FORCING & THE CONTINUUM HYPOTHESIS

Third Lecture: 18 October 2025

Recap from Lecture II:

Def. Δ_0 is the smallest class of formulas which

1. contains all atomic fmlas
2. closed under bdd. quant.
3. closed under $\wedge, \vee, \neg, \rightarrow$.

Def. A fmla is Σ_1 if it is of the form $\exists v_1 \dots \exists v_n \varphi$ for $\varphi \in \Delta_0$.
And Π_1 if it is of the form $\forall v_1 \dots \forall v_n \varphi$ for $\varphi \in \Delta_0$.

Def. If T is any theory, we define $\Delta_0^T, \Sigma_1^T, \Pi_1^T$ to be the class of fmlas that are equivalent in T to a $\Delta_0, \Sigma_1, \Pi_1$ fmla.

Theorem Δ_0^T-formulas are absolute for transitive models T.

Proof. By induction on the formula complexity of Δ_0 formulas.

1. Atomic formulas: substructure lemma.
2. Prop. connectives: as in the proof of the substructure lemma
3. Enough to show $\varphi \rightsquigarrow \exists x \in y\, \varphi$.

If $M \models \exists x \in y\, \varphi \implies M \models \varphi(x, y)$
By IH $\varphi(x, y)$ is true
So $\exists x \in y\, \varphi(x, y)$ is true.
If $\exists x \in y\, \varphi$ is true for $y \in M$, then $x \in M$ by trs
Thus $M \models \exists x \in y\, \varphi$.

q.e.d.

Remark. This proof only shows it for Δ_0, not Δ_0^T.
So: if φ is Δ_0^T, thus $T \vdash \varphi \leftrightarrow \psi$ where ψ is Δ_0.
Then if $M, N \models T$, ψ is absolute by proof, but in both M, N, ψ is eq. to φ.

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Proof. By induction on the formula complexity of Δ_0 formulas.

1. Atomic formulas: substructure lemma
2. Prop. connectives: as in the proof of the substructure lemma
3. Enough to show $\varphi \rightsquigarrow \exists x \in y\, \varphi$.

If $M \models \exists x \in y\, \varphi \Rightarrow M \models \varphi(x,y)$

By IH $\varphi(x,y)$ is true

So $\exists x \in y\, \varphi(x,y)$ is true.

If $\exists x \in y\, \varphi$ is true for $y \in M$, then $x \in M$ by trs

Thus $M \models \exists x \in y\, \varphi$. q.e.d.

Corollary

Σ_1^T fmlas are upwards absolute for trs models of T

Π_1^T fmlas are downwards absolute for trs models of T

More generally: if ψ is absolute, then $\exists x \psi$ is upw. absolute & $\forall x \psi$ is downwards absolute

Proof. Let $\varphi = \exists x \psi$ for ψ Δ_0 and $M \subseteq N$ transitive and $M, N \models T$.

$$
\begin{aligned}
M \models \varphi &\iff M \models \exists x \psi \\
&\iff \text{there is } m \in M \text{ s.t. } M \models \psi(m) \\
\text{UPWARDS} \quad &\underset{M \subseteq N}{\Longrightarrow} \text{there is } m \in N \text{ s.t. } M \models \psi(m) \\
&\underset{\text{since } \psi\ \Delta_0}{\iff} \text{there is } m \in N \text{ s.t. } N \models \psi(m) \\
&\iff N \models \exists x \psi \iff N \models \varphi.
\end{aligned}
$$

Similar for Π_1. q.e.d.

Def. We call a formula $\underline{\underline{\Delta_1^T}}$ if it is both Σ_1^T and Π_1^T.

Typical Δ_1^T expression: "the unique x s.t. ..."

Corollary Δ_1^T formulas are absolute for trs models of T.

(*) More generally, if φ and ψ are absolute and $T \vdash \exists x \varphi \leftrightarrow \forall x \psi$, then both $\exists x \varphi$ and $\forall x \psi$ are absolute for trs models of T.

Call this "the Δ_1 trick"!

Absolute operations

Let M be any $\mathcal{L}$-structure, $\mathcal{L}$ is the l.o.st.

$F : M^n \longrightarrow M$ is called an *n-ary operation on M*

Def. F is *definable by Φ* if

$$F(x_1, \dots, x_n) = x \iff \Phi(x_1, \dots, x_n, x).$$

We say F is *absolute for M* if it's definable by Φ which is absolute for M.

Obs. 1 If F, G are absolute for M, then so is $F \circ G$.

Obs. 2 (Bounded quant. by absolute operations)

If F is absolute and φ is absolute for M, then so is

$$\exists x \in F(y)\, \varphi$$

$$\iff \exists x \in z \left(\Phi(y, z) \wedge \varphi \right) \quad (*)$$

and

$$\varphi(F(x))$$

$$\iff \exists y \left(\Phi(x, y) \wedge \varphi(y) \right) \quad (**)$$

1. The formula $(*)$ is just bounded quantification applied to absolute formulas, so absolute.
2. The formula $(**)$ has an unbounded quantifier, but we can use the "Δ_1-trick" (see page 2) since it is eq. to $\forall y \left(\Phi(x, y) \longrightarrow \varphi(y) \right)$.

Absoluteness of operations defined by (transfinite) recursion

IMPRECISE STATEMENT	PRECISE STATEMENT
If R is defined by transfinite recursion from absolute operations, then R is absolute (for transitive models of a sufficiently strong T).	Let F, G, H be operations and T prove L1, L2 for F, G, H ($T \subseteq ZFC$). If F, G, H are absolute for the models of T, then so is R.

Note that for each F, G, H there is a finite $T \subseteq ZFC$ that satisfies this.

Recursion equations:

$$(*)\left[\begin{array}{lcl} R(0,\vec{x}) & = & F(\vec{x}) \\ R(\alpha+1,\vec{x}) & = & G(\alpha, R(\alpha,\vec{x}),\vec{x}) \quad \forall\alpha \\ R(\lambda,\vec{x}) & = & H(\lambda, \{R(\alpha,\vec{x}) ; \alpha<\lambda\}, \vec{x}) \\ & & \text{for all } \lambda \text{ limit} \end{array}\right.$$

Reminder:

The RECURSION THEOREM states that ZFC proves that for each F, G, H R is uniquely defined.

How do we prove it:

(a) "Attempts" are fragments of R satisfying $(*)$ on their domain.

(b) L1: any attempts r, r' agree everywhere

(c) L2: $\forall(\alpha,\vec{x})$ there is attempt r s.t. $(\alpha,\vec{x}) \in \mathrm{dom}(r)$

(d) $R(\alpha,\vec{x})$ is the unique value of $r(\alpha,\vec{x})$ for some att. r with $(\alpha,\vec{x}) \in \mathrm{dom}(r)$

Proof. Observe that R is defined by Δ_1^T formula. q.e.d

EXAMPLES OF ABSOLUTE FORMULAE & OPERATIONS

(for transitive models)

1. $x \in y$ atomic
2. $x = y$ atomic
3. $x \subseteq y$ $[\forall z \in x (z \in y)]$
4. $x = \emptyset$ done already
5. $x = \{y\}$ $[\underset{1.}{y \in x} \wedge \underset{\text{BDD Q}}{\forall z \in x} \underset{2.}{(z = y)}]$

So $x \longmapsto \{x\}$ is an absolute operation.

6. $x = \{y_0, y_1\}$ Similar.
7. $x = (y_0, y_1) = \{\{y_0\}, \{y_0, y_1\}\}$

5., 6. + Concatenation

8. $x = y_0 \cup y_1$ $\underset{3.}{y_0 \subseteq x} \wedge \underset{3.}{y_1 \subseteq x} \wedge \underset{\text{BDD Q}}{\forall z \in x} (\underset{1.}{z \in y_0} \vee \underset{1.}{z \in y_1})$
9. $x = y_0 \cap y_1$ Similar
10. $x = y_0 \setminus y_1$ Similar
11. $x = \bigcup y$ Similar
12. $x = y \cup \{y\}$ Concat. : 5 + 8.
13. x is a trs set $[\forall y \in x \forall z \in y (z \in x)]$
14. x is an ordered pair
15. $x = y_0 \times y_1$

(14., 15.) ES#1

16. x is a relation $[x \subseteq y_0 \times y_1 \times \dots \times y_{n-1}]$

17. x is a function

18. $x = \mathrm{dom}(f)$

19. $x = \mathrm{ran}(f)$

20. x is injective

21. x is surjective

22. x is bijective

23. x is an ordinal

[$\Longleftrightarrow$ x is transitive $\wedge$ $\in$ is a linear order on x]

This is enough since in transitive models, Foundation holds!

24. x is a successor ordinal

[e.g., $\exists z \in x \, (x = z \cup \{z\}) \wedge$ x is ordinal]

25. x is limit ordinal

26. $x = \omega$ [the unique limit ordinal that has no non-zero limit ordinals as member]

27. All recursively defined operations on ω, such as $+$, $\cdot$, exponentiation etc.