LECTURE I

Tuesday, 14 October 2025

Forcing & the Continuum Hypothesis (M16)

Professor Benedikt Löwe

The method of forcing is one of the most important model constructions in set theory and its versatility is the reason for the plethora of independence results in set theory. It was developed to solve one of the most important foundational problems of the 20th century: determining the cardinality of the set of real numbers. Cantor's *Continuum Hypothesis* asserts that the cardinality of the set of real numbers has the smallest possible value:

> Every infinite set of reals is either equinumerous with the set of natural numbers or equinumerous with the set of all real numbers. Equivalently, $2^{\aleph_0} = \aleph_1$. (CH)

When he presented his list of twenty-three problems for the 20th century at the *International Congress of Mathematicians* in Paris in 1900, David Hilbert listed the the question whether the Continuum Hypothesis is true as the very first problem on his list. It turned out that this question cannot be solved on the basis of ZFC: in 1938, Kurt Gödel showed that CH cannot be disproved in ZFC; in 1963, Paul Cohen invented the method of forcing to show that CH cannot be proved in ZFC.

In this course we shall study the basics of the method of forcing in order to present Cohen's proof. The course will discuss:

1. models of set theory (in particular, transitive models, the Lévy reflection theorem, and the theory of absoluteness);
2. the construction of the generic extension of a countable transitive model (in particular, the syntactic and semantic forcing relation, the forcing theorem, and the generic model theorem);
3. showing relative consistency proofs with the method of forcing;
4. applications to cardinal arithmetic (in particular, preservation results and the consistency of $\neg$CH and related results).

David Hilbert

Hilbert in 1912

Born	23 January 1862 Königsberg or Wehlau, Kingdom of Prussia
Died	14 February 1943 (aged 81) Göttingen, Nazi Germany
Education	University of Königsberg (PhD)

Mathematische Probleme.

Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900.

Von

D. Hilbert.

Wer von uns würde nicht gern den Schleier lüften, unter dem die Zukunft verborgen liegt, um einen Blick zu werfen auf die bevorstehenden Fortschritte unsrer Wissenschaft und in die Geheimnisse ihrer Entwickelung während der künftigen Jahrhunderte! Welche besonderen Ziele werden es sein, denen die führenden mathematischen Geister der kommenden Geschlechter nachstreben? welche neuen Methoden und neuen Thatsachen werden die neuen Jahrhunderte entdecken — auf dem weiten und reichen Felde mathematischen Denkens?

<u>The Continuum Problem</u>

WHAT IS THE SIZE OF $\mathbb{R}$?

Which uncountable cardinal is $|\mathbb{R}|$?

CH — Continuum Hypothesis: The smallest possible: $2^{\aleph_0} = \aleph_1$.

Kurt Gödel

Gödel c. 1926

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary (now Brno, Czech Republic)
Died	January 14, 1978 (aged 71) Princeton, New Jersey, U.S.

Paul Cohen

American mathematician

Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

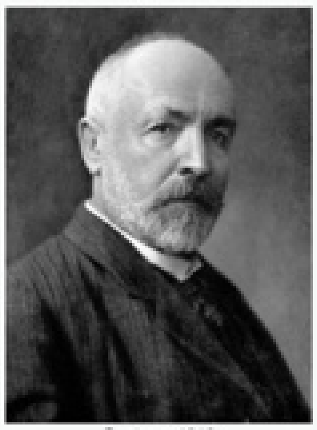
Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

1938 (Gödel): We can't refute CH.

1962 (Cohen): We can't prove CH.

Georg Cantor



Cantor, c.1910

Born Georg Ferdinand Ludwig Philipp Cantor
3 March 1845
Saint Petersburg, Russian Empire

Died 6 January 1918 (aged 72)
Halle, Province of Saxony, German Empire

Friedrich Moritz Hartogs

Friedrich Hartogs

Born 20 May 1874
Brussels

Died 18 August 1943 (aged 69)
Munich

CANTOR'S THEOREM

$\forall X \exists Y$ there is no surjection from X to Y

More specifically
$Y := \wp(X)$,
the power set of X

If κ is a cardinal, we write

$$2^{\kappa} := |\wp(\kappa)|$$

Know $\kappa < 2^{\kappa}$.

HARTOGS'S THEOREM

$\forall X \exists \alpha$ α ordinal and there is no injection from α to X

More specifically
$\alpha := \aleph(X)$,
the Hartogs aleph of X

What is $\aleph(X)$?

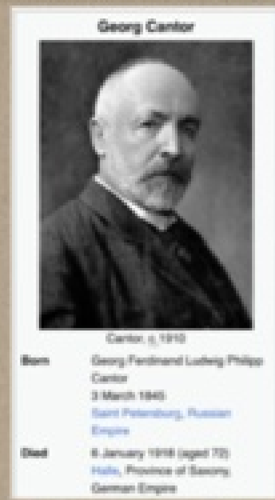
Consider
$W := \{(A,<) ; (A,<)$ is a wellorder and $A \subseteq X\}$

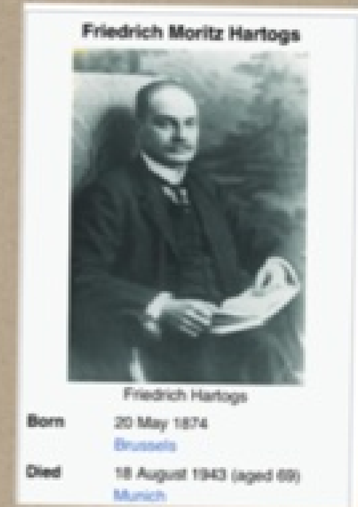
$\cong$ isomorphism

$W/\cong$ has a wellorder $<$: being isomorphic to a proper initial segment

$\aleph(X)$ is the unique ordinal isomorphic to $(W/\cong, <)$.

$\aleph(X)$ is the smallest ordinal $> |X|$.





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BETH

$$\beth_0 := \mathbb{N}$$
$$\beth_{\alpha+1} := 2^{\beth_\alpha}$$
$$\beth_\lambda := \bigcup_{\alpha<\lambda} \beth_\alpha$$
(λ limit)

$$\aleph_0 := \mathbb{N}$$
$$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$$
$$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha \quad \lambda \text{ limit}$$

ALEPH HIERARCHY

Obviously: $\forall \alpha \ \aleph_\alpha \leq \beth_\alpha$.

Since $|\mathbb{R}| = |\wp(\mathbb{N})| = 2^{\aleph_0} = 2^{\beth_0} = \beth_1$, we can define

CH: $\aleph_1 = \beth_1$

GCH: $\forall \alpha \ \aleph_\alpha = \beth_\alpha$.

Generalised Continuum Hypothesis

$\mathrm{ZFC} \nvdash \neg \mathrm{CH}$ $\qquad$ $\mathrm{ZFC} \nvdash \mathrm{CH}$

"CH is independent of ZFC".

This can't be what G. & C. actually proved due to the incompleteness theorem. Instead:

RELATIVE CONSISTENCY PROOFS:

$$\mathrm{Cons}(\mathrm{ZFC}) \Longrightarrow \mathrm{ZFC} \nvdash \neg \mathrm{CH}$$

[eq. $\mathrm{Cons}(\mathrm{ZFC}+\mathrm{CH})$]

$$\mathrm{Cons}(\mathrm{ZFC}) \Longrightarrow \mathrm{ZFC} \nvdash \mathrm{CH}.$$

[eq. $\mathrm{Cons}(\mathrm{ZFC}+\neg\mathrm{CH})$]

By Completeness, equivalent to
if there is a model of ZFC,
then there is a model of ZFC+CH.

If there is $M \models \mathrm{ZFC}$,
then there is $N \models \mathrm{ZFC}+\neg\mathrm{CH}$.

Gödel's Completeness Theorem

$$\Phi \nvdash \varphi \iff \text{there is a model } M \models \Phi + \neg\varphi$$

Gödel's Incompleteness Theorem

$$\mathrm{ZFC} \vdash \mathrm{Cons}(\mathrm{ZFC}) \iff \neg\mathrm{Cons}(\mathrm{ZFC}) \iff \forall\varphi\ \mathrm{ZFC} \vdash \varphi$$

An analogy from algebra

$\mathcal{L}$ language of fields

$\{+, -, 0, \cdot, {}^{-1}, 1\}$

Φ is the set of axioms of fields of char. zero.

$\overline{\varphi}(x) := \quad x \cdot x = 1 + 1.$

"x is a square root of 2"

$\varphi := \quad \exists x \, \overline{\varphi}(x)$

Claim φ is independent of Φ:

$\Phi \nvdash \varphi$ and $\Phi \nvdash \neg\varphi$.

Examples $\mathbb{Q} \models \neg\varphi \qquad \mathbb{Q}(\sqrt{2}) \models \varphi.$

Note that if $K \supseteq \mathbb{Q}(\sqrt{2})$, then

$K \models \overline{\varphi}(\sqrt{2}).$

Q Why is checking $K \models \overline{\Phi}$ so easy in the case of fields and why does $\overline{\varphi}$ persist?

Some model-theoretic basics

Let $\mathcal{L}$ be any language and M and N are $\mathcal{L}$-structures. Then M is a **SUBSTRUCTURE** of N if

(a) $M \subseteq N$ [as sets]

(b) $c^M = c^N$ for every constant symbol c

(c) $R^M = R^N \cap M^k$ for every k-ary relation symbol R

(d) $f^M = f^N \restriction M^k$ for every k-ary function symbol f

Def. If M and N are structures s.t. $M \subseteq N$ (as sets) and φ is a formula in n free variables, we say φ is absolute between M & N if

$$\forall x_1, \dots x_n \in M \quad M \models \varphi(x_1, \dots, x_n) \iff N \models \varphi(x_1, \dots, x_n).$$

Theorem If M is a substructure on N, then all quantifier-free formulas are absolute between M and N.

Why is this different in set theory?

Language of set theory: $\{\in\}$

In particular, no symbols for:

$$\emptyset, \cap, \cup, \setminus, \mathbb{N}, \ldots$$

All of these are defined in terms of $\in$!

Ex.

$$\underbrace{x = \emptyset}_{\varepsilon_0(x)} \;:\Longleftrightarrow\; \forall z\,(z \notin x)$$

$\neg z \in x$

This has a quantifier!

$$\varepsilon_1(x) := \forall z\,(z \in x \longleftrightarrow \varepsilon_0(z))$$

"$x = \{\emptyset\}$"

CLAIM ε_0 and ε_1 are not absolute for substructures in the language of set theory.

Take $M \models ZFC$, take $m \in M$ s.t. $M \models \varepsilon_0(m)$

Take $n \in M$ s.t. $M \models \varepsilon_1(n)$.

Define $N := M \setminus \{m\}$.

Then $N \models \varepsilon_0(n)$.

Thus $M \models \neg\varepsilon_0(n)$ & $N \models \varepsilon_0(n)$.