FORCING & THE CONTINUUM HYPOTHESIS

Lecture VI
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Gödel's Constructible Universe

Kurt Gödel

Gödel c. 1926

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THEOREM Lectures VI + VII

If $M \models ZFC$, then there is $L \subseteq M$ transitive containing all ordinals s.t.

1. $L \models ZFC$
2. $L \models GCH$
3. L is minimal in the sense that if $N \subseteq M$ transitive with all ordinals and $N \models ZFC$, then $L \subseteq N$.

SUMMARY OF ABSOLUTENESS

Formulas absolute for trs models (of T)

1. Δ_0 formulas
2. Closed under bdd quantification
3. Closed under quantification bdd by absolute operations ← PROVIDED T PROVES THE EX. & UNIQUENESS OF OPERATION
4. Closed under transfinite recursive definitions involving absolute operations ← PROVIDED T PROVES THE RELEVANT INSTANCE OF THE REC. THEOREM

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CONVENTION

$T \subseteq ZFC$ is **sufficiently strong (s.s.)** if T proves the absoluteness of all formulas relevant for the proof.

If some $T' \subseteq ZFC$ proves the absoluteness of φ, then by compactness there is a finite $T \subseteq ZFC$ that does.

So we always find <u>finite</u> s.s. theories.

ABSOLUTENESS OF TRUTH IN A MODEL

Lecture IV:

CONSEQUENCES FOR THE EXISTENCE OF TRANSITIVE MODELS OF ZFC

ZFC ⊬ ¬CH   ZFC ⊬ CH

Completeness

Con(T) ⟺ ∃M M ⊨ T

Incompleteness

T ⊬ Con(T)

In particular, for T that satisfies Incompleteness and is consistent:

$T^* := T + Con(T)$ is strictly stronger than T

Con(T) is considered as an encoded statement. This encoding usually happens (following Gödel) in arithmetic (e.g., prime factorization).

So: Con(T) = ∀n∈ω n is not the code of a T-proof of 0=1.

In particular: it's arithmetical!

Thus: it is absolute for transitive models.

This will mean that "there is a tr. model M ⊨ ZFC" is very strong indeed.

Formulas can be encoded as sets (or even numbers).

!

(See ES#1.)

Note that Fml, i.e., the set of (well-formed) formulas is defined by recursion.

So: Fml is absolute for tr. models of a s.s. T.

What is truth?

Tarski's recursive definition of truth defines a relation

$$M \models \varphi$$

between $\mathcal{L}$-structures M and $\varphi \in \mathrm{Fml}$ which is recursively defined.

Critical step in that definition:

$$M, I \models \exists x \varphi \text{ iff there is } a \in M \text{ s.t. } M, I\frac{a}{x} \models \varphi$$

So, if M is a set, then this is bdd. quantification, so Tarski's definition of $M \models \varphi$ is absolute for trs models of a s.s.T.

Remark Note Tarski's Undefinability of Truth: there is no formula Φ s.t.

$$N \models \varphi \iff N \models \Phi(\varphi).$$

So in the above, it's important that N is a set in the model where is interpret truth.

Gödel's Constructible Universe

Analogy is the von Neumann universe:

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \wp(V_\alpha)$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$$
$$V := \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$$

<u>Idea</u> Instead of $\wp(X)$, only use the sets really required by Separation, i.e., the definable ones.

Fix set X, $\varphi \in \mathrm{Fml}$, $p \in X^{<\omega}$ (parameters).

Define $D(\varphi, p, X) := \{ z \in X ; X \models \varphi(p, z) \}$

– the subset of X defined by φ over X with parameters p

Remark: often called "the definable power set of X":

$$\mathcal{D}(X) := \{ D(\varphi, p, X) ; \varphi \in \mathrm{Fml}, p \in X^{<\omega} \}$$

By the above, both D & $\mathcal{D}$ are absolute operations for trs models of a s.s. T.

<u>Remarks</u>
1. $\mathcal{D}(X) \subseteq \wp(X)$.
2. If X is trs, then so is $\mathcal{D}(X)$.

Define L

The constructible hierarchy.

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$
$$L := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha .$$

Observe:

1. L is a hierarchy as in LRT (ES#1)
2. Define a notion of constructible rank: if $x \in L$, then
$$\rho_L(x) := \text{the unique } \alpha \text{ s.t. } x \in L_{\alpha+1} \setminus L_\alpha .$$
3. Compare L-/V-hierarchies: By induction, for all $n \in \mathbb{N}$
$$L_n = V_n,$$
so $L_\omega = V_\omega$.
But $L_{\omega+1} = \mathcal{D}(L_\omega)$ is countable and $V_{\omega+1} = \mathcal{P}(V_\omega)$ is uncountable, so $L_{\omega+1} \neq V_{\omega+1}$.
4. Since L is recursively defined, it is absolute for trs models of a s.s.T.

What does this mean?

There is a formula Γ s.t. $\Gamma(\alpha, x)$ iff $x = L_\alpha$ and Γ is absolute for trs models of a s.s.T.

Corollary If N is any transitive set with $\alpha \in N$ and $N \models T$ [i.e., the s.s. theory for absoluteness of Γ],

then $L_\alpha \in N$, so $L_\alpha \subseteq N$.

Together: $\bigcup_{\alpha \in N} L_\alpha \subseteq N$.

Thus: if N contains all ordinals, then $L \subseteq N$.

This establishes the minimality of Gödel's theorem. [#3 in the theorem on p. 1]

Goal #1: $L \models ZFC$.

AXIOMS OF ZF(C)

STRUCTURAL AXIOMS

EXTENSIONALITY

$$\forall x \forall y \, (\forall w \, (w \in x \leftrightarrow w \in y) \longrightarrow x = y)$$

FOUNDATION

$$\forall x \, (\exists v (v \in x) \longrightarrow \exists m (m \in x \wedge \forall w \, \neg (w \in m \wedge w \in x)))$$

INFINITY

$$\exists i \, \exists e (\forall v \, \neg v \in e \wedge e \in i) \wedge \forall x \, (x \in i \longrightarrow \exists s \, (s \in i \wedge \forall w (w \in s \longleftrightarrow w \in x \vee w = x)))$$

FUNCTIONAL AXIOMS

PAIRING

$$\forall x \forall y \exists p \quad \forall w \, (w \in p \leftrightarrow w = x \vee w = y)$$

UNION

$$\forall x \exists u \quad \forall w \, (w \in u \longleftrightarrow \exists z (z \in x \wedge w \in z))$$

POWERSET

$$\forall x \exists p \quad \forall w \, (w \in p \leftrightarrow \forall v (v \in w \rightarrow v \in x))$$

SEPARATION φ

$$\forall \vec{p} \, \forall x \exists s \;\; \forall w \, (w \in s \leftrightarrow w \in x \wedge \varphi(w, \vec{p}))$$

REPLACEMENT φ

$$\forall \vec{p} \quad \forall x \forall y \forall z (\varphi(x, y, \vec{p}) \wedge \varphi(x, z, \vec{p}) \rightarrow y = z) \longrightarrow \forall x \exists r \; \forall w (w \in r \leftrightarrow \exists y (y \in x \wedge \varphi(y, w, \vec{p})))$$

[PLUS THE AXIOM OF CHOICE]

THE STRUCTURAL AXIOMS

Extensionality, Foundation, Infinity

Lecture II :

The extent of absoluteness for transitive models

Lemma If M is a transitive set, then $(M, \in) \models$ Extensionality + Foundation.

Since $x = \omega$ is absolute and ω satisfies the conditions of the Axiom of Infinity, we have that for all $\alpha > \omega$, $L_\alpha \models$ Infinity.

PAIRING & UNION

Pairing $\forall x \forall y \exists p \, \forall z \, (z \in p \leftrightarrow z = x \vee z = y)$
Union $\forall x \, \exists u \, \forall z \, (z \in u \leftrightarrow \exists y \in z \, (z \in y))$

The operations $x, y \longmapsto \{x, y\}$, $x \longmapsto \bigcup x$ are absolute operations for a.s.s.T

So, I only need to prove:

$$\forall x, y \in L \, (\{x, y\} \in L \text{ and } \bigcup x \in L).$$

So, we only need to show $\{x, y\}, \bigcup x \in \mathcal{D}(L_\alpha)$ for some $\alpha \longrightarrow$ Lecture VII