XV

Fifteenth & penultimate Lecture of

FORCING & THE CONTINUUM HYPOTHESIS

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THEOREM (Cohen 1962)

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Born	April 2, 1934 Long Branch, New Jersey, U.S.
Died	March 23, 2007 (aged 72) Stanford, California, U.S.
Alma mater	University of Chicago (MS, PhD)
Known for	Cohen forcing Continuum hypothesis
Awards	Bôcher Prize (1964) Fields Medal (1966) National Medal of Science (1967)

$$\mathrm{Cons}(ZFC) \Longrightarrow \mathrm{Cons}(ZFC + \neg CH)$$

Proof. 1. By compactness, it's enough to show for each $T \subseteq ZFC$ finite that there is a model $N \models T + \neg CH$.

2. Find appropriate $T^* \subseteq ZFC$ finite and ctm $M \models T^*$.

[We determine T^* by "what is needed for steps 3., 4. and possibly later"]

3. Consider $\mathbb{P} = Fn(\omega_2^M \times \omega, 2) \in M$ and G $\mathbb{P}$-gen./M. EXISTS by L IX, p3.

4. Force $M[G] \models T$. (Lectures X – XIII).

5. Note: $M[G] \models 2^{\aleph_0} \geq \omega_2^M$. Lecture XIII, p9.

6. $\mathbb{P}$ has ccc (XIV, p7)

7. ccc forcings preserve cardinals, so $\omega_2^M = \omega_2^{M[G]}$.

8. 5. + 7. + 4. $\Longrightarrow$ $M[G] \models \neg CH + T$.

q.e.d.

Why isn't this the end of FCH?

① There are more questions.

② Forcing is a general technique doing much more than $\neg CH$.

<u>Remark</u> There is nothing special about $\aleph_2$ in all of these!

If $\mathbb{P} = Fn(\omega_n^M \times \omega, 2)$, the same proof shows $M[G] \models 2^{\aleph_0} \geq \omega_n^M$.

$\underset{c.c.c.}{\Longrightarrow}$ $M[G] \models 2^{\aleph_0} \geq \aleph_n$.

We can blow up the continuum to be as large as we want!

<u>Q</u>. What is the size of $2^{\aleph_0}$ in $M[G]$?

More concretely; Is $2^{\aleph_0} = \aleph_2$ consistent?

Or: can we show that forcing with the <u>canonical</u> forcing $Fn(\omega_2^M \times \omega, 2)$ guarantees that $2^{\aleph_0} = \aleph_2$?

A. Trivially, if $M \models 2^{\aleph_0} > \aleph_2$, then that will still be true in $M[G]$ (note that the forcing preserves cardinals; and absoluteness of existence of injections).

Thus, the answer must depend on what is true in M.

<u>Q2.</u> What if $M \models CH$?

THE METHOD OF NICE NAMES

$A \subseteq \mathbb{P}$ is a <u>maximal antichain</u> if it's an antichain and $\forall p \notin A$ $\{p\} \cup A$ is not an antichain.

Let $\mathbb{P}$ be any forcing, λ any ordinal. Then

τ is a <u>nice name for subsets of λ</u>

if there is a family $\mathfrak{A} = \{A_\alpha ; \alpha < \lambda\}$ s.t. A_α is a max. antichain s.t.

$$\tau = \tau_{\mathfrak{A}} = \{(\check{\alpha}, p) ; p \in A_\alpha\}.$$

Obviously, $\mathrm{val}(\tau_{\mathfrak{A}}, G) \subseteq \lambda$.

NICE NAME THEOREM

Every subset of λ has a nice name.
[i.e., if $x \in M[G]$, then there is a nice name τ s.t. $x = \mathrm{val}(\tau, G)$.]

<u>Proof.</u> Fix $x = \mathrm{val}(\mu, G)$ for some (not nice) μ.
Fix $\alpha < \lambda$. Build a maximal antichain A_α of $\mathbb{P}$ s.t. $\forall p \in A_\alpha$ $p \Vdash \check{\alpha} \in \mu$.

This an AC/ZL proof in M, so $A_\alpha \in M$ and also $\mathfrak{A} = \{A_\alpha ; \alpha < \lambda\} \in M$.

<u>Claim</u> : $\mathrm{val}(\tau_{\mathfrak{A}}, G) = \mathrm{val}(\mu, G)$.

$\mathrm{val}(\tau_{\mathfrak{A}}, G) = \mathrm{val}(\mu, G)$

$\subseteq$ If $\alpha \in \mathrm{val}(\tau_{\mathfrak{A}}, G)$, then by def. there is $p \in G$ s.t. $(\check{\alpha}, p) \in \tau_{\mathfrak{A}}$, so $(\check{\alpha}, p) \in A_\alpha$, so

$$p \Vdash \check{\alpha} \in \mu.$$

$\alpha \in \mathrm{val}(\mu, G)$.

(*)

Remember from previous lectures (& ES#3)
If G generic, E arbitrary if $G \cap E = \emptyset$, then $\exists p \in G \, \forall r \in E \; p \perp r$.

CORRECTED FROM EARLIER VERSION

$\supseteq$ If $\alpha \in \mathrm{val}(\mu, G)$, so by FT find

$$q \in G \quad q \Vdash \check{\alpha} \in \mu.$$

Claim w.l.o.g., I could have taken $q \in G \cap A_\alpha$.

[If $G \cap A_\alpha = \emptyset$, then by (*) $\exists p \in G \, \forall r \in A_\alpha \; p \perp r$.

CORRECTED FROM EARLIER VERSION

So if $q \Vdash \check{\alpha} \in \mu$ and $q \in G$, find $\bar{q} \leq q, p$ with $\bar{q} \in G$ and then $A_\alpha \cup \{\bar{q}\}$ is a larger antichain with the property that all of its elts force $\check{\alpha} \in \mu$. Contradiction! So $G \cap A_\alpha \neq \emptyset$.]

Clarification.

But then $(\check{\alpha}, q) \in \tau_{\mathfrak{A}}$

$\alpha \in \mathrm{val}(\tau_{\mathfrak{A}}, G)$.

q.e.d.

Remark Counting names in general is hard / impossible (Example (28) on ES#2)

In contrast, counting nice names is easy.

Fix $\lambda, \mathbb{P}$. Define $\mu := |\mathbb{P}|$.
Let κ be such that $\mathbb{P}$ has the κ-c.c.

Thus there are at most $\mu^{<\kappa}$ many antichains.

So there are at most

$$(\mu^{<\kappa})^\lambda$$

many nice names.

Corollary If $\mathbb{P} \in M$ $M \models \mathbb{P}$ has the κ-c.c. + $|\mathbb{P}| = \mu$

Then define ν to be s.t. $M \models (\mu^{<\kappa})^\lambda = \nu$.

Then $M[G] \models 2^\lambda \leq \nu$.

So, need to calculate $(\mu^{<\kappa})^\lambda$ in M.

Concrete example $\mathbb{P} = \mathrm{Fn}(\omega_2^M \times \omega, 2)$

$$\mu = |\mathbb{P}| = \omega_2^M$$

$$\kappa = \omega_1^M$$ [Note: c.c.c. = $\aleph_1$-c.c.]

$$\lambda = \omega$$

$$(\mu^{<\kappa})^\lambda = \left({\omega_2^M}^{\omega}\right)^\omega = (\omega_2^M)^{\omega\cdot\omega} = (\omega_2^M)^\omega.$$

So, the question is:

What is $\aleph_2^{\aleph_0}$?

Hausdorff's Formula

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$$

So
$$\aleph_0^{\aleph_0} = 2^{\aleph_0}$$
$$\aleph_1^{\aleph_0} = \aleph_1 \cdot \aleph_0^{\aleph_0} = 2^{\aleph_0}$$
$$\underline{\aleph_2^{\aleph_0}} = \aleph_2 \cdot \aleph_1^{\aleph_0} = \underline{\aleph_2 \cdot 2^{\aleph_0}}$$
$$= \max(\aleph_2, 2^{\aleph_0})$$

Felix Hausdorff

Born	November 8, 1868 Breslau, Kingdom of Prussia (now Wrocław, Poland)
Died	January 26, 1942 (aged 73) Bonn, Germany

Cor. If $M \models 2^{\aleph_0} \leq \aleph_2$, then
$$M[G] \models 2^{\aleph_0} = \aleph_2.$$
for G $\mathrm{Fn}(\omega_2^M \times \omega, 2)$-gen./$M$

Again, there is nothing special about $\aleph_2$: the same argument gives models of $2^{\aleph_0} = \aleph_n$. Something interesting happens at $\aleph_\omega$!!!