FORCING & THE CONTINUUM HYPOTHESIS
FOURTH LECTURE
IV: 21 Oct 2025

RECAP

(1) Quantifier-free fmlas absolute for substructures

BUT: almost no (interesting) fmlas are quantifier-free

(2) Δ_0 formulas and their closure under concatenation & transfinite recursion absolute for **transitive models**

Lecture III:

EXAMPLES OF ABSOLUTE FORMULAE & OPERATIONS (for transitive models)

1. $x \in y$ atomic
2. $x = y$ atomic
3. $x \subseteq y$ $[\forall z \in x (z \in y)]$
4. $x = \emptyset$ done already
5. $x = \{y\}$ $[y \in x \wedge \forall z \in x (z = y)]$ — 1., bdd Q, 2.

   So $x \mapsto \{x\}$ is an absolute operation.
6. $x = \{y_0, y_1\}$ Similar.
7. $x = (y_0, y_1) = \{\{y_0\}, \{y_0, y_1\}\}$ — 5, 6 + Concatenation
8. $x = y_0 \cup y_1$ $y_0 \subseteq x \wedge y_1 \subseteq x \wedge \forall z \in x (z \in y_0 \vee z \in y_1)$ — 3., 3., bdd Q, 1., 1.
9. $x = y_0 \cap y_1$ Similar
10. $x = y_0 \setminus y_1$ Similar
11. $x = \bigcup y$ Similar
12. $x = y \cup \{y\}$ Concat.: 5 + 8.
13. x is a trs set $[\forall y \in x \forall z \in y (z \in x)]$
14. x is an ordered pair } ES#1
15. $x = y_0 \times y_1$ } ES#1
16. x is a relation $[x \subseteq y_0 \times y_1 \times \dots \times y_{n-1}]$
17. x is a function
18. $x = \mathrm{dom}(f)$
19. $x = \mathrm{ran}(f)$
20. x is injective
21. x is surjective
22. x is bijective
23. x is an ordinal

    [x is transitive $\wedge$ $\in$ is a linear order on x]

    This is enough since in transitive models, Foundation holds!
24. x is a successor ordinal

    [e.g., $\exists z \in x (x = z \cup \{z\}) \wedge$ x is ordinal]
25. x is limit ordinal
26. $x = \omega$ [the unique limit ordinal that has no non-zero limit ordinals as members]
27. All recursively defined operations on ω, such as $+$, $\cdot$, exponentiation etc.

Remark. 27. is almost too good to be true.

Def. The *arithmetical formulas* are the smallest class of fmlas containing arithmetical equalities and inequalities closed under prop. connectives and the bounded quantifiers $\exists n \in \omega$ and $\forall n \in \omega$.

OBSERVE All arithmetical fmlas are absolute for trs models.

CONSEQUENCES FOR THE EXISTENCE OF TRANSITIVE MODELS OF ZFC

Completeness

$\mathrm{Cons}(T) \iff \exists M\ M \models T$

Incompleteness

$T \nvdash \mathrm{Cons}(T)$

$\mathrm{ZFC} \nvdash \neg\mathrm{CH}$ $\mathrm{ZFC} \nvdash \mathrm{CH}$

"CH is independent of ZFC".

This can't be what G. & C. actually proved due to the incompleteness theorem. Instead:

RELATIVE CONSISTENCY PROOFS.

$\mathrm{Cons}(\mathrm{ZFC}) \Longrightarrow \mathrm{ZFC} \nvdash \neg\mathrm{CH}$ [eq. $\mathrm{Cons}(\mathrm{ZFC}+\mathrm{CH})$]

$\mathrm{Cons}(\mathrm{ZFC}) \Longrightarrow \mathrm{ZFC} \nvdash \mathrm{CH}$. [eq. $\mathrm{Cons}(\mathrm{ZFC}+\neg\mathrm{CH})$]

By Completeness, equivalent to if there is a model of ZFC, then there is a model of ZFC+CH.

if there is $M \models \mathrm{ZFC}$, then there is $N \models \mathrm{ZFC}+\neg\mathrm{CH}$

Gödel's Completeness Theorem

$\Phi \nvdash \varphi$

$\Longleftrightarrow$

there is a model $M \models \Phi + \neg\varphi$

Gödel's Incompleteness Theorem

$\mathrm{ZFC} \vdash \mathrm{Cons}(\mathrm{ZFC})$

$\Longleftrightarrow$

$\neg\mathrm{Cons}(\mathrm{ZFC})$

$\Longleftrightarrow$

$\forall\varphi\ \mathrm{ZFC} \vdash \varphi$

In particular, for T that satisfies Incompleteness and is consistent:

$T^* := T + \mathrm{Cons}(T)$ is strictly stronger than T

$\mathrm{Cons}(T)$ is considered as an encoded statement: This encoding usually happens (following Gödel) in arithmetic (e.g., prime factorisation).

So. $\mathrm{Cons}(T) = \forall n \in \omega$ n is not the code of a T-proof of $0=1$.

In particular: It's arithmetical!

Thus: it is absolute for transitive models!

This will mean that

"there is a trs model $M \models \mathrm{ZFC}$"

is very strong indeed.

Def. $T^{(0)} := T$

$T^{(n+1)} := \left(T^{(n)}\right)^*$

$ZFC^{\oplus} := \overbrace{\text{ZFC+}}$ there is a tms model of ZFC

Clearly, $ZFC^{\oplus} \vdash ZFC^*$.

Prop. $ZFC^{\oplus} \vdash ZFC^{(n)}$ for all n.

Proof. This is a bootstrapping argument

$\rightsquigarrow$ induction.

Suppose $ZFC^{\oplus}$ holds, then by remark

$ZFC^* = ZFC + Cons(ZFC)$ holds.

Let M be tms s.t. $M \models ZFC$ by $ZFC^{\oplus}$.

By absoluteness of $Cons(ZFC)$ for M, we get:

$$M \models Cons(ZFC)$$

$$M \models ZFC^*.$$

So, together we prove $\exists M\ M \models ZFC^*$

$\Longleftrightarrow Cons(ZFC^*)$

That's again absolute, so

by absoluteness:

$$M \models Cons(ZFC^*)$$

$$M \models ZFC^{**} = ZFC^{(2)}$$

Etc. q.e.d.

MAIN THEOREM FOR TODAY

For every $T \subseteq ZFC$ finite, there is a transitive model $M \models T$.

The main tool for proving MT is:

Lévy Reflection Theorem

If $\alpha \mapsto Z_\alpha$ is a hierarchy and φ is any formula, then

$$\forall \gamma \, \exists \vartheta > \gamma \text{ s.t.}$$

φ is absolute between Z_ϑ and Z.

Def. $\alpha \mapsto Z_\alpha$ s.t.

(a) Z_α is a transitive set

(b) $\mathrm{Ord} \cap Z_\alpha = \alpha$

(c) $\alpha < \beta \implies Z_\alpha \subseteq Z_\beta$

(d) λ limit $\implies Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

is called a hierarchy.

We write Z for the proper class that is the union of the Z_α.

Proof of MT from LRT

Consider the von Neumann hierarchy

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \wp(V_\alpha)$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha.$$

This is a hierarchy (ES#1)

and V is the set theoretic universe.

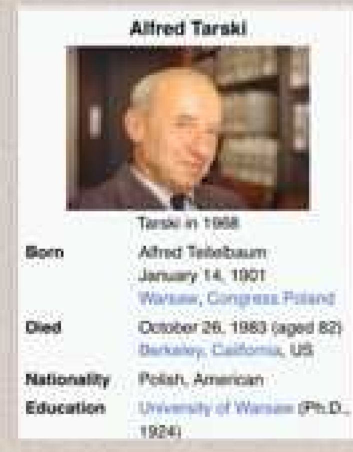
Fix $T \subseteq ZFC$ finite. Form $\varphi := \bigwedge_{\psi \in T} \psi$. Then φ is eq. to T.

Then φ is true (since $T \subseteq ZFC$) and by LRT find ϑ s.t. $V_\vartheta \models \varphi$. This is transitive.

q.e.d.

The TARSKI-VAUGHT-TEST

TVT



Suppose $\mathcal{L}$ is any language, M and N are $\mathcal{L}$-structures such that M is a substructure of N, and Φ is any set of $\mathcal{L}$-formulas closed under subformulas. Then all formulas in Φ are absolute between M and N if and only if for any formula $\exists x\varphi \in \Phi$ where φ has $n+1$ free variables including x, we have that

> if $m_1, ..., m_n \in M$ and $N \models \exists x\varphi(x, m_1, ..., m_n)$, then there is some $m \in M$ such that $N \models \varphi(m, m_1, ..., m_n)$.

ES#1.

Proof of LRT. Fix φ, $\alpha \longmapsto Z_\alpha$, and γ.
Let $\Phi :=$ the set of subformulas of φ.
(This is a finite set.)

For each $\exists x\psi \in \Phi$ and $\vec{p} = (p_1, \dots, p_n)$, let

$$o(\psi, \vec{p}) := \begin{cases} \text{the least } \alpha \text{ s.t. } \exists y \in Z_\alpha \; Z \models \psi(y, \vec{p}) & \text{if exists} \\ 0 & \text{o/w} \end{cases}$$

$$o(\vec{p}) := \max_{\exists x\psi \in \Phi} o(\psi, \vec{p})$$ [using Φ finite]

$$\vartheta_0 := \gamma + 1.$$

$$\vartheta_{i+1} := \max\{\vartheta_i + 1, \sup\{o(\vec{p}); \vec{p} \in Z_{\vartheta_i}^{<\omega}\}\}$$

$$\vartheta := \sup_{i \in \mathbb{N}} \vartheta_i$$

Note that ϑ is a limit ordinal; $Z_\vartheta = \bigcup_{i \in \mathbb{N}} Z_{\vartheta_i}$

So, any $\vec{p} \in Z_\vartheta^{<\omega}$ has some n s.t. $\vec{p} \in Z_{\vartheta_n}^{<\omega}$, so $\exists x\psi(x, \vec{p})$ has a witness in $Z_{\vartheta_{n+1}}$. q.e.d.

NEXT AIM:

Improve the Main Theorem to:
For all $T \subseteq ZFC$ finite, there is a countable transitive M s.t. $M \models T$.

First thing that you have in mind:

LÖWENHEIM-SKOLEM

This clearly gives us a countable model, but ... is is transitive ?!?

__LÖWENHEIM-SKOLEM.__
If N is any model and Φ any set of formulas, find $M \subseteq N$ countable s.t. M is a substr. of N and all fmlas in Φ are absolute between M and N.

[This is a minor refinement of the usual LöSko]

HW: Think about how to use TVT to prove this version of LöSko.

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