Forcing & the Continuum Hypothesis

Lecture VII

30 October 2025

TODAY 2pm BEVIN ROOM @ Churchill College : Foundations PhD Advice

We are in the middle of the proof of Gödel's 1938 theorem:

THEOREM Lectures VI + VII

If $M \models ZFC$, then there is $L \subseteq M$ transitive containing all ordinals s.t.

1. $L \models ZFC$
2. $L \models GCH$
3. L is minimal in the sense that if $N \subseteq M$ transitive with all ordinals and $N \models ZFC$, then $L \subseteq N$.

We defined L by

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

The definition of L is absolute for s.s. T; this gives MINIMALITY (3.)

Now working on 1.

Lecture VI: Axioms of ZF

Axioms of ZF

STRUCTURAL AXIOMS

EXTENSIONALITY
$\forall x \forall y (\forall w (w \in x \leftrightarrow w \in y) \rightarrow x = y)$

FOUNDATION
$\forall x (\exists v (v \in x) \rightarrow \exists w (w \in x \wedge \forall u \neg (u \in x \wedge u \in w)))$

INFINITY
$\exists I \exists e (\forall v \neg v \in e \wedge e \in I) \wedge \forall x (x \in I \rightarrow \exists s (s \in I \wedge \forall w (w \in s \leftrightarrow w \in x \vee w = x)))$

FUNCTIONAL AXIOMS

PAIRING
$\forall x \forall y \exists p \; \forall w (w \in p \leftrightarrow w = x \vee w = y)$

UNION
$\forall x \exists u \; \forall w (w \in u \leftrightarrow \exists z (z \in x \wedge w \in z))$

POWERSET
$\forall x \exists p \; \forall w (w \in p \leftrightarrow \forall v (v \in w \rightarrow v \in x))$

SEPARATION φ
$\forall \vec{p} \forall x \exists s \; \forall w (w \in s \leftrightarrow w \in x \wedge \varphi(w, \vec{p}))$

REPLACEMENT φ
$\forall \vec{p} \; \forall x \forall y \forall z (\varphi(x, y, \vec{p}) \wedge \varphi(x, z, \vec{p}) \rightarrow y = z) \rightarrow \forall x \exists r \; \forall w (w \in r \leftrightarrow \exists y (y \in x \wedge \varphi(y, w, \vec{p})))$

Lecture VI

THE STRUCTURAL AXIOMS

Extensionality, Foundation, Infinity

Lecture II:

The extent of absoluteness for transitive models

Lemma: If M is a transitive set, then $(M, \in) \models$ Extensionality + Foundation.

Since $x = \omega$ is absolute and ω satisfies the conditions of the Axiom of Infinity, we have that for all $\alpha > \omega$, $L_\alpha \models$ Infinity.

Now: the Functional Axioms
Pairing, Union, Powerset, Separation, Replacement

PAIRING & UNION

Pairing $\forall x \forall y \exists p \forall z (z \in p \leftrightarrow z = x \vee z = y)$
Union $\forall x \exists u \forall z (z \in u \leftrightarrow \exists y \in z (z \in y))$

The operations $x, y \longmapsto \{x, y\}$, $x \longmapsto \bigcup x$ are absolute operations for a.c.s.T.

So, I only need to prove:

$\forall x, y \in L (\{x, y\} \in L$ and $\bigcup x \in L)$

So, we only need to show $\{x, y\}, \bigcup x \in \mathcal{D}(L_\alpha)$ for some α $\longrightarrow$ Lecture VII

Lecture VI

Find $\{x, y\}$ in L:

Assume $x, y \in L$, so take α s.t. $x, y \in L_\alpha$

Consider:

$\varphi(z, x, y) := z = x \vee z = y$

Form $\mathcal{D}(\varphi, (x, y), L_\alpha) =$

$\{z \in L_\alpha ; L_\alpha \models \varphi(z, x, y)\} =$

$\{z \in L_\alpha ; L_\alpha \models z = x \vee z = y\} =$

$\{z \in L_\alpha ; z = x \vee z = y\} = \{x, y\}.$

So $\{x, y\} \in \mathcal{D}(L_\alpha) \subseteq L$. Same proof for Union.

Powerset

$$\forall x\, \exists p\, \forall z\, (z \in p \longleftrightarrow z \subseteq x)$$

What is a candidate for the power set of x in L?
Call it p.

Clearly (by abs. of $z \subseteq x$), $p \subseteq \mathcal{P}(x)$.
Also clearly $p \subseteq L$. POWER SET

So, a candidate could be $p := \mathcal{P}(x) \cap L$.

Obviously, if $p \in L$, then p satisfies the conditions of the power set axiom in L.

Is $p \in L$?

Define $\Omega := \{ \rho_L(z) ;\ \substack{z \subseteq x \\ z \in L} \}$

By Replacement, there is some ϑ s.t. $\Omega \subseteq \vartheta$, so $p = \mathcal{P}(x) \cap L \subseteq L_\vartheta$.

If $\varphi(z,x) := z \subseteq x$, then

$$p = \mathcal{D}(\varphi, x, L_\vartheta), \text{ so } p \in L_{\vartheta+1} \subseteq L.$$

<u>Remark</u>. We have no good bound on ϑ at the moment since it was given by the Replacement Axiom.
We shall need to improve on that later.

Separation$_\varphi$

$$\forall x \, \forall p \, \exists s \, \forall z \, (z \in s \leftrightarrow z \in x \wedge \varphi(z,p))$$

Which set do we need:

$$\{ z \in x ; L \models \varphi(z,p) \}$$

the same.

Attempt Assume $x \in L_\alpha$.

Take formula $\psi(z,x,p) := z \in x \wedge \varphi(z,p)$

$$D(\psi, (x,p), L_\alpha) =$$

$$\{ z \in L_\alpha ; L_\alpha \models z \in x \wedge \varphi(z,p) \}$$

$$= \{ z \in x ; L_\alpha \models \varphi(z,p) \}$$

Problem: Only works if φ is absolute between L_α and L.

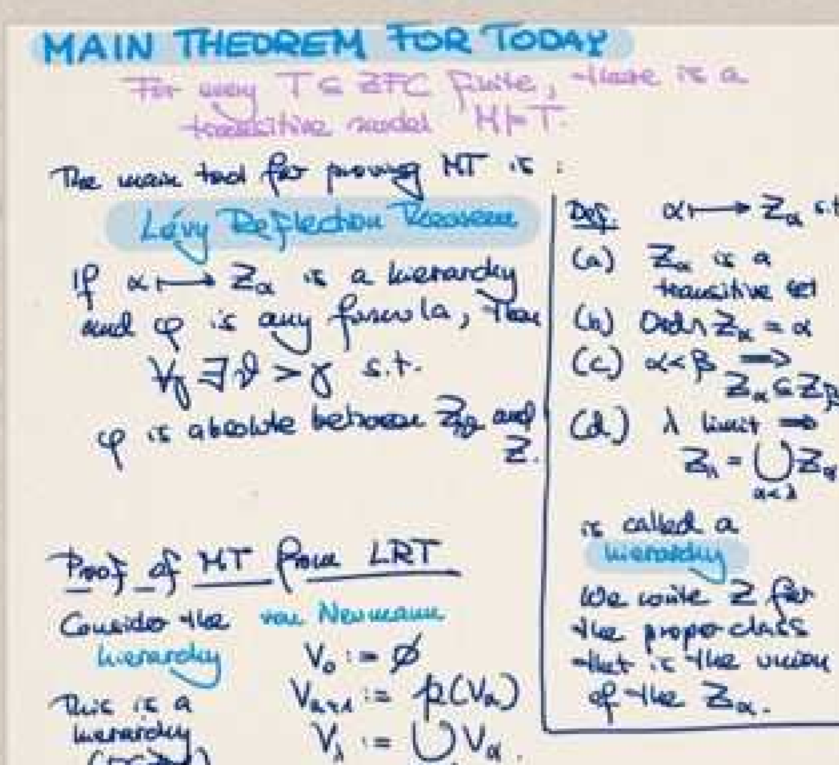
LRT to the rescue:

Find $\vartheta > \alpha$ s.t. φ is abs. between L_ϑ and L and define $D(\psi, (x,p), L_\vartheta) = \{ z \in x ; L_\vartheta \models \varphi(z,p) \}$

$$= \{ z \in x ; L \models \varphi(z,p) \}$$

Replacement on ES#2: Same proof ideas as power set & separation COMBINED.

Lévy Reflection Theorem (from Lecture IV)



MAIN THEOREM FOR TODAY

For every $T \subseteq ZFC$ finite, there is a transitive model $M \models T$.

The main tool for proving MT is:

Lévy Reflection Theorem

If $\alpha \longmapsto Z_\alpha$ is a hierarchy and φ is any formula, then $\forall \gamma \, \exists \vartheta > \gamma$ s.t. φ is absolute between Z_ϑ and Z.

Def. $\alpha \longmapsto Z_\alpha$ s.t.

(a) Z_α is a transitive set

(b) $\mathrm{Ord} \cap Z_\alpha = \alpha$

(c) $\alpha < \beta \Rightarrow Z_\alpha \subseteq Z_\beta$

(d) λ limit $\Rightarrow Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

is called a hierarchy.

We write Z for the proper class that is the union of the Z_α.

Proof of MT from LRT

Consider the von Neumann hierarchy

$V_0 := \emptyset$

$V_{\alpha+1} := \wp(V_\alpha)$

$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$.

This is a hierarchy (ES#1)

Remarks on the Axiom of Choice

So far, AC was not used.
We therefore get: in each model M of ZF, there is an inner model $L \models ZF$. (plus MINIMALITY)

However, even if AC is not true in M, we can provide a wellorder of L:

Fix some wellorder $<_\omega$ on L_ω of ordertype ω.
Assume that $<_\alpha$ is a wellorder of L_α, define (lexicographically) a wellorder of

$$\mathrm{Fml} \times L_\alpha^{<\omega}$$

[par abus de langage write $<_\alpha$ for this]

and then write for $x \in L_{\alpha+1}$

$$w(x) := \min_{<_\alpha} \{ (\varphi, p) ; x = D(\varphi, p, L_\alpha) \}$$

Make this into an end-extension of $<_\alpha$ by

$$x <_{\alpha+1} y :\iff \begin{array}{l} x, y \in L_\alpha \wedge x <_\alpha y \\ \text{OR } x \in L_\alpha \wedge y \in L_{\alpha+1} \setminus L_\alpha \\ \text{OR } x, y \notin L_\alpha \wedge w(x) < w(y). \end{array}$$

Because the orders end-extend,

$$< := \bigcup_{\alpha \in \mathrm{Ord}} <_\alpha \quad \text{is a wellorder of } L.$$

However, this was a recursive definition, so it's absolute and so

$$L \models \; < \text{ is a global wellorder of } L.$$

<u>COROLLARY</u> $\mathrm{Cons}(ZF) \Rightarrow \mathrm{Cons}(ZFC)$.

Now we have established 1. + 3. (ZFC and Minimality) and can proceed to 2. (GCH).

A closer look at Minimality

If T is s.s., then for each transitive $X \models T$, we have $\bigcup_{\alpha \in X} L_\alpha \subseteq X$.

A converse

AXIOM OF CONSTRUCTIBILITY

$$\forall x \exists \alpha (\alpha \text{ is an ordinal } \wedge x \in L_\alpha)$$

often written as $V = L$.

Prop If T is s.s., X transitive s.t. $X \models T \wedge V = L$, then $\bigcup_{\alpha \in X} L_\alpha = X$.

Proof. $\subseteq$ is minimality as above; $\supseteq$ is $V=L$.

THEOREM GÖDEL'S CONDENSATION LEMMA

There is a sentence σ (the condensation sentence) s.t. for all trs X

if $X \models \sigma$, then there is α s.t. $X = L_\alpha$.

Proof. Take the s.s. T (finite) as in Proposition and write $\sigma := \bigwedge T \wedge V = L \wedge$ there is no largest ordinal

If X trs, $X \cap \mathrm{Ord} = \lambda$ for some λ.
So if $X \models \sigma \implies \lambda$ is a limit ordinal.
By Prop $X = \bigcup_{\alpha < \lambda} L_\alpha = L_\lambda$ (since λ limit). q.e.d.