Forcing & the Continuum Hypothesis
Michaelmas Term 2025
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #3

Examples Classes.

Examples Class #3. Monday 19 January 2026, 3:30–5:30pm, **MR2**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it in on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

(29) We call $p \in \mathbb{P}$ an *atom* if there are no $q, r \leq p$ such that $q \perp r$. We call $\mathbb{P}$ *non-atomic* (sometimes called *splitting*) if it has no atoms. Prove that $\mathbb{P}$ is non-atomic if and only if for all G that are $\mathbb{P}$-generic over M, we have $G \notin M$.

(30) Presentation Example. Let M be a ctm, $\mathbb{P} \in M$, and G be $\mathbb{P}$-generic over M. Assume the Forcing Theorem. Complete the proof of the powerset axiom from p. 6 of Lecture XI. Show that if σ and μ are names such that $\mathrm{val}(\mu, G) \subseteq \mathrm{val}(\sigma, G)$ and $\mu^* := \{(\tau, p)\,;\, \tau \in \mathrm{dom}(\sigma) \wedge p \Vdash \tau \in \mu\}$, then $\mathrm{val}(\mu, G) = \mathrm{val}(\mu^*, G)$. Why does this complete the proof of the powerset axiom?

(31) Let M be a ctm, $\mathbb{P} \in M$, and G be $\mathbb{P}$-generic over M. Furthermore, let $E \subseteq \mathbb{P}$, $E \in M$, and $p, q \in \mathbb{P}$. Prove the following.

(i) If E is dense below p and $q \leq p$, then E is dense below q.

(ii) If $\{r\,;\, E \text{ is dense below } r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \varnothing$ or there is $q \in G$ such that for all $r \in E$, we have $r \perp q$.

(iv) If $p \in G$ and E is dense below p, then $G \cap E \neq \varnothing$.

(32) Complete the proof of the Forcing Theorem given in the lectures by doing the induction steps for $\wedge$ and $\exists$.

(33) Let M be a ctm and $\mathbb{P} \in M$ be a non-atomic partial order (cf. (29)). Let $M_0 := M$ and iterative define a family of ctms. Assume M_i is defined, pick some G_i is $\mathbb{P}$-generic over M_i, and let $M_{i+1} := M_i[G_i]$. Prove that $\bigcup_{i \in \mathbb{N}} M_i$ is not a model of ZFC.

(34) Let κ be an infinite cardinal and $\vartheta > \kappa$ a regular cardinal such that for all $\alpha < \vartheta$, we have that $|\alpha^{<\kappa}| < \vartheta$. Show the Δ *System Lemma*: If A is any family of sets of size $< \kappa$ such that $|A| = \vartheta$, then there is a Δ-system $D \subseteq A$ such that $|D| = \vartheta$.

(35) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i\colon \mathbb{P} \to \mathbb{Q}$ is called a *complete embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility in both directions, i.e., $p \perp_{\mathbb{P}} p'$ if and only if $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) for all $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ such that for all $p' \leq_{\mathbb{P}} p$, we have that $i(p')$ and q are compatible in $\mathbb{Q}$.

Suppose that M is a ctm, $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding with $i, \mathbb{P}, \mathbb{Q} \in M$ and let H be $\mathbb{Q}$-generic over M. Show that $G := \{p \in \mathbb{P}\,;\, i(p) \in H\}$ is $\mathbb{P}$-generic over M and that $M[G] \subseteq M[H]$.

(36) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility, i.e., if $p \perp_{\mathbb{P}} p'$, then $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) the image of $\mathbb{P}$ under i is dense in $\mathbb{Q}$.

Show that every dense embedding is a complete embedding.

(37) We say that $\mathbb{P}$ *preserves cofinalities* if for every $\mathbb{P}$-generic filter G over M and every limit ordinal $\lambda \in M$, we have that $\mathrm{cf}(\lambda)^M = \mathrm{cf}(\lambda)^{M[G]}$. Prove that if $\mathbb{P}$ preserves cofinalities, then it preserves cardinals.

(38) Prove Hausdorff's formula: for any α and β, we have that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$.

(39) Let M be a ctm, $\mathbb{P} := \mathrm{Fn}(\aleph_\omega^M \times \omega, 2) \in M$, and G $\mathbb{P}$-generic over M. Analyse upper and lower bounds for $2^{\aleph_0}$ in $M[G]$ in the style of Lectures XIV & XV. Note that it is not possible that $2^{\aleph_0} = \aleph_\omega$, so not everything can go through exactly as we did it in Lecture XV. Explain what is different.

[*The supplementary material on cofinality may be useful.*]

(40) Let M be a ctm, $\mathbb{P}_n := \mathrm{Fn}(\omega, \omega_n^M) \in M$, and G_n be $\mathbb{P}_n$-generic over M. Let N be a ctm such that $\{G_n\,;\, n \in \omega\} \subseteq N$. Show that $\aleph_\omega^M$ is countable in N.

(41) Presentation Example. Show that if ZFC is consistent, then so is $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_{\omega_1}$.

(42) Let M be a ctm such that $M \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, $\mathbb{P} := \mathrm{Fn}(\omega, \omega_1^M) \in M$, and G $\mathbb{P}$-generic over M. Show that $M[G] \models \mathsf{CH}$.

(43) Let M be a ctm, $\lambda \leq \kappa$ be cardinals in M, $\mathbb{P} := \mathrm{Fn}(\kappa \times \lambda^+, 2, \lambda^+) \in M$, and G $\mathbb{P}$-generic over M. Show that in $M[G]$ there is a surjection from λ^+ onto $\wp(\lambda) \cap M$. Use this to show that if $M \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, then forcing with $\mathrm{Fn}(\omega_3^M \times \omega_1^M, 2, \omega_1^M)$ over M does not give a model of $2^{\aleph_1} = \aleph_3$.