X

Tenth Lecture of
FORCING & THE CONTINUUM HYPOTHESIS
13 November 2025

! Lecture $\underline{\text{XI}}$ will be on Saturday
15 November 2025 !!!

! <u>FOUNDATIONS STUDENTS</u>

Progress interviews
this afternoon
$14^{30} - 16^{00}$
Churchill College
43B South Court

RECAP

$(\mathbb{P}, \leq, \mathbb{1})$ forcing

$F \subseteq \mathbb{P}$ filter — all elts compatible, upwards closed

$\mathcal{D}$ collection of dense sets

$\rightsquigarrow$ $\mathcal{D}$-generic filters

Lecture $\underline{\text{IX}}$

<u>Theorem</u> If $\mathcal{D}$ is countable, then there is a $\mathcal{D}$-generic filter.

<u>Proof.</u> $\mathcal{D} = \{D_i \,;\, i \in \mathbb{N}\}$. Let $p_0 \in \mathbb{P}$ be arbitrary. Pick recursively

p_{i+1} s.t. $p_{i+1} \leq p_i$ and $p_{i+1} \in D_i$
(possible by density of D_i)

Then $\mathcal{B} := \{p_i \,;\, i \in \mathbb{N}\}$ is a filter base.
Then the filter generated by $\mathcal{B}$ is $\mathcal{D}$-generic.
q.e.d.

RECAP Main example $\mathbb{P} = \mathrm{Fn}(X, Y)$
finite partial fns from X to Y.
If $F \subseteq \mathbb{P}$ is sufficiently generic, then
$\bigcup F : X \longrightarrow Y$ is a surjection.

Convention I shall say that M is a ctm (countable transitive model) if it's a countable transitive model of ZFC or of $T \subseteq$ ZFC finite where T is the currently relevant fragment of ZFC.

Suppose I want to show for some trs models $M \subseteq N$ that $N \models \varphi$ for some $\varphi \in$ ZFC. Suppose that I know that if $M \models$ ZFC, then $N \models \varphi$. By compactness, find $T_\varphi \subseteq$ ZFC finite s.t. $M \models T_\varphi$ suffices.

Similarly, if $T \subseteq$ ZFC finite, then
$T^* := \bigcup_{\varphi \in T} T_\varphi$ is enough:
if $M \models T^*$, then $N \models T$.

Def. If M is a ctm and $(\mathbb{P}, \leq, \mathbb{1}) \in M$, then $\mathcal{D}_M := \{D \subseteq \mathbb{P} ; D \in M\}$ is countable. Therefore, by Thm, there is a $\mathcal{D}$-generic filter. Call this a $\mathbb{P}$-generic filter over M.

Explore the notion of

$\mathbb{P}$-generic over M:

(1) If M ctm, G is $\mathbb{P}$-generic over M (which exists by Thm), then we can apply our knowledge from example.

E.g., if $X, Y \in M$, then

$$\mathbb{P} = \mathrm{Fn}(X, Y) \in M,$$

so if G is $\mathbb{P}$-generic,

$$\bigcup G : X \longrightarrow Y$$

is surjection from X to Y.

[Check that all of the sets needed for that are in $\mathcal{D}_M$.]

(2) Remember $N_f = \{p \in P;\ \exists x\ p(x) \neq f(x)\}$

if $f \in M$, $N_f \in \mathcal{D}_M$, so $\bigcup G \neq f$.

Thus $\bigcup G \notin M$.

(3) Example: $X := \mathbb{N}$

$Y := \alpha$ where

$M \models \alpha$ is the least uncttble ordinal

Then $M \models$ there is no surj. from X to Y, but $\bigcup G$ is such a surjection.

FURTHER RECAP

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NAMES Fix $\mathbb{P}$ and define the $\mathbb{P}$-names by recursion:

$$\mathrm{Name}_0^{\mathbb{P}} := \emptyset$$
$$\mathrm{Name}_{\alpha+1}^{\mathbb{P}} := \{\tau \,;\, \tau \subseteq \mathrm{Name}_\alpha^{\mathbb{P}} \times \mathbb{P}\} \cup \mathrm{Name}_\alpha^{\mathbb{P}}$$
$$\mathrm{Name}_\lambda^{\mathbb{P}} := \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}}$$
$$\mathrm{Name}^{\mathbb{P}} := \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Name}_\alpha^{\mathbb{P}}$$

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THE VALUE OF A NAME

Fix $F \subseteq \mathbb{P}$, $\tau \in \mathrm{Name}^{\mathbb{P}}$ and define recursively

$$\mathrm{val}(\tau, F) := \{\mathrm{val}(\sigma, F) \,;\, \exists p \in F \; (\sigma, p) \in \tau\}.$$

Important remark

Note that $\mathrm{Name}^{\mathbb{P}}$ and $\mathrm{val}(\tau, F)$ are recursive definitions, so absolute for trc models (containing τ, F).

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The extension of a model M by F

is

$$M[F] := \{\mathrm{val}(\tau, F) \,;\, \tau \in \mathrm{Name}^{\mathbb{P}} \cap M\}.$$

Goals: Find conditions on F s.t.

1. $M[F]$ is a trc model
2. $M \subseteq M[F]$
3. $F \in M[F]$
4. $M[F] \models \mathrm{ZFC}$.

Lecture $\overline{X}$ (for 1–3)

This is going to be the hardest result of this lecture course and will take several lectures.

THE (GENERIC) EXTENSION

Let M be a ctm, $\mathbb{P} \in M$, and $F \subseteq \mathbb{P}$. Then
$$M[F] := \{ val(\tau, F) ; \tau \in Name^{\mathbb{P}} \cap M \}$$

Observations

(1) $Name^{\mathbb{P}} \cap M$ is countable, thus $M[F]$ is countable.

(2) By definition, it's transitive.

(3) Therefore (by Lecture II, p5) $M[F] \models$ Extensionality + Foundation.

Goals :

$$\boxed{\begin{array}{c} M \subseteq M[F] \\ F \in M[F] \end{array}}$$

Note that this implies (due to absoluteness of $Name^{\mathbb{P}}$ & $val(\tau, F)$) that

if N is a tm model with $M \subseteq N$ & $F \in N$, then
$$M[F] \subseteq N.$$

CANONICAL NAMES

$M \subseteq M[F]$
$F \in M[F]$

If $x \in M$, define

$$\check{x} := \{(\check{y}, \mathbb{1});\ y \in x\}$$

pronounced as "x check"

Note: This is a recursive definition by $\in$-recursion.

the canonical name for x.

Define $\Gamma := \{(\check{p}, p);\ p \in \mathbb{P}\}$

the canonical name for the generic object.

Lemma 1 $\mathrm{val}(\check{x}, F) = x$ if $\mathbb{1} \in F$.

[obvious by $\in$-induction.]

Corollary $M \subseteq M[F]$ if $\mathbb{1} \in F$.

Lemma 2 $\mathrm{val}(\Gamma, F) = F$ if $\mathbb{1} \in F$.

$$\Big[\ \mathrm{val}(\Gamma, F) = \{\mathrm{val}(\check{p}, F);\ (\check{p}, p) \in \Gamma \ \&\ p \in F\}$$
$$= \{\mathrm{val}(\check{p}, F);\ p \in F\}$$
$$\overset{L1}{=} \{p;\ p \in F\} = F.\ \Big]$$

Corollary $F \in M[F]$.

Next goal: $M[F] \models ZFC$!!

Reminder

AXIOMS OF ZF

STRUCTURAL AXIOMS

All of the structural axioms are fine: Ext/Found follow from transitivity; infinity follows from the fact that $\omega \in M$ and $M \subseteq M[F]$.

EXTENSIONALITY

$$\forall x \forall y \left(\forall w (w \in x \leftrightarrow w \in y) \longrightarrow x = y \right)$$

FOUNDATION

$$\forall x \left(\exists v (v \in x) \longrightarrow \exists m (m \in x \wedge \forall w \neg (w \in m \wedge w \in x)) \right)$$

INFINITY

$$\exists i \, \exists e (\forall v \neg v \in e \wedge e \in i) \wedge \forall x (x \in i \longrightarrow \exists s (s \in i \wedge \forall w (w \in s \leftrightarrow w \in x \vee w = x)))$$

FUNCTIONAL AXIOMS

PAIRING

$$\forall x \forall y \exists p \quad \forall w \left(w \in p \leftrightarrow w = x \vee w = y \right)$$

UNION

$$\forall x \exists u \quad \forall w \left(w \in u \longleftrightarrow \exists z (z \in x \wedge w \in z) \right)$$

POWERSET

$$\forall x \exists p \quad \forall w \left(w \in p \leftrightarrow \forall v (v \in w \rightarrow v \in x) \right)$$

SEPARATION φ

$$\forall \vec{p} \, \forall x \exists s \quad \forall w \left(w \in s \leftrightarrow w \in x \wedge \varphi(w, \vec{p}) \right)$$

REPLACEMENT φ

$$\forall \vec{p} \quad \forall x \forall y \forall z \left(\varphi(x,y,\vec{p}) \wedge \varphi(x,z,\vec{p}) \rightarrow y = z \right) \longrightarrow \forall x \exists r \; \forall w \left(w \in r \leftrightarrow \exists y (y \in x \wedge \varphi(y, w, \vec{p}) \right)$$

On to the functional axioms !!

Start with PAIRING

Given $\sigma, \tau \in \mathrm{Name}^{\mathbb{P}}$, find a name for

$$\{\mathrm{val}(\sigma, F), \mathrm{val}(\tau, F)\} = \pi$$

Idea : $\mathrm{up}(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$

unordered pair

This is the obvious name for the pair.

Obviously :

$$\mathrm{val}(\mathrm{up}(\sigma, \tau), F) = \pi.$$

if $\mathbb{1} \in F$.

Then UNION

Given $\sigma \in \mathrm{Name}^{\mathbb{P}}$, find a name for

$$\bigcup \mathrm{val}(\sigma, F).$$

Define $u_\sigma := \{(\sigma', r) ; \exists \tau, p, q$ s.t. $(\tau, p) \in \sigma, (\sigma', q) \in \tau, r \leq p, q\}$

Show : if F is a filter, then

$$\mathrm{val}(u_\sigma, F) = \bigcup \mathrm{val}(\sigma, F).$$

[Check this !]

This is going to be much harder for

Separation & Replacement.

Let's look at Separation:

Fix $x = \mathrm{val}(\sigma, F)$, want

$$A_\varphi = \{ z \in x ; \varphi(z, \cancel{\vec{p}}) \}$$

ignore parameters for now

<u>Idea for a name</u>

$$\tau_\varphi := \{ (\tau', p) ; \text{ there is } q \text{ s.t. } (\tau', q) \in \sigma, \; p \leq q \text{ and } \text{"}p \text{ makes } \varphi \text{ true for } \tau'\text{"} \; [\text{"}p \text{ forces } \varphi \text{ for } \tau'\text{"}] \}$$

With this:

$\mathrm{val}(\tau', F) \in \mathrm{val}(\tau_\varphi, F)$

$\iff$

$\mathrm{val}(\tau', F) \in \mathrm{val}(\sigma, F)$ and "p forces φ for τ'".

$\iff$

$M[F] \models \varphi(\mathrm{val}(\tau', F))$

If we can
(a) define a notion of "forces"
(b) prove this equivalence
(c) have the notion of "forces" definable in M

this solves the problem.