XII TWELFTH LECTURE
Forcing & the Continuum Hypothesis

18 NOVEMBER 2025

PROOF OF THE FORCING THEOREM

THE FORCING THEOREM

Let M ctm, $\mathbb{P} \in M$.

Then there is an absolutely definable and downwards closed (i.e., if $p \Vdash \varphi$, $q \leq p \Rightarrow q \Vdash \varphi$) relation $\Vdash \subseteq \mathbb{P} \times \mathrm{Sent}_{\mathbb{P},M}$ (known as the forcing relation) with the properties:

if G is $\mathbb{P}$-generic over M and $\varphi \in \mathcal{L}_{\mathbb{P},M}$ with k free variables and $\tau_1, \dots, \tau_k \in \mathrm{Name}^{\mathbb{P}} \cap M$ then

TFAE

(i) $M[G] \models \varphi(\mathrm{val}(\tau_1, G), \dots, \mathrm{val}(\tau_k, G))$

(ii) $\exists p \in G \; M \models p \Vdash \varphi(\tau_1, \dots, \tau_k)$.

Lecture XI

Now to the proof of the FT:

Def. $D \subseteq \mathbb{P}$ is <u>dense below p</u> if $\forall q \le p \; \exists r \le q \;\; r \in D$.

ES#3

<u>Lemma</u> If G is $\mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then

(i) If E is dense below p, $q \le p$, then E is dense below q.

(ii) If $\{r;\, E \text{ is dense below } r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G \; \forall r \in E \;\; r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.

<u>Preview</u>:

1. Define $\Vdash$.
   This is going to a double recursion.
   First define it on atomic fmlas by rec. on the rank of the names.
   Then define it for arb. fmlas by recursion on fmla complexity.
2. If done well: absolute.
3. Proofs about it: by induction.

THE FORCING RELATION

Fix $p \in \mathbb{P}$, $\tau_0, \tau_1 \in \mathrm{Name}^{\mathbb{P}}$.

$p \Vdash \tau_0 = \tau_1$ $:\Longleftrightarrow$

$\forall (\pi_0, s_0) \in \tau_0$

$D_{\pi_0,s_0} := \boxed{\{q \leq p \,;\, q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1)\}}$

is dense below p

and

$\forall (\pi_1, s_1) \in \tau_1$

$\{q \leq p \,;\, q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \, (q \leq s_0 \wedge q \Vdash \pi_0 = \pi_1)\}$ is dense below p

In order to reduce writing effort by 50%:

$p \Vdash \tau_0 \subseteq \tau_1$ $:\Longleftrightarrow$ $\forall (\pi_0, s_0) \in \tau_0$ D_{π_0,s_0} is dense below p

$p \Vdash \tau_0 \in \tau_1$ $:\Longleftrightarrow$

$\{q \leq p \,;\, \exists (\pi, s) \in \tau_1 \, (q \leq s \wedge q \Vdash \pi = \tau_0)\}$ is dense below p

$p \Vdash \varphi \wedge \psi$ $:\Longleftrightarrow$ $p \Vdash \varphi$ and $p \Vdash \psi$.

$p \Vdash \neg\varphi$ $:\Longleftrightarrow$ $\forall q \leq p$ $q \not\Vdash \varphi$.

$p \Vdash \exists x \varphi(x)$ $:\Longleftrightarrow$ $\{r \,;\, \exists \sigma \; r \Vdash \varphi(\sigma)\}$ is dense below p.

REMARKS & ROADMAP

1. This is a (double) recursive definition using only absolute concepts, so it is absolute between transitive models.
2. Using our Lemma from page 2, we can see that $\Vdash$ is downwards closed.
3. There is a close relation between this definition and Kripke semantics for intuitionistic logic. $\rightsquigarrow$ L26 : Logic & Computability

ROADMAP FOR THE PROOF OF THE FORCING THEOREM:

1. 1. Prove $\mathrm{val}(\tau_0, G) \subseteq \mathrm{val}(\tau_1, G) \iff \exists p \in G \; p \Vdash \tau_0 \subseteq \tau_1$ by induction on rank of τ_0, τ_1
   - 1a. $[\subseteq, \Rightarrow]$
   - 1b. $[\subseteq, \Leftarrow]$
2. 2. Prove $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G) \iff \exists p \in G \; p \Vdash \tau_0 \in \tau_1$ assuming 1.
   - 2a $[\in, \Rightarrow]$
   - 2b $[\in, \Leftarrow]$

3. Prove that if φ, ψ satisfy FT, then so does $\varphi \wedge \psi$. ← ES#3

4. 4. Prove that if φ satisfies FT, then so does $\neg\varphi$.

5. Prove that if φ satisfies FT, then so does $\exists x \varphi(x)$. ← ES#3

FT : Proof of 4.

Assume $M[G] \models \varphi \iff \exists p \in G \; p \Vdash \varphi$

WTS : $M[G] \models \neg\varphi \iff \exists p \in G \; p \Vdash \neg\varphi$.

ROADMAP FOR THE PROOF OF THE FORCING THEOREM:

1. Prove $\mathrm{val}(\tau_0,G) \subseteq \mathrm{val}(\tau_1,G) \iff \exists p \in G \; p \Vdash \tau_0 \subseteq \tau_1$ by induction on rank of τ_0, τ_1
   - 1a. $[\subseteq, \Rightarrow]$
   - 1b. $[\subseteq, \Leftarrow]$
2. Prove $\mathrm{val}(\tau_0,G) \in \mathrm{val}(\tau_1,G) \iff \exists p \in G \; p \Vdash \tau_0 \in \tau_1$ assuming 1.
   - 2a. $[\in, \Rightarrow]$
   - 2b. $[\in, \Leftarrow]$
3. Prove that if φ, ψ satisfy FT, then so does $\varphi \wedge \psi$.
4. Prove that if φ satisfies FT, then so does $\neg\varphi$.
5. Prove that if φ satisfies FT, then so does $\exists x \varphi(x)$.

$[\neg, \Rightarrow]$ Consider

$$D := \{ p \,;\, p \Vdash \varphi \text{ or } p \Vdash \neg\varphi \}.$$

By definition of $\Vdash \neg\varphi$, this set is dense.

Thus, there is $p \in D \cap G$.

By assumption $M[G] \not\models \varphi$, so $p \not\Vdash \varphi$ by IH.

So $p \Vdash \neg\varphi$.

$[\neg, \Leftarrow]$ Suppose $p \in G$ $p \Vdash \neg\varphi$.

By def., $\forall q \leq p \;\; q \not\Vdash \varphi$. $(*)$

Towards a contradiction, assume $M[G] \models \varphi$.

By IH, find $r \in G$ s.t. $r \Vdash \varphi$. $(**)$

Find $q \leq r, p$ with $q \in G$.

But then

$q \not\Vdash \varphi$ by $(*)$

and

$q \Vdash \varphi$ by $(**)$ + downwards closure of $\Vdash$.

q.e.d.

Fix τ_0, τ_1.

$$D_{\pi_0,s_0} := \{ q \leq p \,;\, q \leq s_0 \rightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1) \}$$

$p \Vdash \tau_0 \subseteq \tau_1 :\iff \forall (\pi_0, s_0) \in \tau_0 \;\; D_{\pi_0,s_0}$ is dense below p

$p \Vdash \tau_0 \in \tau_1 :\iff \{ q \leq p \,;\, \exists (\pi, s) \in \tau_1 \, (q \leq s \wedge q \Vdash \pi = \tau_0) \}$ is dense below p

$p \Vdash \varphi \wedge \psi :\iff p \Vdash \varphi$ and $p \Vdash \psi$

$p \Vdash \neg\varphi :\iff \forall q \leq p \;\; q \not\Vdash \varphi$

$p \Vdash \exists x \varphi(x) :\iff \{ r \,;\, \exists \sigma \; r \Vdash \varphi(\sigma) \}$ is dense below p

FT: Proof of 2.

WTS $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$

$$\Longleftrightarrow$$

$$\exists p \in G \;\; p \Vdash \tau_0 \in \tau_1$$

ROADMAP FOR THE PROOF OF THE FORCING THEOREM:

1. Prove $\mathrm{val}(\tau_0,G) = \mathrm{val}(\tau_1,G) \Longleftrightarrow \exists p \in G \; p \Vdash \tau_0 = \tau_1$ by induction on rank of τ_0, τ_1
   - 1a. $[=, \Rightarrow]$
   - 1b. $[=, \Leftarrow]$
2. Prove $\mathrm{val}(\tau_0,G) \in \mathrm{val}(\tau_1,G) \Longleftrightarrow \exists p \in G \; p \Vdash \tau_0 \in \tau_1$ assuming 1.
   - 2a. $[\in, \Rightarrow]$
   - 2b. $[\in, \Leftarrow]$
3. Prove that if φ, ψ satisfy FT, then so does $\varphi \wedge \psi$.
4. Prove that if φ satisfies FT, then so does $\neg\varphi$.
5. Prove that if φ satisfies FT, then so does $\exists x \varphi(x)$.

$[\in, \Rightarrow]$ Assume $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$,
then there is $(\pi, s) \in \tau_1$ s.t. $s \in G$ & $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$.
By FT for $=$, find $r \in G$ s.t. $r \Vdash \pi = \tau_0$.

Find $p \leq s, r$ s.t. $p \in G$.

Need to show: D is dense below p. So $p \Vdash \tau_0 \in \tau_1$.

[Clearly the case, picking (π, s) as the witness.]

$[\in, \Leftarrow]$ Have $p \Vdash \tau_0 \in \tau_1$, so D is dense below p. and $p \in G$.

So by Lemma (iv), find $q \in G \cap D$.

But then there is $(\pi, s) \in \tau_1$ with $q \leq s$ ($\Rightarrow s \in G$) $\Rightarrow \mathrm{val}(\pi, G) \in \mathrm{val}(\tau_1, G)$

s.t. $q \Vdash \pi = \tau_0$

IH: $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$

$\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$.

q.e.d.

Fix τ_0, τ_1.

$D_{\pi_0, s_0} := \{ q \leq p ;\; q \leq s_0 \rightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1) \}$

$p \Vdash \tau_0 \subseteq \tau_1 :\Longleftrightarrow \forall (\pi_0, s_0) \in \tau_0 \; D_{\pi_0, s_0}$ is dense below p

$p \Vdash \tau_0 \in \tau_1 :\Longleftrightarrow \{ q \leq p ;\; \exists (\pi, s) \in \tau_1 \, (q \leq s \wedge q \Vdash \pi = \tau_0) \}$ is dense below p $=: D$

$p \Vdash \varphi \wedge \psi :\Longleftrightarrow p \Vdash \varphi$ and $p \Vdash \psi$

$p \Vdash \neg\varphi :\Longleftrightarrow \forall q \leq p \;\; q \not\Vdash \varphi$

$p \Vdash \exists x \varphi(x) :\Longleftrightarrow \{ r ;\; \exists \sigma \; r \Vdash \varphi(\sigma) \}$ is dense below p

<u>Lemma</u> If G is $\mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then

(i) If E is dense below p, $q \leq p$, then E is dense below q.

(ii) If $\{ r ;\; E \text{ is dense below } r \}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G \; \forall r \in E \;\; r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.

FT: Proof of 1.

ROADMAP FOR THE PROOF OF THE FORCING THEOREM:

1. Prove $\mathrm{val}(\tau_0,G) \subseteq \mathrm{val}(\tau_1,G) \iff \exists p \in G\ p \Vdash \tau_0 \subseteq \tau_1$ by induction on rank of τ_0, τ_1
   - 1a. $[\subseteq, \Rightarrow]$
   - 1b. $[\subseteq, \Leftarrow]$
2. Prove $\mathrm{val}(\tau_0,G) \in \mathrm{val}(\tau_1,G) \iff \exists p \in G\ p \Vdash \tau_0 \in \tau_1$ assuming 1.
   - 2a $[\in, \Rightarrow]$
   - 2b $[\in, \Leftarrow]$
3. Prove that if φ, ψ satisfy FT, then so does $\varphi \wedge \psi$.
4. Prove that if φ satisfies FT, then so does $\neg\varphi$.
5. Prove that if φ satisfies FT, then so does $\exists x \varphi(x)$.

$[\subseteq, \Leftarrow]$ Assume $p \in G$ s.t. $p \Vdash \tau_0 \subseteq \tau_1$.

WTS $\mathrm{val}(\tau_0,G) \subseteq \mathrm{val}(\tau_1,G)$.

Fix $x = \mathrm{val}(\pi_0,G) \in \mathrm{val}(\tau_0,G)$ with $(\pi_0,s_0) \in \tau_0$ and $s_0 \in G$.

By assumption, D_{π_0,s_0} is dense below p. Thus find $q \leq s_0, p$ s.t. $q \in G$. By Lemma (i), D_{π_0,s_0} is dense below q.

By Lemma (iv), find $r \leq q$ s.t. $r \in G \cap D_{\pi_0,s_0}$.

$r \leq s_0$

Thus (because $r \leq s_0$), by def. of D_{π_0,s_0}, we find $(\pi_1,s_1) \in \tau_1$ s.t. $r \leq s_1$ and $r \Vdash \pi_0 = \pi_1$.

$\Rightarrow s_1 \in G$

By IH, $\mathrm{val}(\pi_0,G) = \mathrm{val}(\pi_1,G)$.

Thus $\mathrm{val}(\pi_0,G) \in \mathrm{val}(\tau_1,G)$.

q.e.d. ($\Leftarrow$)

This completes 90% of the roadmap. Only part of the proof missing is $[\subseteq, \Rightarrow]$ which will be completed in Lecture XIII.

<u>Lemma</u>: If G is $\mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then

(i) If E is dense below p, $q \leq p$, then E is dense below q.

(ii) If $\{r;\ E$ is dense below $r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G\ \forall r \in E\ r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.

Fix τ_0, τ_1.

$D_{\pi_0,s_0} := \{q \leq p;\ q \leq s_0 \rightarrow \exists (\pi_1,s_1) \in \tau_1\ (q \leq s_1 \wedge q \Vdash \pi_0 = \pi_1)\}$

$p \Vdash \tau_0 \subseteq \tau_1 :\iff \forall (\pi_0,s_0) \in \tau_0\ D_{\pi_0,s_0}$ is dense below p

$p \Vdash \tau_0 \in \tau_1 :\iff \{q \leq p;\ \exists (\pi,s) \in \tau_1\ (q \leq s \wedge q \Vdash \pi = \tau_0)\}$ is dense below p

$p \Vdash \varphi \wedge \psi :\iff p \Vdash \varphi$ and $p \Vdash \psi$

$p \Vdash \neg\varphi :\iff \forall q \leq p\ q \not\Vdash \varphi$

$p \Vdash \exists x \varphi(x) :\iff \{r;\ \exists \sigma\ r \Vdash \varphi(\sigma)\}$ is dense below p