XIV Fourteenth Lecture of FORCING & CH

25 November 2025

Recap.

$M \rightsquigarrow M[G]$

ctm of ZFC
$\mathbb{P} \in M$

ctm of ZFC
$G \subseteq \mathbb{P}$
$\mathbb{P}$-generic over M

if $\mathbb{P} = \mathrm{Fn}(X \times \omega, 2)$: $f := \bigcup G$

$$\hat{f}(x) := \{ n \in \omega ; f(x,n) = 1 \}$$

$\hat{f} : X \longrightarrow \wp(\mathbb{N})$ is an injection

Thus if $X = \omega_2^M$, then

$$M[G] \models |X| \geq \omega_2^M .$$

Does this refute CH in $M[G]$?

Only if $\omega_2^M = \omega_2^{M[G]}$. And that needs $\omega_1^M = \omega_1^{M[G]}$.

Lecture XIII, p. 7

$\rightsquigarrow$ PRESERVATION OF CARDINALS

WARM-UP: What does it mean to collapse ω_1^M?

This means: "ω_1^M is not a cardinal in $M[G]$".

It means that there is $f : \mathbb{N} \longrightarrow \omega_1^M$ that is a surjection. Clearly, $f \in M[G] \setminus M$.

Let τ be a name for it:

$$M[G] \models \underbrace{\text{val}(\tau, G) \text{ is a surjection from } \mathbb{N} \text{ to } \omega_1^M}_{\varphi}.$$

By FT, find $p \in G$ s.t. $p \Vdash \varphi$.

For each $\alpha < \omega_1^M$ there is n s.t. $M[G] \models f(n) = \alpha$, so there is some $q_\alpha \leq p$ s.t. $q_\alpha \Vdash \tau(\check{n}) = \check{\alpha}$. (*)

Suggestive shorthand for "the function named by τ has value α at n".

Fix $n \in \mathbb{N}$ and consider

$$F_n := \{\alpha < \omega_1^M \,;\, \exists q \leq p \;\; q \Vdash \tau(\check{n}) = \check{\alpha}\}$$

Note that if $q \Vdash \tau(\check{n}) = \check{\alpha}$ and $q \Vdash \tau(\check{n}) = \check{\beta}$, then $\alpha = \beta$.

Therefore $|F_n| \leq |\mathbb{P}|$. Note that $F_n \in M$ and $M \models |F_n| \leq |\mathbb{P}|$.

By (*), $\operatorname{ran}(f) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

If $M \models \mathbb{P}$ is countable, then $M \models \bigcup_{n \in \mathbb{N}} F_n$ is countable so $\omega_1^M \not\subseteq \bigcup_{n \in \mathbb{N}} F_n$, so f is not surjective.

(ANTI-) CHAIN CONDITIONS

$A \subseteq \mathbb{P}$ was called antichain if for $a, b \in A$
$a \neq b \Rightarrow a \perp b$.

Def. $\mathbb{P}$ has the κ-chain condition (κ-c.c.) if any antichain A in $\mathbb{P}$ has size $< \kappa$.

Remark Should be called "antichain condition".

If $\kappa = \aleph_1$, the $\aleph_1$-chain condition is usually called the countable chain condition (c.c.c.)

THEOREM Suppose $M \models \kappa$ is a cardinal & $\mathbb{P}$ has the κ-c.c.

Suppose $f \in M[G]$ s.t. $M[G] \models f : A \longrightarrow B$ for $A, B \in M$.

Then there is a function $F : A \longrightarrow \mathcal{P}(B)$ s.t. $F \in M$ with

(1) $\forall a \in A \;\; f(a) \in F(a)$

(2) $M \models \forall a \in A \;\; |F(a)| < \kappa$.

Proof. We follow the ideas of the warm-up

Let τ be a name for f and $p \in \mathbb{P}$ $\;\; p \Vdash \tau : \check{A} \longrightarrow \check{B}$.

τ name for $f: A \longrightarrow B$

$p \Vdash \tau : \check{A} \longrightarrow \check{B}$

Now define

$F(a) := \{ b \in B ; \exists q \leq p \ q \Vdash \tau(\check{a}) = \check{b} \}$ Clearly, $F \in M$.

Clearly, $f(a) \in F(a)$ by the FT.

For each $b \in F(a)$, consider $\{ q \leq p ; q \Vdash \tau(\check{a}) = \check{b} \} \neq \emptyset$ by definition

Using AC in M, pick an element

$$q_b \in \{ q \leq p ; q \Vdash \tau(\check{a}) = \check{b} \}.$$

Write $Q_a := \{ q_b ; b \in F(a) \} \in M$.

<u>Claim</u> Q_a is an antichain.

[If $b \neq b'$, then $q_b \Vdash \tau(\check{a}) = \check{b}$, $q_{b'} \Vdash \tau(\check{a}) = \check{b}'$, so they have to be incompatible.]

But now $M \models \mathbb{P}$ has κ-c.c.

$\Longrightarrow M \models Q_a$ has size $< \kappa$.

But the function $b \longmapsto q_b$ is an injection from $F(a)$ to Q_a.

q.e.d.

Insert on cofinality and regularity

λ limit ordinal

$C \subseteq \lambda$ **cofinal** if $\forall \alpha < \lambda \, \exists \gamma \in C \; \gamma > \alpha$.

$cf(\lambda) := \min\{|C|; C \text{ is cofinal}\}$

Clearly, $\omega \leq cf(\lambda) \leq \lambda$.

Note: $cf(cf(\lambda)) = cf(\lambda)$ and $cf(\lambda)$ is always a cardinal.

κ is called **regular** if $cf(\kappa) = \kappa$

singular otherwise.

Note: κ regular $\iff$ $\forall \lambda < \kappa$ κ is not the union of λ many sets of cardinality $< \kappa$.

$cf(\aleph_\lambda) = cf(\lambda)$

AC $\Longrightarrow$ all successor cardinals are regular

Corollary: If $M \models \kappa$ is regular and $\mathbb{P}$ has the κ-c.c. then $M[G] \models \kappa$ is a cardinal.

Proof. Suppose not. So find $\lambda < \kappa$ and $f : \lambda \longrightarrow \kappa$ surjection.

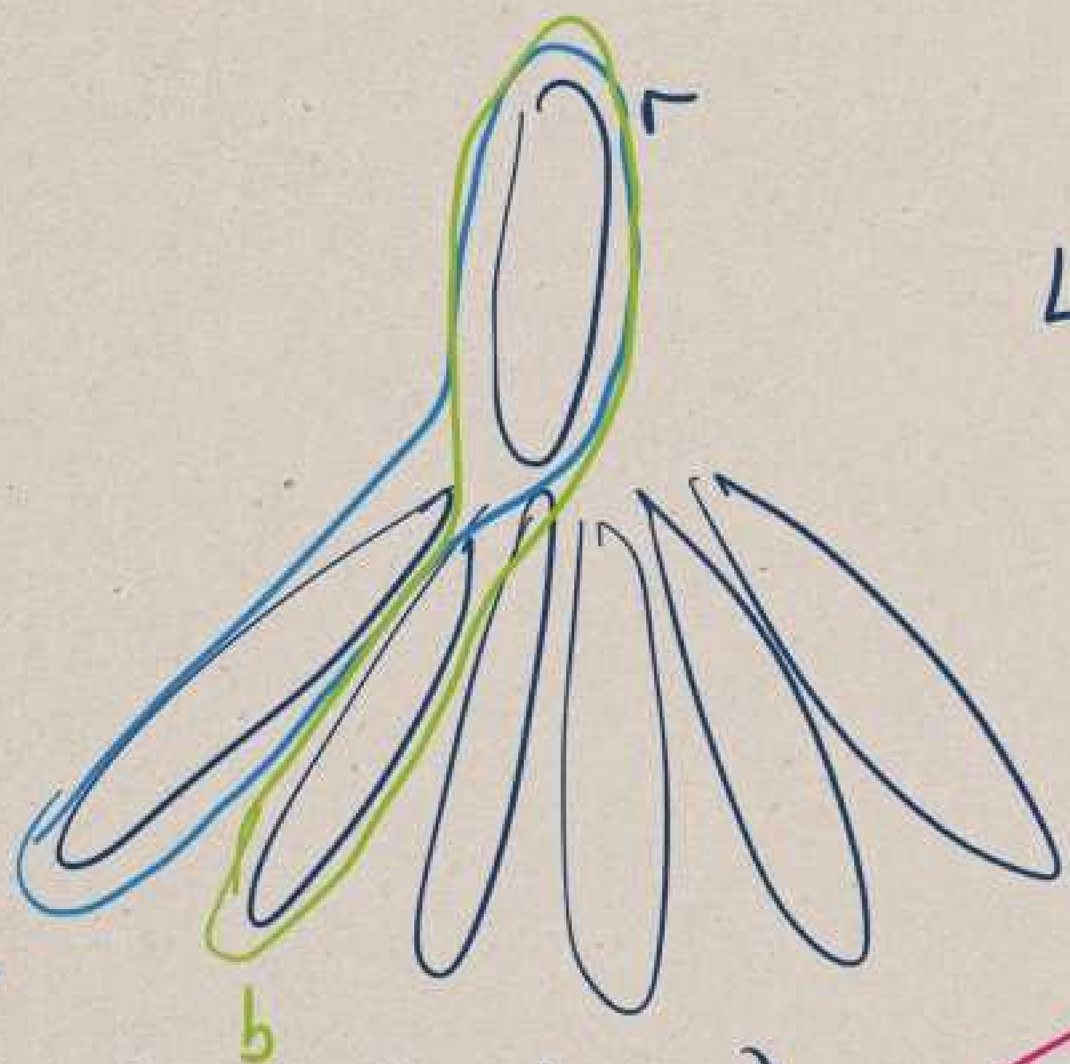
By Theorem, find $F : \lambda \longrightarrow \wp(\kappa)$, $F \in M$ s.t. $M \models |F(\alpha)| < \kappa$ for all $\alpha < \lambda$. and $f(\alpha) \in F(\alpha)$.

So $ran(f) \subseteq \bigcup_{\alpha < \lambda} F(\alpha)$

This contradicts $M \models \kappa$ is regular. q.e.d.

The Δ-SYSTEM LEMMA

<u>Def.</u> If $\mathcal{A}$ is any collection of sets, we call it a Δ-system if there is a set r called the root s.t.
$\forall a, b \in \mathcal{A} \quad a \neq b \Longrightarrow a \cap b = r$.



Looks like a river delta.

ES#3

<u>ΔSL</u> (Δ-system lemma)
If κ is infinite, $\vartheta > \kappa$ regular and
$\forall \alpha < \vartheta \quad |\alpha^{<\kappa}| < \vartheta$
If $\mathcal{A}$ is any family of sets of size $< \kappa$ s.t.
$|\mathcal{A}| = \vartheta$, then
there is $\mathcal{D} \subseteq \mathcal{A}$ s.t. $|\mathcal{D}| = \vartheta$ and $\mathcal{D}$ is a Δ-system.

Special case:

$\kappa = \omega$ and $\vartheta = \omega_1$:

Every uncountable family $\mathfrak{A}$ of finite sets contains an uncountable Δ-system.

<u>Lemma</u> For any X, $\mathrm{Fn}(X,2)$ has the c.c.c.

<u>Proof.</u> Suppose $A \subseteq \mathbb{P}$ is uncountable. NTS it's not an antichain, i.e. find $a, b \in A$ compatible.

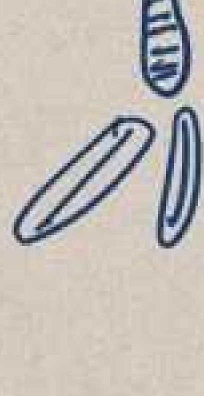
Fix $a \in A$, consider $d_a := \mathrm{dom}(a) \subseteq X$ finite

and $S := \{ d_a ; a \in A \}$.

This is an uncountable set of finite sets, so by ΔSL find $D \subseteq S$ uncountable Δ-system, say with root r.

By pigeonhole, find $a \neq b \in A$ s.t. $a \restriction r = b \restriction r$. But now $d_a \cap d_b = r$, and so a and b are compatible.

q.e.d.



r

$$\left[\; p(x) := \begin{cases} a(x) & \text{if } x \in \mathrm{dom}(a) \\ b(x) & \text{if } x \in \mathrm{dom}(b) \end{cases} \right.$$

is welldefined and $p \leq a, b$.]