XIII Thirteenth Lecture
Forcing & the CH

20 November 2025

THE FORCING THEOREM

Let M ctm, $\mathbb{P} \in M$.
Then there is an absolute definable and downwards closed (i.e., if $p \Vdash \varphi$, $q \le p \Rightarrow q \Vdash \varphi$) relation $\Vdash \subseteq \mathbb{P} \times \mathrm{Sent}_{\mathbb{P},M}$ (known as the forcing relation) with the properties:

if G is $\mathbb{P}$-generic over M and $\varphi \in \mathcal{L}_{\mathbb{P},M}$ with k free variables and $\tau_1, \dots, \tau_k \in \mathrm{Name}^{\mathbb{P}} \cap M$ then TFAE

(i) $M[G] \models \varphi(\mathrm{val}(\tau_1, G), \dots, \mathrm{val}(\tau_k, G))$

(ii) $\exists p \in G \; M \models p \Vdash \varphi(\tau_1, \dots, \tau_k)$.

We are 90% done with the proof of the FT.

ROADMAP FOR THE PROOF OF THE FORCING THEOREM:

1. Prove $\mathrm{val}(\tau_0, G) \subseteq \mathrm{val}(\tau_1, G) \iff \exists p \in G \; p \Vdash \tau_0 \subseteq \tau_1$ by induction on rank of τ_0, τ_1

10% → 1a. $[\subseteq, \Rightarrow]$

XII ✓ 1b. $[\subseteq, \Leftarrow]$

XII ✓ 2. Prove $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G) \iff \exists p \in G \; p \Vdash \tau_0 \in \tau_1$ assuming 1.

2a $[\in, \Rightarrow]$

2b $[\in, \Leftarrow]$

ES#3 ✓ 3. Prove that if φ, ψ satisfy FT, then so does $\varphi \wedge \psi$.

XII ✓ 4. Prove that if φ satisfies FT, then so does $\neg\varphi$.

ES#3 ✓ 5. Prove that if φ satisfies FT, then so does $\exists x \varphi(x)$.

Final 10% : $[\subseteq, \Longrightarrow]$

$$\mathrm{val}(\tau_0, G) \subseteq \mathrm{val}(\tau_1, G) \Longrightarrow \exists p \in G \ p \Vdash \tau_0 \subseteq \tau_1$$

Fix $r \in \mathbb{P}$, $(\pi_0, s_0) \in \tau_0$

$$\varphi_{r,\pi_0,s_0} :\Longleftrightarrow r \leq s_0 \wedge \forall (\pi_1, s_1) \in \tau_1 \ \forall q \ [q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1 \longrightarrow q \perp r]$$

"a version of the negation of the condition for D_{π_0,s_0}"

$$\varphi_r :\Longleftrightarrow \exists (\pi_0, s_0) \in \tau_0 \ \varphi_{r,\pi_0,s_0}$$

<u>Claim 1</u> If D_{π_0,s_0} is not dense below p, then there is $r \leq p$ s.t. φ_{r,π_0,s_0}.

[Check!]

<u>Corollary</u> $D := \{p \,;\, p \Vdash \tau_0 \subseteq \tau_1 \text{ or } \varphi_p\}$ is dense.

<u>Claim 2</u> If $r \in G$, then φ_r is false.

[Towards a contradiction, assume φ_r true, so find $(\pi_0, s_0) \in \tau_0$ s.t. φ_{r,π_0,s_0} is true.

$\Rightarrow r \leq s_0$
$\Rightarrow s_0 \in G$
$\Rightarrow \mathrm{val}(\pi_0, G) \in \mathrm{val}(\tau_0, G)$

Thus, by assumption, $\mathrm{val}(\pi_0, G) \in \mathrm{val}(\tau_1, G)$.
So by def. find $(\pi_1, s_1) \in \tau_1$ and $s_1 \in G$ and $\mathrm{val}(\pi_1, G) = \mathrm{val}(\pi_0, G)$.
By FT, find $q_0 \in G$ $q_0 \Vdash \pi_0 = \pi_1$.
Find $q \leq q_0, s_1$, $q \in G$.
Then $q \perp r$
Contradiction!

Fix τ_0, τ_1.

$$D_{\pi_0,s_0} := \{q \leq p \,;\, q \leq s_0 \to \exists (\pi_1, s_1) \in \tau_1 \ (q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1)\}$$

$p \Vdash \tau_0 \subseteq \tau_1 :\Longleftrightarrow \forall (\pi_0, s_0) \in \tau_0 \ D_{\pi_0,s_0}$ is dense below p

$p \Vdash \tau_0 \in \tau_1 :\Leftrightarrow \{q \leq p \,;\, \exists (\pi, s) \in \tau_1 \ (q \leq s \wedge q \Vdash \pi = \tau_0)\}$ is dense below p

$p \Vdash \varphi \wedge \psi :\Leftrightarrow p \Vdash \varphi$ and $p \Vdash \psi$

$p \Vdash \neg\varphi :\Leftrightarrow \forall q \leq p \ q \not\Vdash \varphi$

$p \Vdash \exists x \varphi(x) :\Leftrightarrow \{r \,;\, \exists \sigma \ r \Vdash \varphi(\sigma)\}$ is dense below p

Therefore, using Corollary to Claim 1, there is

$$r \in D \cap G$$

so either $r \Vdash \tau_0 \subseteq \tau_1$ or φ_r.

But by Claim 2, φ_r is false, so

$$r \Vdash \tau_0 \subseteq \tau_1.$$

q.e.d. (FT)

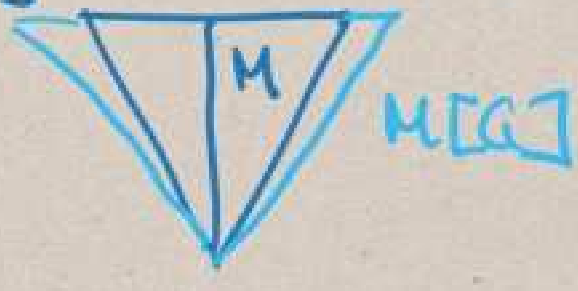
We have reached the summit: let's have a view from the peak:

1. Have FT.
2. FT $\Longrightarrow$ for each $T \subseteq ZFC$ finite, there is $T^* \subseteq ZFC$ finite with M ctm of T^*, then $M[G]$ ctm of T
3. By the compactness trick, 2. gives us a means of proving relative consistency theorems: $\mathrm{Cons}(ZFC) \Rightarrow \mathrm{Cons}(ZFC + \varphi)$.

PROVIDED that we can prove $M[G] \models \varphi$, where φ is our formula of interest, i.e., here $\neg CH$.

Remark.
Note that $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[G]$

By definition $\mathrm{rank}(\mathrm{val}(\tau, G)) \leq \mathrm{rank}(\tau)$
[Easy induction proof.]



Remember our **MAIN EXAMPLE** from Lecture IX:

MAIN EXAMPLE

Let X, Y be sets and

$\mathbb{P} := \mathrm{Fn}(X, Y) := \{p\,;\ p$ is a finite function with $\mathrm{dom}(p) \subseteq X$ and $\mathrm{ran}(p) \subseteq Y\}$

$p \leq q :\iff p \supseteq q.$

$\mathbb{1} := \emptyset.$

Note: $p \perp q \iff \exists x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ s.t. $p(x) \neq q(x)$.

Lemma 1 F filter (ideal) $\Rightarrow \bigcup F$ is a function $= \{(x,y)\,;\ \exists p \in F\ (x,y) \in p\}$

Lemma 2 $D_x := \{p\,;\ x \in \mathrm{dom}(p)\}$ This is dense

$\mathcal{D}_0 := \{D_x\,;\ x \in X\}$

F is $\mathcal{D}_0$-generic $\Rightarrow \mathrm{dom}(\bigcup F) = X$.

Lemma 3 $R_y := \{p\,;\ y \in \mathrm{ran}(p)\}$.

If X is infinite, R_y is dense

$\mathcal{D}_1 := \{R_y\,;\ y \in Y\}$

F is $\mathcal{D}_1$-generic & X is infinite $\Rightarrow \mathrm{ran}(\bigcup F) = Y$.

Lemma 4 Fix $f : X \to Y$. $N_f := \{p\,;\ \exists x \in \mathrm{dom}(p)\ f(x) \neq p(x)\}$

If X inf., N_f is dense.

F is $\{N_f\}$-generic & X is infinite $\Rightarrow \bigcup F \neq f$.

Rephrased in terms of a G that is $\mathbb{P}$-generic over M:

If $X, Y \in M$, then $\mathbb{P} = \mathrm{Fn}(X, Y) \in M$ and $\mathcal{D}_0, \mathcal{D}_1 \subseteq M$, so if G is $\mathbb{P}$-gen./M, then $f := \bigcup G$ is a surjection from X to Y.

If $g \in M$, then $N_g := \{p\,;\ \exists x\ g(x) \neq p(x)\} \in M$, so $f \neq g$. Since g was arbitrary, this means $f \notin M$.

APPLICATION 1:
COLLAPSING CARDINALS

Let $X := \omega$, $Y := \alpha$ where
$M \models \alpha$ is the first unctble cardinal
[Usual notation: $\aleph_1^M$ or ω_1^M, and similarly
$\omega_2^M, \omega_3^M, \omega_4^M$ etc.
$\omega_1^{M[G]}, \omega_2^{M[G]}, \omega_3^{M[G]}$ etc.]

Let $\mathbb{P} = Fn(\omega, \alpha)$. By the last page,
if G is $\mathbb{P}$-gen./M & $f := \bigcup G$, then

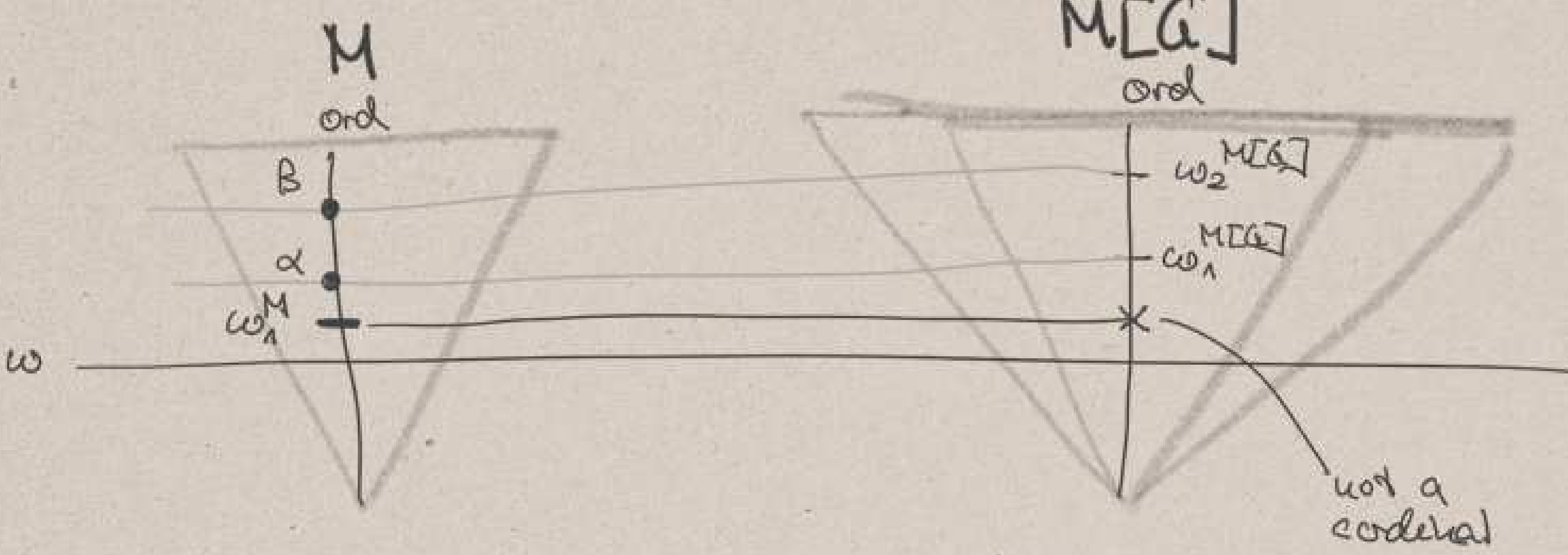
$$f : \omega \twoheadrightarrow \alpha$$

and clearly $f \notin M$.
Known as collapse forcing.
In particular: $M \models \alpha$ is a cardinal
$M[G] \not\models \alpha$ is a cardinal.

Consequence "κ is a cardinal" is not absolute for transitive models.
[not even w/ the same ordinals]

"$\forall \lambda < \kappa \, \forall f : \lambda \to \kappa$ f is not a surjection"
is Π_1, so it's downwards absolute.

Example of something that's downwards, but not upwards absolute.



α, β must be cardinals in M

Possibly M has more cardinals.

If there are no M-cardinals between ω_1^M and α, then $\alpha = \omega_2^M$.

APPLICATION 2: $V \neq L$

Just for the sake of being specific,
say $\mathbb{P} = Fn(\omega, \omega)$,
then if G is $\mathbb{P}$-gen. $/M$, $M[G] \supsetneq M$.
Say $f \in M[G] \setminus M$.

<u>Claim</u> $M[G] \models V \neq L$.

[if $M[G] \models V = L$
$\forall x \exists \alpha (x \in L_\alpha)$
Apply this to f and get
$M[G] \models \exists \alpha \; f \in L_\alpha$.
But L_α is absolute, so $L_\alpha \in M$.
But M is transitive, so $f \in M$.
Contradiction!]

With the compactness trick, this yields

$$\mathrm{Cons}(ZFC) \Longrightarrow \mathrm{Cons}(ZFC + V \neq L).$$

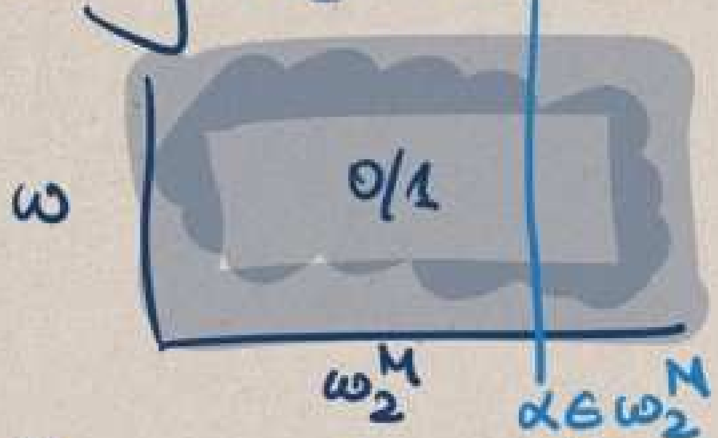
APPLICATION 3 : $\neg$CH

Consider

$$\mathbb{P} = \mathrm{Fn}(\omega_2^M \times \omega, 2)$$

and let G be $\mathbb{P}$-gen./M and $f := \bigcup G$.

f is a binary $\omega_2^M \times \omega$ matrix



Each column can be interpreted as a subset of ω.

Conside for $\alpha \neq \beta < \omega_2^M$

$$D_{\alpha,\beta} := \{ p ; \exists n \; p(\alpha,n) \neq p(\beta,n) \}.$$

This is dense and $\in M$.

<u>Therefore</u> $\hat{f}(\alpha) := \{ n ; f(\alpha,n) = 1 \}$

is an injection from ω_2^M into $\mathcal{P}(\omega)$.

This means : $M[G] \models 2^{\aleph_0} \geq |\omega_2^M|$.

$\neg$CH is : $M[G] \models 2^{\aleph_0} = \omega_2^{M[G]}$

Therefore, if $\omega_1^M = \omega_1^{M[G]}$ and $\omega_2^M = \omega_2^{M[G]}$,

this gives $M[G] \models \neg$CH.