XI ELEVENTH LECTURE
Forcing & the Continuum Hypothesis

Saturday 15 November 2025

What we have done so far:

M ctm ; $\mathbb{P} \in M$; $F \subseteq \mathbb{P}$

$\rightsquigarrow \mathrm{Name}^{\mathbb{P}}$

$\rightsquigarrow \mathrm{val}(\tau, F)$ for $\tau \in \mathrm{Name}^{\mathbb{P}}$

$\rightsquigarrow M[F] = \{\mathrm{val}(\tau, F) ; \tau \in \mathrm{Name}^{\mathbb{P}} \cap M\}$

$M[F]$ is a ctble transitive model; $M \subseteq M[F]$

$F \in M[F]$

AXIOMS OF ZF

STRUCTURAL AXIOMS

- EXTENSIONALITY $\forall x \forall y (\forall w (w \in x \leftrightarrow w \in y) \longrightarrow x = y)$ ✓ DONE
- FOUNDATION $\forall x (\exists v (v \in x) \longrightarrow \exists w (w \in x \wedge \forall u \neg (u \in w \wedge u \in x)))$ ✓ DONE
- INFINITY $\exists i \exists e (\forall v \neg v \in e \wedge e \in i) \wedge \forall x (x \in i \longrightarrow \exists s (s \in i \wedge \forall w (w \in s \leftrightarrow w \in x \vee w = x)))$ ✓ DONE

FUNCTIONAL AXIOMS

- PAIRING $\forall x \forall y \exists p \; \forall w (w \in p \leftrightarrow w = x \vee w = y)$ ✓ DONE
- UNION $\forall x \exists u \; \forall w (w \in u \leftrightarrow \exists z (z \in x \wedge w \in z))$ ✓ DONE
- POWERSET $\forall x \exists p \; \forall w (w \in p \leftrightarrow \forall v (v \in w \rightarrow v \in x))$
- SEPARATION φ $\forall \vec{p} \forall x \exists s \; \forall w (w \in s \leftrightarrow w \in x \wedge \varphi(w, \vec{p}))$
- REPLACEMENT φ $\forall \vec{p} \; \forall x \forall y \forall z (\varphi(x, y, \vec{p}) \wedge \varphi(x, z, \vec{p}) \rightarrow y = z) \longrightarrow \forall x \exists r \; \forall w (w \in r \leftrightarrow \exists y (y \in x \wedge \varphi(y, w, \vec{p})))$

still to be done

Lecture X:

Let's look at Separation:

Fix $x = \mathrm{val}(\sigma, F)$, want

$$A_\varphi = \{ z \in x ;\ \varphi(z, \cancel{\vec{x}}) \}$$

ignore parameters for now

<u>Idea for a name</u>

$$\tau_\varphi := \{ (\tau', p) ;\ \text{there is } q \text{ s.t. } (\tau', q) \in \sigma,\ p \leq q \text{ and } \text{“}p \text{ makes } \varphi \text{ true for } \tau'\text{”}\ [\text{“}p \text{ forces } \varphi \text{ for } \tau'\text{”}] \}$$

With this:

$$\mathrm{val}(\tau', F) \in \mathrm{val}(\tau_\varphi, F)$$
$$\Longleftrightarrow$$
$$\mathrm{val}(\tau', F) \in \mathrm{val}(\sigma, F) \text{ and } \text{“}p \text{ forces } \varphi \text{ for } \tau'\text{”}.$$
$$\Longleftrightarrow$$
$$M[F] \models \varphi(\mathrm{val}(\tau', F))$$

The Forcing Language

Let $\mathcal{L}_\in$ be the language of set theory, fix M ctm, $\mathbb{P} \in M$.

Add all elements of $\mathrm{Name}^{\mathbb{P}} \cap M$ as constant symbols to obtain

$$\mathcal{L}_{\mathbb{P}, M}$$ the forcing language

We write $\mathrm{Sent}_{\mathbb{P}, M}$ for the set of sentences in the forcing language.

THE FORCING THEOREM

Let M ctm, $\mathbb{P} \in M$.
Then there is an absolute definable and downwards closed (i.e., if $p \Vdash \varphi$, $q \leq p \Rightarrow q \Vdash \varphi$) relation $\Vdash \subseteq \mathbb{P} \times \mathrm{Sent}_{\mathbb{P},M}$ (known as the forcing relation) with the properties:

if G is $\mathbb{P}$-generic over M and $\varphi \in \mathcal{L}_{\mathbb{P},M}$ with k free variables and $\tau_1, \ldots, \tau_k \in \mathrm{Name}^{\mathbb{P}} \cap M$ then TFAE

(i) $M[G] \models \varphi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_k, G))$

(ii) $\exists p \in G \; M \models p \Vdash \varphi(\tau_1, \ldots, \tau_k)$.

We say "p forces φ" for $p \Vdash \varphi$.

Remark. The equivalence is what is usually called "controlling truth in $M[G]$ from M".

We'll prove the FT in Lecture XII.

Separation in M[G]

ASSUME FT

Have $x = \mathrm{val}(\sigma, G)$.

Want

$$A\varphi = \{ z \in x ;\ M[G] \models \varphi(z, \cancel{\vec{x}}) \}$$

OMIT FOR READABILITY!

Let's look at Separation:

Fix $x = \mathrm{val}(\sigma, F)$, want

$$A\varphi = \{ z \in x ;\ \varphi(z, \cancel{\vec{x}}) \}$$

ignore parameters for now

Idea for a name

$$\tau\varphi := \{ (\tau', p) ;\ \text{there is } q \text{ s.t. } (\tau', q) \in \sigma,\ p \leq q \text{ and "} p \text{ makes } \varphi \text{ true for } \tau' \text{"} \ [\text{"} p \text{ forces } \varphi \text{ for } \tau' \text{"}] \}$$

With this:

$\mathrm{val}(\tau', F) \in \mathrm{val}(\tau\varphi, F)$

$\Longleftrightarrow$

$\mathrm{val}(\tau', F) \in \mathrm{val}(\sigma, F)$ and "p forces φ for τ'".

$\Longleftrightarrow$

$M[F] \models \varphi(\mathrm{val}(\tau', F))$

$$\tau\varphi = \{ (\tau', p) ;\ \exists q\ (\tau', q) \in \sigma \wedge p \leq q \wedge p \Vdash \varphi(\tau') \}$$

<u>Claim</u> $\mathrm{val}(\tau\varphi, G) = A\varphi$.

<u>Pf.</u> "$\supseteq$": Suppose $z = \mathrm{val}(\tau, G)$

$z \in A\varphi \Longrightarrow z \in x \wedge M[G] \models \varphi(z)$

$\Longrightarrow \mathrm{val}(\tau, G) \in \mathrm{val}(\sigma, G) \wedge M[G] \models \varphi(z)$

$\Longrightarrow \exists q \in G\ (\tau, q) \in \sigma \wedge M[G] \models \varphi(z)$

$\overset{FT}{\Longrightarrow} \underline{\exists q \in G}\ (\tau, q) \in \sigma \wedge \underline{\exists p \in G}\ p \Vdash \varphi(\tau)$

$\underset{\substack{G \text{ filter} \\ + \Vdash \text{ is downwards closed}}}{\Longrightarrow} \exists r \in G\ r \leq q \wedge r \Vdash \varphi(\tau)$.

$\Longrightarrow (\tau, r) \in \tau\varphi \wedge r \in G$

$\Longrightarrow \mathrm{val}(\tau, G) \in \mathrm{val}(\tau\varphi, G)$.

"$\subseteq$" $\quad \mathrm{val}(\tau_\varphi, G) \subseteq A_\varphi$.

$\mathrm{val}(\tau, G) = z \in \mathrm{val}(\tau_\varphi, G)$

$\Longrightarrow \exists p \in G \ (\tau, p) \in \tau_\varphi$.

$\Longrightarrow \exists p \in G \ \exists q \ \underline{p \leq q} \wedge (\tau, q) \in \sigma \wedge p \Vdash \varphi(\tau)$

$\Rightarrow q \in G$

FT

$\Longrightarrow$

$\mathrm{val}(\tau, G) \in \mathrm{val}(\sigma, G) \ \wedge \ M[G] \models \varphi(z)$

$z \in x$

$\Longrightarrow z \in A_\varphi$.

q.e.d.

[modulo FT].

Powerset in M[G]

Assume FT; fix $x = \mathrm{val}(\sigma, G)$.

<u>Notation</u> $\mathrm{dom}(\sigma) := \{\tau;\ \exists p\ (\tau, p) \in \sigma\}$

Define $\pi := \{(\tau, 1\!\!1);\ \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\}$

<u>Claim</u> If $y \in M[G]$, $y \subseteq x$, then

$$y \in \mathrm{val}(\pi, G).$$

With Separation already in place, we can now separate the powerset of x from $\mathrm{val}(\pi, G)$.

<u>Proof idea</u> of <u>claim</u>

Prove that if $y = \mathrm{val}(\mu, G) \subseteq x$ and $\mu^* := \{(\tau, p);\ \tau \in \mathrm{dom}(\sigma) \wedge p \Vdash \tau \in \mu\}$ then

$$\boxed{\mathrm{val}(\mu, G) = \mathrm{val}(\mu^*, G).} \quad \text{ES\#3}$$

That shows the claim, since clearly $(\mu^*, 1\!\!1) \in \pi$.

q.e.d.

Replacement in M[G]

REMARK
Doing [XI], I claimed that we do not need functionality. But we do!! (*)

By Separation, it's enough to show the following:

If $x = \mathrm{val}(\sigma, G)$ and φ is a functional formula there there is $R \in M[G]$ s.t.

$$(*) \quad M[G] \models \forall y \in x \, \exists z \in R \, \varphi(y, z, \cancel{\vec{v}})$$

omit for readability

Pf. In M, find α s.t. $\mathrm{dom}(\sigma) \subseteq V_\alpha$.
and write

$$\psi(p, \pi) := \underbrace{\exists \mu \; p \Vdash \varphi(\pi, \mu)}_{\text{by the FT, this is just a regular sentence in } \mathcal{L}_\in.}$$

Still working in M, use LRT to find $\vartheta > \alpha$ s.t. ψ is absolute between $\underset{V_\vartheta^M}{\overset{\shortparallel}{V_\vartheta}}$ and $\underset{M}{\overset{\shortparallel}{V}}$

Define $\varrho := \{(\mu, \mathbb{1}) ; \mu \in V_\vartheta\}$ and $R := \mathrm{val}(\varrho, G)$.

Prove (*): Fix $y = \mathrm{val}(\pi, G) \in x$ and
$z = \mathrm{val}(\mu, G)$ s.t. $M[G] \models \varphi(y, z)$.

By FT, find $p \in G$ s.t. $M \models p \Vdash \varphi(\pi, \mu)$, i.e. $M \models \psi(p, \pi)$.
By absoluteness, $V_\vartheta \models \psi(p, \pi)$, so ex. $\mu^* \in V_\vartheta$ s.t. $p \Vdash \varphi(\pi, \mu^*)$. By functionality of φ (*), we have $\mathrm{val}(\mu^*, G) = \mathrm{val}(\mu, G)$.

Thus $(\mu^*, \mathbb{1}) \in \varrho$, so $\mathrm{val}(\mu^*, G) = \mathrm{val}(\mu, G) \in \mathrm{val}(\varrho, G) = R$.
q.e.d.

Remark on AC

If $\mathrm{val}(\sigma, G) \subseteq \{\mathrm{val}(\tau, G) ; \tau \in \mathrm{dom}(\sigma)\}$

But by AC in M, find injection from

$\mathrm{dom}(\sigma) \longrightarrow \alpha \in \mathrm{Ord}$,

so $\mathrm{val}(\sigma, G)$ is wellordered as a subset of a wellordered set.

Summary

If M ctm of ZFC, then

$M[G]$ is a ctm of ZFC.

[The generic model theorem.]

By the trick mentioned in lecture X, this works with finite fragments $T, T^* \subseteq \mathrm{ZFC}$.

Now to the proof of the FT:

Def. $D \subseteq \mathbb{P}$ is dense below p
if $\forall q \leq p \, \exists r \leq q \; r \in D$.

ES#3

Lemma If G is $\mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then

(i) If E is dense below p, $q \leq p$, then E is dense below q.

(ii) If $\{r ; E \text{ is dense below } r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G \, \forall r \in E \; r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.