VIII Eighth Lecture of FORCING & THE CONTINUUM HYPOTHESIS

4 NOVEMBER 2025

Lecture VI

THEOREM Lectures VI + VII

If $M \models ZFC$, then there is $L \subseteq M$ transitive containing all ordinals s.t.

1. $L \models ZFC$
2. $L \models GCH$
3. L is minimal in the sense that if $N \subseteq M$ transitive with all ordinals and $N \models ZFC$, then $L \subseteq N$.

We are done with 1. + 3.
Now doing 2. : $L \models CH$

Lecture VII :

THEOREM GÖDEL'S CONDENSATION LEMMA

There is a sentence σ (the condensation sentence) s.t. for all trs X

if $X \models \sigma$, then there is α s.t. $X = L_\alpha$.

Proof. Take the s.s. T (finite) as in Proposition and write $\sigma := \bigwedge T \wedge V = L \wedge$ there is no largest ordinal

If X trs, $X \cap \mathrm{Ord} = \lambda$ for some λ.
So if $X \models \sigma \implies \lambda$ is a limit ordinal.
By Prop $X = \bigcup_{\alpha < \lambda} L_\alpha = L_\lambda$ (since λ limit). q.e.d.

Lemma If $\alpha \geq \omega$, then $|L_\alpha| = |\alpha|$.

Proof. By induction on α. Clearly, L_ω is countable

$\alpha \longmapsto \alpha+1$ Assume $|L_\alpha| = |\alpha|$.

$$|\alpha+1| \leq |L_{\alpha+1}| \leq \aleph_0 \cdot |L_\alpha^{<\omega}| = \aleph_0 \cdot |L_\alpha| = \aleph_0 \cdot |\alpha| = |\alpha| = |\alpha+1|$$

λ limit Assume for all $\alpha < \lambda$, $|L_\alpha| = |\alpha| \leq |\lambda|$.

$$|\lambda| \leq |L_\lambda| = \Big|\bigcup_{\alpha<\lambda} L_\alpha\Big| \leq |\lambda| \cdot |\lambda| = |\lambda|.$$

Hessenberg's theorem

q.e.d.

Remark Note how different this is in the V-hierarchy:

$$|V_{\omega+n}| = \beth_n.$$

Main idea of the proof of CH

Note that if $X \in L_\alpha$, then $X \subseteq L_\alpha$, so

$$|X| \leq |L_\alpha| = |\alpha|.$$

So proving CH is the same as identifying where the L-power set of $\mathbb{N}$

$$\wp(\mathbb{N}) \cap L$$

lies in the L-hierarchy.

In particular, if $\wp(\mathbb{N}) \cap L \subseteq L_{\omega_1}$, that implies $L \models CH$.

<u>Claim</u> For every $x \subseteq \mathbb{N}$, $x \in L$, there is some $\alpha < \omega_1$ s.t. $x \in L_\alpha$.

<u>Proof.</u> W.l.o.g. work in L, so all defined terms such as ω_1, $\mathcal{P}(\mathbb{N})$ etc. refer to the objects in L.

[In order to prove Claim, I need to find ctble trs. A s.t. $x \in A$ and $A \models \sigma$.]

Define $B_x := \mathbb{N} \cup \{x\}$. Note that this is a transitive set.

Find ϑ large enough by LRT s.t.

$x \in L_\vartheta$ and σ is absolute between L_ϑ and L.

Because $L \models \sigma$, we get $L_\vartheta \models \sigma$.

Using our Lösko technique, build

$$B_x \subseteq M \subseteq L_\vartheta \quad \text{s.t.}$$

σ is absolute between M and L_ϑ, so $M \models \sigma$. and M is countable

Form the Mostowski collapse $N \cong M$.

[N is transitive
  N is countable
  $N \models \sigma$

Since the Mostowski collapse is the identity on transitive sets, $x \in N$.

Since $N \models \sigma$ is transitive, by Condensation Lemma, there is α s.t. $N = L_\alpha$.

Since N is countable, $\alpha < \omega_1$.

CLAIM.

q.e.d.

Remark

Almost literally the same proof shows

$L \models GCH$:

Claim: If $x \subseteq \kappa$, then there is $|\alpha| \leq \kappa$ s.t. $x \in L_\alpha$.

Proof just has

$$B_x := \kappa \cup \{x\}$$

and M & N are size κ.

FORCING

[Technique of Outer Models]

Paul Cohen

American mathematician

Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

Theorem (COHEN 1962)

If M is a countable transitive model of ZFC, then there is a ctble transitive $N \supseteq M$ s.t. $N \models ZFC + \neg CH$.

Problem

We have seen that

$ZFC + Cons(ZFC) \nvdash$ there are trs models of ZFC.

So, the above theorem may not be useful for proving

$(*)\quad Cons(ZFC) \longrightarrow Cons(ZFC + \neg CH)$

Therefore, we are going to prove a stronger variant of the above theorem:

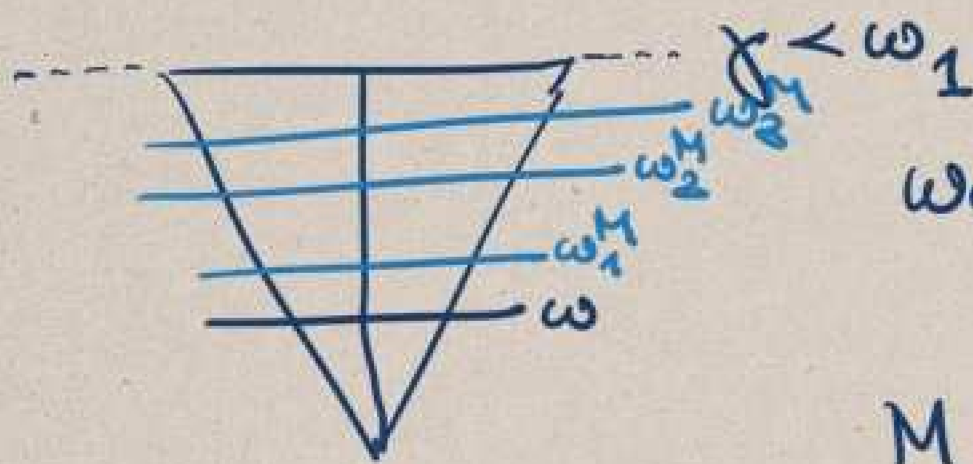
Theorem (**) If $T \subseteq ZFC$ is finite, then there is $T^* \subseteq ZFC$ finite such that:

whenever $M \models T^*$ ctble trs, then there is $N \supseteq M$ ctble trs s.t. $N \models T + \neg CH$.

Musings about how to prove (**).

[To ease the conceptual load, think rather of $M \models ZFC \rightsquigarrow N \supseteq M$, $N \models ZFC + \neg CH$.]

$M \models ZFC$ countable trs



We find in M α, β s.t.

$$M \models \alpha = \aleph_1 \wedge \beta = \aleph_2.$$

Let's assume $M \models CH$. We find in M $P \subseteq \wp(\mathbb{N})$ s.t. $M \models P$ is the powerset of $\mathbb{N}$ and some $f \in M$ s.t. f is a bijection between α and P.

Find OUTSIDE OF M some $g: \beta \longrightarrow \wp(\mathbb{N})$ injection.

Clearly $g \notin M$.

Take $X :=$ the transitive closure of $\underbrace{M \cup \{g\}}_{\text{ctble}}$ and form a (ctble) transitive model of ZFC containing X, $N \supseteq M$.

Then in N, we have $|\wp(\mathbb{N})| \geq |\beta|$.

But: WE HAVE NO CONTROL OVER α and β AND WHETHER $N \models \alpha = \aleph_1 \wedge \beta = \aleph_2$.

Let's prove

$$(**) \Longrightarrow (*) .$$

[Assume ZFC is consistent.
And towards contradiction that
$ZFC + \neg CH$ is not.
So find $T \subseteq ZFC$ finite s.t. $T + \neg CH$
is inconsistent.
Take T^* as in the statement of $(**)$.
The Löwenheim-Skolem-Mostowski technique gives us a
ctble trs model $M \models T^*$.
Now $(**)$ says:
there is (ctble trs) $N \supseteq M$ s.t.
$N \models T + \neg CH$.

Contradiction!

q.e.d.]