IX Ninth Lecture
Forcing & the Continuum Hypothesis

11 November 2025

we'll schedule another Saturday replacement lecture.

The Method of Outer Models

showing $\mathrm{Cons}(\mathrm{ZFC}) \Rightarrow \mathrm{Cons}(\mathrm{ZFC}+\varphi)$.

Lecture VIII

Theorem (**) If $T \subseteq \mathrm{ZFC}$ is finite, then there is $T^* \subseteq \mathrm{ZFC}$ finite such that: whenever $M \models T^*$ ctble trs, then there is $\underset{\text{ctble trs}}{N \supseteq M}$ s.t. $N \models T + [\neg\mathrm{CH}]$

φ

Practical problem:

If $M \models \mathrm{ZFC}$ is ctble transitive and $f \notin M$ witnesses φ, we can form a ctble trs $N \supseteq M$ with $f \in N$, but how do we control what is true in N ?!?

MAIN TASK Build N s.t. $M \subseteq N$ & $f \in N$ in a controlled way so that we can determine truth values of sentences in N!

The METHOD OF FORCING

__Def.__ We call $(\mathbb{P}, \leq, \mathbb{1})$ a forcing partial order (forcing) if $(\mathbb{P}, \leq)$ is a partial order and $\mathbb{1} \in \mathbb{P}$ is its maximum element.

The elements of $\mathbb{P}$ are called (forcing) conditions. We interpret them as "forcing things to be true".

__Interpret__ $p \leq q$ as "p is stronger than q"

Fixing notation:

$p, q \in \mathbb{P}$ are compatible if $\exists r$ s.t. $r \leq p, q$.

Otherwise incompatible: $p \perp q$.

$A \subseteq \mathbb{P}$ is an antichain if $\forall a, b \in A$ $a \perp b$.

$D \subseteq \mathbb{P}$ is called dense if $\forall p \in \mathbb{P} \exists d \in D$ $d \leq p$.

Shelah broke with this convention and writes his forcings the other way round.

JERUSALEM CONVENTION

Shelah in 2005

$B \subseteq \mathbb{P}$ is called a filter base if $\forall p, q \in B\ \exists r \in B\ \ r \leq p, q$

$F \subseteq \mathbb{P}$ is called a filter if F is a filter base and $\forall p, q\ \ p \leq q\ \&\ p \in F \longrightarrow q \in F$.

If B is a filter base, then

$$\{ p \in \mathbb{P} ;\ \exists q \in B\ \ p \geq q \}$$

is a filter: the filter generated by B.

If $\mathcal{D}$ is any family of dense sets, we call F <u>$\mathcal{D}$-generic</u> if

① F is a filter
② $\forall D \in \mathcal{D}\ \ F \cap D \neq \emptyset$.

<u>Theorem</u> If $\mathcal{D}$ is countable, then there is a $\mathcal{D}$-generic filter.

<u>Proof.</u> $\mathcal{D} = \{ D_i ;\ i \in \mathbb{N} \}$. Let $p_0 \in \mathbb{P}$ be arbitrary. Pick recursively

p_{i+1} s.t. $p_{i+1} \leq p_i$ and $p_{i+1} \in D_i$ (possible by density of D_i)

Then $B := \{ p_i ;\ i \in \mathbb{N} \}$ is a filter base. Then the filter generated by B is $\mathcal{D}$-generic.

q.e.d.

MAIN EXAMPLE

Let X, Y be sets and

$$\mathbb{P} := \mathrm{Fn}(X, Y) := \{ p \,;\, p \text{ is a finite function with } \mathrm{dom}(p) \subseteq X \text{ and } \mathrm{ran}(p) \subseteq Y \}$$

$$p \leq q \;:\Longleftrightarrow\; p \supseteq q.$$

$$\mathbb{1} := \emptyset.$$

<u>Note</u>: $p \perp q \Longleftrightarrow \exists x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ s.t. $p(x) \neq q(x)$.

<u>Lemma 1</u> F filter (base) $\Longrightarrow \bigcup F$ is a function $= \{(x,y) \,;\, \exists p \in F \; (x,y) \in p\}$

<u>Lemma 2</u> $D_x := \{ p \,;\, x \in \mathrm{dom}(p) \}$ This is dense.

$\mathcal{D}_0 := \{ D_x \,;\, x \in X \}$

F is $\mathcal{D}_0$-generic $\Longrightarrow \mathrm{dom}(\bigcup F) = X$.

<u>Lemma 3</u> $R_y := \{ p \,;\, y \in \mathrm{ran}(p) \}$. If X is infinite, R_y is dense

$\mathcal{D}_1 := \{ R_y \,;\, y \in Y \}$

F is $\mathcal{D}_1$-generic & X is infinite $\Longrightarrow \mathrm{ran}(\bigcup F) = Y$.

<u>Lemma 4</u> Fix $f : X \to Y$. $N_f := \{ p \,;\, \exists x \in \mathrm{dom}(p) \; f(x) \neq p(x) \}$

If X inf., N_f is dense.

F is $\{N_f\}$-generic & X is infinite $\Longrightarrow \bigcup F \neq f$.

NAMES

Fix $\mathbb{P}$ and define the $\mathbb{P}$-names by recursion:

$$\mathrm{Name}_0^{\mathbb{P}} := \emptyset$$

$$\mathrm{Name}_{\alpha+1}^{\mathbb{P}} := \{\tau \,;\, \tau \subseteq \mathrm{Name}_\alpha^{\mathbb{P}} \times \mathbb{P}\} \cup \mathrm{Name}_\alpha^{\mathbb{P}}$$

$$\mathrm{Name}_\lambda^{\mathbb{P}} := \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}}$$

$$\mathrm{Name}^{\mathbb{P}} := \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Name}_\alpha^{\mathbb{P}}$$

Interpretation

If $(\tau', p) \in \tau$, then p guarantees that the set named by τ' is in the set named by τ

Note This is a generalization of the von Neumann hierarchy $\mathbb{P} = \{\mathbb{1}\}$.
Then the map
$$h(\tau) := \{h(\tau') \,;\, (\tau', \mathbb{1}) \in \tau\}$$
is an "isomorphism" between $\mathrm{Name}^{\mathbb{P}}$ and V.

Examples

$\mathrm{Name}_1^{\mathbb{P}} = \{\emptyset\}$. In particular, $\emptyset$ is a name.

$$\mathrm{Name}_2^{\mathbb{P}} = \{\tau \,;\, \tau \subseteq \{\emptyset\} \times \mathbb{P}\}$$
$$= \{\tau \,;\, \tau \subseteq \{(\emptyset, p) \,;\, p \in \mathbb{P}\}\}$$

E.g., $\tau_p := \{(\emptyset, p)\}$.

At level 3: $\tau_{p,q} := \{(\tau_p, q)\}$

THE VALUE OF A NAME

Fix $F \subseteq \mathbb{P}$ and define recursively $\tau \in \mathrm{Name}^{\mathbb{P}}$

$$\mathrm{val}(\tau, F) := \{ \mathrm{val}(\sigma, F) ; \exists p \in F \ (\sigma, p) \in \tau \}.$$

Important remark

Note that $\mathrm{Name}^{\mathbb{P}}$ and $\mathrm{val}(\tau, F)$ are recursive definitions, so absolute for tr models (containing τ, F).

Examples From last page $\emptyset, \tau_p, \tau_{p,q} \in \mathrm{Name}^{\mathbb{P}}$.

$\mathrm{val}(\emptyset, F) = \emptyset$ for any F.

$$\mathrm{val}(\underset{\{(\emptyset, p)\}}{\underset{\shortparallel}{\tau_p}}, F) = \begin{cases} \emptyset & \text{if } p \notin F \\ \{\emptyset\} & \text{if } p \in F \end{cases}$$

$$\mathrm{val}(\underset{\{(\tau_p, q)\}}{\underset{\shortparallel}{\tau_{pq}}}, F) = \begin{cases} \emptyset & \text{if } q \notin F \\ \{\emptyset\} & \text{if } q \in F \ \& \ p \notin F \\ \{\{\emptyset\}\} & \text{if } p, q \in F \end{cases}$$

The relationship between p, q affects this value.

E.g., ① if $q \leq p$ & F is a filter, then $\{\emptyset\}$ is impossible

② if $q = \mathbb{1}$ & $F \neq \emptyset$ is a filter, then $\emptyset$ is impossible.

③ if $p \perp q$ & F is a filter, then $\{\{\emptyset\}\}$ is imp.

The extension of a model M by F

is

$$M[F] := \{ \mathrm{val}(\tau, F) ; \tau \in \mathrm{Name}^{\mathbb{P}} \cap M \}.$$

Goals :

Find conditions on F s.t.

$$\Longrightarrow \left\{ \begin{array}{l} \text{① } M[F] \text{ is a trs model} \\ \text{② } M \subseteq M[F] \\ \text{③ } F \in M[F] \\ \text{④ } M[F] \models \mathrm{ZFC}. \end{array} \right.$$

Lecture X

This is going to be the hardest result of this lecture course and will take several lectures.