Forcing & the Continuum Hypothesis

23 October 2025

Fifth Lecture: V

RECAP

$ZFC^* \nvdash ZFC^{\oplus}$

So, we cannot assume $ZFC^{\oplus}$.

However, $ZFC \vdash T^{\oplus}$ for every finite $T \subseteq ZFC$.

Lecture IV

NEXT AIM:

Improve the Main Theorem to:

For all $T \subseteq ZFC$ finite, there is a countable transitive M s.t. $M \models T$.

First thing that you have in mind:

LÖWENHEIM-SKOLEM

This clearly gives us a countable model, but ... is it transitive ?!?

LÖWENHEIM-SKOLEM.

If N is any model and Φ any set of formulas, find $M \subseteq N$ countable s.t. M is a substr. of N and all fmlas in Φ are absolute between M and N.

[This is a minor refinement of the usual LöSko]

Q Is the model obtained by LöSko transitive?

PROOF OF LÖWENHEIM-SKOLEM

LÖWENHEIM-SKOLEM.
If N is any model and Φ any set of formulas, find $M \subseteq N$ countable s.t. M is a substr. of N and all fmlas in Φ are absolute between M and N.

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Proof.

Let $\exists x \psi \in \Phi$

$$w(\psi, \vec{p}) := \begin{cases} x & \text{if } N \models \psi(x, \vec{p}) \quad \text{if such an } x \text{ exists} \\ \emptyset & \text{o/w} \end{cases}$$

NOTE: This is not a proper definition, since x is not specified.
It's a use of AC $\longrightarrow \{x ; N \models \psi(x, \vec{p})\} \neq \emptyset$, so use AC to pick one of these.

$M_0 := \emptyset$

$M_{n+1} := M_n \cup \{w(\psi, \vec{p}) ; \exists x \psi \in \Phi \wedge \vec{p} \in M_n^{<\omega}\}$

$M := \bigcup_{n \in \mathbb{N}} M_n$

(1) By same argument as before, all fmlas in Φ are abs. between M & N (TVT).

(2) By induction, all M_n are ctble
$[|M_{n+1}| \leq |M_n| + \aleph_0 \cdot |M_n^{<\omega}|.]$

(3) So M is ctble as a ctble union of ctble sets.

q.e.d.

TARSKI-VAUGHT TEST

Suppose $\mathcal{L}$ is any language, M and N are $\mathcal{L}$-structures such that M is a substructure of N, and Φ is any set of $\mathcal{L}$-formulas closed under subformulas. Then all formulas in Φ are absolute between M and N if and only if for any formula $\exists x \varphi \in \Phi$ where φ has $n+1$ free variables including x, we have that

> if $m_1, \ldots, m_n \in M$ and $N \models \exists x \varphi(x, m_1, \ldots, m_n)$, then there is some $m \in M$ such that $N \models \varphi(m, m_1, \ldots, m_n)$.

Is the M we obtained in that construction transitive?

A: In general, no! [More details on ES#1.]

By LRT, we found ϑ s.t. $V_\vartheta \models T$.
By LöSko, we found $M \subseteq V_\vartheta$ countable s.t. $M \models T$.

Suppose $\vartheta > \aleph_1$, so $\aleph_1 \in V_\vartheta$.
Let $\varphi(x) := x$ is the smallest uncountable ordinal.

<u>CLAIM</u> $V_\vartheta \models \varphi(\aleph_1)$.

Why?
- $\aleph_1$ is ordinal in V_ϑ since "ordinal" is absolute
- Countability is Σ_1, so upwards absolute. So $V_\vartheta \models \aleph_1$ is uncountable.
- If $\alpha < \aleph_1$, its countability is witnessed by $f : \omega \to \alpha$ surjective. $\rightsquigarrow f \subseteq \omega \times \alpha$, so $f \in V_{\alpha+2} \subseteq V_\vartheta$

  $\Longrightarrow V_\vartheta \models \alpha$ is countable.

In part., $V_\vartheta \models \exists x\, \varphi(x)$.
In the LöSko construction, we added $w(\varphi, \emptyset)$ to M_1, so $w(\varphi, \emptyset)$ is an elt of M.
But $\aleph_1$ is the only witness to $\exists x \varphi(x)$, so $\aleph_1 \in M$. But $\aleph_1 \not\subseteq M$, since $\aleph_1$ is uncountable, so M is not transitive.

Situation $T \subseteq ZFC$
Found $V_\theta \models T$ and $M \subseteq V_\theta$ countable s.t. $M \models T$

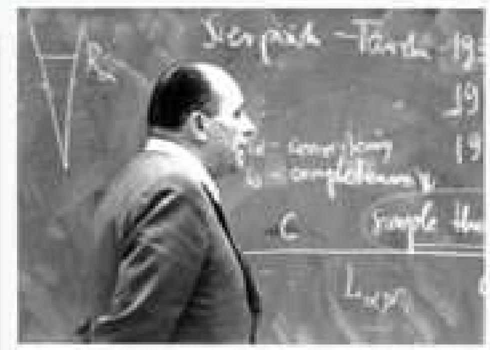
Leader: Logic & Set Theory NOTES §5

The analogue of 'subset collapse' is:

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x\ r\ y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

Andrzej Mostowski



Mostowski in 1973

Born	1 November 1913 Lemberg, Austria-Hungary
Died	22 August 1975 (aged 61) Vancouver, British Columbia, Canada
Nationality	Polish
Alma mater	University of Warsaw
Known for	Kleene–Mostowski hierarchy Mostowski collapse lemma Mostowski model

We have $M \models T$.

- $\in$ on M is well-founded (since Foundation is true)
- W.l.o.g. assume that the axiom of extensionality is in T, then $\in$ on M is extensional.

So by MOSTOWSKI, there is transitive set N s.t. $(N, \in) \cong (M, \in)$.

$\Rightarrow$ since M is ctble, N is ctble.

By the "Isomorphism Lemma" [isomorphic structures satisfy the same sentences], we get $N \models T$.

Three steps: Lévy + LöSko + Mostowski:
For every $T \subseteq ZFC$ finite, there is a ctble trs model of T.

NON-ABSOLUTENESS

<u>Remember</u> Absoluteness is good since we wish that basic terms ("empty", "ordinal", "function") retain their meaning.
But if <u>EVERYTHING</u> is absolute, we can't prove any independence.

Note: there was an error in the slides during the lecture; it said $N \subseteq M$.

Our situation: $N \cong M \subseteq V_\vartheta \models T$

$$\underset{\text{trs}}{\uparrow} \qquad \underset{\text{not trs}}{\uparrow} \qquad \underset{\text{trs}}{\uparrow}$$

From p3: $V_\vartheta \models \exists x \varphi(x)$ ← is the least uncountable ordinal

Therefore: $N \models \exists x \varphi(x)$

What is $N \cap \mathrm{Ord} = \gamma$ for some ordinal $\gamma < \aleph_1$.

Therefore, if $\alpha \in N$ s.t. $N \models \varphi(\alpha)$, then $\alpha < \gamma < \aleph_1$.

I.e., α is a countable ordinal that is not a cardinal.

<u>Cor</u> Therefore "x is countable" and "x is a cardinal" are not absolute between N and V_ϑ.

Σ_1 Π_1

The first MAIN THEOREM of FCH

THEOREM (Gödel 1938)

If $M \models ZFC$, then there is an $N \subseteq M$, N transitive in M s.t.

$N \models ZFC + CH$.

"Technique of Inner Models" [Plan for Lectures $\underline{V}$ – $\underline{VII}$.]

Cor. If ZFC is consistent, $ZFC \nvdash \neg CH$.

Idea. (1) CH is true in the "minimal model" of ZFC in M.

(2) Minimality achieved by throwing in everything that's necessary to make axioms true.

(3) Some axioms have an algebraic flavour, e.g., Pairing and you just throw in the required set.

(4) But what about, say, Separation?

Suppose S structure and $x \in S$ s.t. there no separation instance, so no $t \in S$ s.t.

$$S \models \forall z \in x \, (z \in t \leftrightarrow \varphi(z,x))$$

The right set is $\{z \in x ; S \models \varphi(z,x)\} =: t$

But if $S^* := S \cup \{t\}$. But if φ is not absolute betw. S & S^*, this is not a witness.