Forcing & the Continuum Hypothesis
Michaelmas Term 2025
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #2

Examples Classes.

Examples Class #2. Monday 17 November 2025, 3:30–5:30pm, **MR3**.
Examples Class #3. Monday 19 January 2026, 3:30–5:30pm, **MR2**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it in on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

(15) An ordinal γ is called a *gamma number* if for all $\alpha, \beta < \gamma$, we have $\alpha + \beta < \gamma$.

- (i) Show that $\gamma > 0$ is a gamma number if and only if there is an ordinal ξ such that $\gamma = \omega^\xi$.
- (ii) Show that if M is a countable transitive model of ZFC and $\gamma := \mathrm{Ord} \cap M$, then γ is a gamma number.
- (iii) Show that there is some $E \in \mathbf{L}_{\omega+1}$ such that (ω, E) is isomorphic to a gamma number $\gamma > \omega$.

(16) Let $\{Z_\alpha\,;\, \alpha \in \mathrm{Ord}\}$ be a hierarchy in the sense of Lecture IV. Assume that for all α there is a γ such that $\mathcal{D}(Z_\alpha) \subseteq Z_\gamma$. Show that $Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$ is a model of the powerset axiom.

(17) Show that the axiom schema of replacement holds in $\mathbf{L}$.

(18) For an ordinal α, we say that α is *ω-productive* if $\wp(\omega) \cap \mathbf{L}_{\alpha+1} \backslash \mathbf{L}_\alpha \neq \varnothing$. Show that

- (i) ω is ω-productive;
- (ii) no ordinal which is uncountable in $\mathbf{L}$ is ω-productive;
- (iii) for every ordinal α which is countable in $\mathbf{L}$, there is an ω-productive $\gamma > \alpha$.

(19) Let x be any transitive set and define by transfinite recursion:

$$\mathbf{L}_0(x) := x, \qquad \mathbf{L}_{\alpha+1}(x) := \mathcal{D}(\mathbf{L}_\alpha(x)), \qquad \mathbf{L}_\lambda(x) := \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha(x) \text{ (for } \lambda \text{ limit).}$$

Show that

- (i) if x is countable and $\alpha \geq \omega$, then $|\mathbf{L}_\alpha(x)| = |\alpha|$; and
- (ii) if x is countable, then $\mathbf{L}(x) \models \mathsf{GCH}$.

(20) Using the notion of a beth fixed point from (1), show that for $\lambda > \omega$, we have that $|\mathbf{V}_\lambda| = |\mathbf{L}_\lambda|$ if and only if λ is a beth fixed point.

(21) Presentation Example. Using the models from (19), let α_x be the ordinal such that $\mathbf{L}(x) \models$"α_x is the least uncountable cardinal". Show that if for all $x \subseteq \omega$, we have that $\alpha_x < \aleph_1$, then $\aleph_1$ is a regular limit cardinal in $\mathbf{L}$.

(22) Assume that ZFC is consistent. Show that there is a finite $T \subseteq$ ZFC such that ZFC does not prove "there is a transitive model of $T + \neg$CH".

(23) Let $(\mathbb{P}, \leq)$ be a partial order and $p \in \mathbb{P}$. Show that

  (a) if D is dense below p and $r \leq p$, then D is dense below r;

  (b) if $\{r\,;\, D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

(24) We say that G is $\mathbb{P}$-*antichain generic over* M if for every maximal $\mathbb{P}$-antichain $A \in M$, we have $A \cap G \neq \varnothing$. We call a set B a $\mathbb{P}$-*bar* if for every $p \in \mathbb{P}$ there is a $b \in B$ such that p and b are compatible. We say that G is $\mathbb{P}$-*bar generic over* M if for every $\mathbb{P}$-bar $B \in M$ we have that $B \cap G \neq \varnothing$.

Let $\mathbb{P} \in M$, and G be a filter over $\mathbb{P}$. Show that the following are equivalent:

  (i) G is $\mathbb{P}$-generic over M,

  (ii) G is $\mathbb{P}$-antichain generic over M, and

  (iii) G is $\mathbb{P}$-bar generic over M.

(25) Presentation Example. Let M be a transitive model of ZFC. Consider the partial order $\mathbb{P}$ of finite binary sequences, ordered by the initial segment relation and assume that G is $\mathbb{P}$-generic over M. Define $c(n) := s(n)$ for some $s \in G$ with $n \in \mathrm{dom}(s)$ and $C := \{n : c(n) = 1\}$.

  (i) Show that $c\colon \mathbb{N} \to \mathbb{N}$ is well-defined.

  (ii) Let $s \in 2^{<\omega}$ be any finite binary sequence. Show that it occurs infinitely often in c.

  (iii) Let $x\colon \mathbb{N} \to \mathbb{N}$. For $k > 0$, an increasing finite or infinite sequence of natural numbers is called a k-*progression in* x if it is of the form $n, n+k, n+2k, n+3k, \ldots$ and $x(m) = 1$ for all m occurring in the sequence. Show that for each $k > 0$, there are arbitrarily long k-progressions in c, but there is no infinite k-progression in c.

  (iv) Let $X \subseteq \mathbb{N}$ be infinite and $X \in M$. Show that both $X \cap C$ and $X \backslash C$ are infinite. (We say that "C splits X".) Why can this not be in general true for $X \notin M$?

(26) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Suppose $\sigma, \tau \in \mathrm{Names}(M, \mathbb{P})$ and that D is a filter on $\mathbb{P}$. Show that $\mathrm{val}(\sigma \cup \tau, D) = \mathrm{val}(\sigma, D) \cup \mathrm{val}(\tau, D)$.

(27) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and $x \in M$. Recursively define $\mathrm{can}(x) := \{(\mathrm{can}(y), p)\,;\, y \in x \wedge p \in \mathbb{P}\}$. Show that if D is a filter, then $\mathrm{val}(\mathrm{can}(x), D) = x$.

(28) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and D is a filter on $\mathbb{P}$ with $D \neq \mathbb{P}$. Show that there is a proper class of names τ such that $\mathrm{val}(\tau, D) = \varnothing$.