Forcing & the Continuum Hypothesis

LECTURE II

16 October 2025

LECTURE III Saturday 18 Oct 10am MR11

RECAP FROM LECTURE I

Def. If M and N are structures s.t. $M \subseteq N$ (as sets) and φ is a formula in n free variables, we say φ is absolute between M & N if

$$\forall x_1, \dots x_n \in M \quad M \models \varphi(x_1, \dots, x_n) \iff N \models \varphi(x_1, \dots, x_n)$$

Theorem If M is a substructure of N then all quantifier free formulas are absolute between M and N.

SUBSTRUCTURE LEMMA

Examples

FIELDS "being $\sqrt{2}$" is absolute for substructures

SET THEORY "being empty" is not absolute for substructures

CLAIM ε_0 and ε_1 are not absolute for substructures in the language of set theory.

Take $M \models ZFC$, take $m \in M$ s.t. $M \models \varepsilon_0(m)$

Take $n \in M$ s.t. $M \models \varepsilon_1(n)$

Define $N := M \setminus \{m\}$.

Then $N \models \varepsilon_0(n)$.

Thus $M \models \neg\varepsilon_0(n)$ & $N \models \varepsilon_0(n)$.

$$M \models ZFC \wedge \varepsilon_0(m) \wedge \varepsilon_1(n) \wedge \neg\varepsilon_0(n)$$

$$M \setminus \{m\} \models \varepsilon_0(n)$$

PROBLEM If simple set-theoretic formulas are non-absolute, how do we control whether models are models of ZFC? Or any more complicated thing?

Preview

GÖDEL

COHEN

$M \models ZFC \rightsquigarrow N \models ZFC + CH$

Technique of INNER MODELS

So: $N \subseteq M$.

$M \models ZFC \rightsquigarrow N \models ZFC + \neg CH$

Technique of OUTER MODELS

[in particular "forcing"]

So: $M \subseteq N$.

In general: $\subseteq, \supseteq$ is not enough to preserve anything.

However: non-absoluteness is CRUCIAL for independence arguments.

Example. If $M \subseteq N$, $M, N \models ZFC$, M and N have the same ordinals and the following formulas are absolute:

$x = \mathbb{N}$, $x = \mathbb{R}$, $|x| \leq |y|$, $|x| = |y|$

Then $M \models CH \iff N \models CH$.

[Why? $M \models \neg CH \iff M \models \exists \alpha$ ordinal $(|\mathbb{N}| < |\alpha| < |\mathbb{R}|)$]

TRANSITIVE MODELS

Suppose (M, E), (N, E) $\mathcal{L}$-structures where $\mathcal{L}$ is the language of set theory.

$M \subseteq N$

Def. M is transitive in N if

$$\forall x, y \in N \text{ if } \left.\begin{array}{l} y \in M \\ x E y \end{array}\right\} \Longrightarrow x \in M.$$

Special case: If N is the set theoretic universe, then M is transitive in N

$$\Longleftrightarrow$$

M is a transitive set.

Note: If M is transitive in N, then it's not possible that for $u \in M$

$$N \models \neg \varepsilon_0(u),$$

but $M \models \varepsilon_0(u)$.

ABSOLUTENESS

Lecture I

Def. If M and N are structures s.t. $M \subseteq N$ (as sets) and φ is a formula in n free variables, we say φ is absolute between M & N if

$$\forall x_1, \dots x_n \in M \quad M \models \varphi(x_1, \dots, x_n) \Leftrightarrow N \models \varphi(x_1, \dots, x_n)$$

$$\forall x_1 \dots x_n \in M \quad M \models \varphi(x_1, \dots, x_n) \Longleftrightarrow N \models \varphi(x_1, \dots, x_n)$$

absolute between M & N

upwards absolute between M & N $\Longrightarrow$

downwards absolute between M & N $\Longleftarrow$

Special case: if N is the set theoretic universe

$$\forall x_1 \dots x_n \in M \quad M \models \varphi(x_1, \dots, x_n) \Longleftrightarrow \varphi(x_1, \dots, x_n) \text{ is true}$$

absolute for M

upwards abs. for M $\Longrightarrow$

downwards abs. for M $\Longleftarrow$

Obs. 1 ($M \subseteq N$) If φ is absolute for M [upw./downw.] and absolute for N, then φ is absolute between [upw./downw.] M and N

Obs. 2 If M is transitive in N, then $\in_0$ is absolute between M and N.

The extent of absoluteness for transitive models

Lemma If M is a transitive set, then $(M, \in) \models$ Extensionality + Foundation.

Proof. Extensionality:

$$\forall x \forall y \big(\forall w (w \in x \leftrightarrow w \in y) \rightarrow x = y\big)$$

Take $x \neq y$, $x, y \in M$.

By (real) extensionality, w.l.o.g. there is

$$z \in x \setminus y.$$

$$x \in M \quad z \in x \xrightarrow[\text{trs}]{} z \in M$$

So $M \models z \in x \wedge z \notin y$

$\Longrightarrow$ $M \models \neg \forall w (w \in x \leftrightarrow w \in y)$.

Foundation

$$\forall x \big(\underbrace{x \neq \emptyset}_{\neg \varepsilon_0(x)} \rightarrow \exists m (m \in x \wedge \forall w (\neg(w \in m \wedge w \in x)))\big)$$

As before $x \in M$, find by (real) foundation an $\in$-minimal $m \in x$.

$$m \in M \quad \text{trs}$$

Check that m is still $\in$-minimal in M.

q.e.d.

Def. A class of formulas Γ is

CLOSED UNDER BOUNDED QUANTIFICATION

if for $\varphi \in \Gamma$, also

$$\exists x \in y\, \varphi \quad [\longleftrightarrow \exists x (x \in y \wedge \varphi)]$$
$$\forall x \in y\, \varphi \quad [\longleftrightarrow \forall x (x \in y \rightarrow \varphi)]$$

are in Γ.

Def. Δ_0 is the smallest class of formulas which

1. contains all atomic fmlas
2. closed under bdd. quant.
3. closed under $\wedge, \vee, \neg, \rightarrow$.

Def. A fmla is Σ_1 if it is of the form $\exists v_1 \ldots \exists v_n\, \varphi$ for $\varphi \in \Delta_0$.

And Π_1 if it is of the form $\forall v_1 \ldots \forall v_n\, \varphi$ for $\varphi \in \Delta_0$.

Def. If T is any theory, we define

$$\Delta_0^T, \Sigma_1^T, \Pi_1^T$$

to be the class of fmlas that are equivalent in T to a $\Delta_0, \Sigma_1, \Pi_1$ fmla.

Example $\quad \varepsilon_0(x) :\Longleftrightarrow \forall w (w \notin x)$

is Δ_0^T where T is predicate logic

[because it's eq. to $\forall w (w \in x \wedge w \neq w)$]

Theorem $\quad \Delta_0^T$-formulas are absolute for transitive models T.

Proof. By induction on the formula complexity of Δ_0 formulas.

1. Atomic formulas: substructure lemma.
2. Prop. connectives: as in the proof of the substructure lemma
3. Enough to show

$$\varphi \rightsquigarrow \exists x \in y \, \varphi.$$

If $M \models \exists x \in y \, \varphi \Rightarrow M \models \varphi(x, y)$

By IH $\varphi(x, y)$ is true

So $\exists x \in y \, \varphi(x, y)$ is true.

If $\exists x \in y \, \varphi$ is true for $y \in M$,

then $x \in M$ by trs

Thus $M \models \exists x \in y \, \varphi$. $\qquad$ q.e.d.