1 ; f(\alpha, \gamma) = 1\}$

$\subseteq \aleph_1$

By usual density argument. $\alpha \neq \beta \implies A_\alpha \neq A_\beta$.

Thus: $2^{\aleph_1} \geq \aleph_3$ in $M[G]$.

$\boxed{\text{BUT}}$ Consider $B_\alpha := A_\alpha \cap \omega$.

Claim $\alpha \neq \beta \implies B_\alpha \neq B_\beta$, so $2^{\aleph_0} \geq \aleph_3$.

[Consider $D_{\alpha\beta} = \{p \in \mathbb{P} ; \exists n \in \omega \ p(\alpha, n) \neq p(\beta, n)\}$

This is dense by the same argument,

so $B_\alpha \neq B_\beta$.]

Remark. Consider ES#3, Example (BS) on complete embeddings and realize that the inclusion map from $\mathrm{Fn}(\omega_3^M \times \omega, 2)$ into $\mathrm{Fn}(\omega_3^M \times \omega_1^M, 2)$ is a complete embedding.

<u>Idea</u> Find $\mathbb{P}$ s.t. $\mathrm{Fn}(\omega_3^M \times \omega, 2)$ does not completely embed into $\mathbb{P}$.

$\rightsquigarrow$ $\mathrm{Fn}(X, Y, \kappa) := \{ p \,;\, p$ is a function
$\mathrm{dom}(p) \subseteq X$
$\mathrm{ran}(p) \subseteq Y$
$|p| < \kappa \}$

Clearly, $\mathrm{Fn}(X, Y) = \mathrm{Fn}(X, Y, \aleph_0)$

And if we compare $\mathrm{Fn}(X, 2, \aleph_0)$ and $\mathrm{Fn}(X, 2, \aleph_1)$, then the former does not completely embed into the latter:

e.g., take $p' : \omega \times \omega \longrightarrow 2$
the constant function 0, then many values are determined to be 0.

<u>Problem</u> $\mathrm{Fn}(\omega_3^M \times \omega_1^M, 2, \aleph_1) = \mathbb{P}$
does not have the ccc anymore.
The small countable functions above create an antichain of size $2^{\aleph_0}$.

What is the chain condition of $\mathbb{P}$?

Lemma $Fn(X,2,\kappa^+)$ has the $(2^\kappa)^+$-c.c.

Proof. Standard ΔSL argument using the generalized form of the ΔSL.

Remember from L $\overline{XIV}$, p. 5:

Corollary If $M \models \kappa$ is regular and $\mathbb{P}$ has the κ-c.c. then $M[G] \models \kappa$ is a cardinal.

Proof. Suppose not. So find $\lambda < \kappa$ and $f : \lambda \longrightarrow \kappa$ surjection.
By Theorem, find $F : \lambda \longrightarrow \wp(\kappa)$, $F \in M$ s.t.
$M \models |F(\alpha)| < \kappa$ for all $\alpha < \lambda$.
and $f(\alpha) \in F(\alpha)$.
So $\operatorname{ran}(f) \subseteq \bigcup_{\alpha<\lambda} F(\alpha)$
This contradicts $M \models \kappa$ is regular. q.e.d.

Thus: in our concrete case we start from a model of CH, so $M \models 2^{\aleph_0} = \aleph_1$ and force with $\mathbb{P} = Fn(X, 2, \underset{\aleph_0^+}{\underset{\shortparallel}{\aleph_1}})$ which has the $(2^{\aleph_0})^+$-c.c., so that's $\aleph_1^+ = \aleph_2$-c.c.

Then all cardinal $\geq \aleph_2$ are preserved.

Critical question: Is $\aleph_1^M = \aleph_1^{M[G]}$?

Def. A forcing $\mathbb{P}$ is called <u>λ-closed</u> if for every $\gamma < \lambda$ and every descending chain $\{p_\xi ; \xi < \gamma\}$ [if $\xi < \xi'$, then $p_{\xi'} < p_\xi$] there is $q \in \mathbb{P}$ s.t. $\forall \xi \; q \leq p_\xi$.

Clearly $Fn(X,2)$ is not $\aleph_1$-closed.

Also: $Fn(X,2,\aleph_1)$ <u>is</u> $\aleph_1$-closed.

[If $\alpha < \omega_1$ and $\{p_\xi ; \xi < \alpha\}$ is a descending chain, then $\bigcup p_\xi =: q \in \mathbb{P}$ (since ctble unions of ctble sets are ctble) and $q \leq p_\xi$.]

<u>**Theorem**</u> If $M \models \mathbb{P}$ is λ-closed, $\alpha < \lambda$, $X \in M$, G $\mathbb{P}$-gen./M and $f : \alpha \to X$ s.t. $f \in M[G]$. Then $f \in M$.

Note that this is a strong preservation theorem:

forcing with $Fn(X,2,\aleph_1)$ can never add a new function $f : \omega \to 2$, so $2^{\aleph_0}$ must remain the same.

Proof of Thm $f: \alpha \longrightarrow X$, $f \in M[G]$

Suppose for contradiction that $f \notin M$.

Consider $\mathcal{B} := \{f : \alpha \to X\} \cap M \in M$.

So in particular $f \notin \mathcal{B}$. Let τ be a name for f.

By FT find $p \in G$ s.t.

$$p \Vdash \tau : \check{\alpha} \longrightarrow \check{X} \wedge \tau \notin \check{\mathcal{B}}.$$

(*) Define decreasing seq. in $\mathbb{P}$:

$$p_0 := p.$$

if $\mu < \alpha$ is a limit ordinal, the fact that $\mathbb{P}$ is λ-closed gives us $p_\mu \leq p_\xi$ for $\xi < \mu$.

Suppose now p_γ is defined, $p_\gamma \leq p$.

Since $p_\gamma \Vdash \tau : \check{\alpha} \longrightarrow \check{X}$, we find $q \leq p_\gamma$ s.t. $q \Vdash \tau(\check{\gamma}) = \check{x}_\gamma$ where $x_\gamma \in X$.

Define $p_{\gamma+1} := q$.

(*) happens in M, and so $\{p_\gamma ; \gamma < \alpha\} \in M$ and $\{x_\gamma ; \gamma < \alpha\}$

Note $h: \gamma \longmapsto x_\gamma$ is a function from α to X !! in $M / \mathcal{B}$.

By λ-closure find $q \leq p_\xi$.

But then $q \Vdash \tau : \check{\alpha} \to \check{X} \wedge \tau \notin \check{\mathcal{B}} \wedge \underbrace{\tau(\check{\gamma}) = \check{x}_\gamma}_{\text{for all } \alpha}$.

$\Leftarrow$ $\mathrm{val}(\tau, H) = h$ if $q \in H \Rightarrow q \Vdash \tau \in \check{\mathcal{B}}$

CONTRADICTION

Cor 1 Consider $\mathbb{P} = \mathrm{Fn}(X, 2, \aleph_1)^M$
G $\mathbb{P}$-gen./M
Then $\wp(\omega) \cap M = \wp(\omega) \cap M[G]$.
[Follows directly from Theorem.]

Cor. 2 $\mathbb{P}$ preserves $\aleph_1$.

Alternatively:
If $f: \omega \to \alpha$ is a surjection, then by ω_1-closure, $f \in M$. Thus $\alpha < \omega_1^M$.

(a) if $x \in y$, then $\varrho(x) < \varrho(y)$;
(b) $\varrho(x) = \sup\{\varrho(y)+1\,;\, y \in x\}$;
(c) $\varrho(\alpha) = \alpha$.

(10) Presentation Example. Let M be a transitive set model of **ZFC** and let $\alpha \in M$ be such that $(M, \in) \models$"α is the least uncountable cardinal". Show that if α is countable, then there is some $A \subseteq \omega$ such that $A \notin M$.

(11) Let $\vartheta \geq \omega + 2$ be an ordinal and assume that $M \subseteq \mathbf{V}_\vartheta$ is a countable elementary submodel constructed using the Tarski-Vaught test, i.e., by iteratively collecting witnesses to all exis-

$\Longrightarrow$ $\mathbb{P}$ preserves all cardinals.

Cor. 3 If $M \models CH$, G $\mathbb{P}$-gen./M for $\mathbb{P} = \mathrm{Fn}(\omega_3^M \times \omega_1^M, 2, \omega_1^M)^M$, then

$$M[G] \models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} \geq \aleph_3.$$

Independent control of $2^{\aleph_0}$ and $2^{\aleph_1}$?
ITERATION

But: the order matters!