Forcing & the Continuum Hypothesis
Michaelmas Term 2025
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

Examples Classes.

Examples Class #1. Monday 3 November 2025, 3:30–5:30pm, **MR3**.
Examples Class #2. Monday 17 November 2025, 3:30–5:30pm, **MR3**.
Examples Class #3. Monday 19 January 2026, 3:30–5:30pm, **MR2**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it in on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

(1) Consider the $\aleph$ and $\beth$ hierarchies and show (in ZFC) that for every infinite X there is an α such that $|X| = \aleph_\alpha$ and that there are cardinals κ and λ such that $\kappa = \aleph_\kappa$ and $\lambda = \beth_\lambda$ (called an *aleph fixed point* and a *beth fixed point*, respectively). Show that the least such cardinals have countable cofinality (i.e., are a union of countably many strictly smaller ordinals).

(2) Prove the *Substructure Lemma* as stated in Lecture I ("quantifier-free formulas are absolute for substructures").

(3) Let M and $E \subseteq M \times M$ be sets such that $(M, E) \models \mathsf{ZFC}$ and $e \in M$ such that $(M, E) \models$"e is empty". Consider $N := M \backslash \{e\}$ as in Lecture I and check which of the axioms of ZFC hold in (N, E) (or, more pedantically, (N, E') where $E' := E \cap (N \times N)$).

(4) Show that the following properties can be expressed by formulas that are absolute between transitive models.

(a) $z = y \times y$;
(b) z is a function;
(c) z is a group;
(d) z is a linear order;
(e) z is a set with exactly two elements.

(5) Presentation Example. Show that if ZFC is consistent, then there is a model $(M, E) \models \mathsf{ZFC}$ such that E is an ill-founded relation. Can you build such an M such that (N, E) is ill-founded where $N := \{x \in M \,;\, (M, E) \models$"$x$ is a natural number"$\}$?

(6) Show that the following properties can be expressed by formulas that are absolute between transitive models.

(a) $z = \omega \cdot 3$;
(b) $z = \omega^2$;
(c) $z = \omega^2 + \omega$.

(7) Prove the *Tarski-Vaught Test* as stated in Lecture IV:

Suppose $\mathcal{L}$ is any language, M and N are $\mathcal{L}$-structures such that M is a substructure of N, and Φ is any set of $\mathcal{L}$-formulas closed under subformulas. Then all formulas in Φ are absolute between M and N if and only if for any formula $\exists x\varphi \in \Phi$ where φ has $n+1$ free variables including x, we have that

> if $m_1, ..., m_n \in M$ and $N \models \exists x\varphi(x, m_1, ..., m_n)$, then there is some $m \in M$ such that $N \models \varphi(m, m_1, ..., m_n)$.

(8) Show that $\alpha \mapsto \mathbf{V}_\alpha$ is a hierarchy in the sense of Lecture IV.

(9) The *Mirimanoff rank* is defined by $\varrho(x) := \min\{\alpha \,;\, x \in \mathbf{V}_{\alpha+1}\}$. Show that for any ordinal α and any sets x and y, the following hold:

(a) if $x \in y$, then $\varrho(x) < \varrho(y)$;
(b) $\varrho(x) = \sup\{\varrho(y) + 1 \,;\, y \in x\}$;
(c) $\varrho(\alpha) = \alpha$.

(10) Presentation Example. Let M be a transitive set model of ZFC and let $\alpha \in M$ be such that $(M, \in) \models$"α is the least uncountable cardinal". Show that if α is countable, then there is some $A \subseteq \omega$ such that $A \notin M$.

(11) Let $\vartheta \geq \omega + 2$ be an ordinal and assume that $M \subseteq \mathbf{V}_\vartheta$ is a countable elementary submodel constructed using the Tarski-Vaught test, i.e., by iteratively collecting witnesses to all existential formulas true in $\mathbf{V}_\vartheta$ (cf. Lecture V). Show that M is not transitive. What happens in the cases $\vartheta = \omega$ and $\vartheta = \omega + 1$?

(12) Let $\Sigma := \{\in, =, \wedge, \vee, \neg, \exists, \forall, (,)\}$ and V be a countable set of variables disjoint from Σ. Fix some bijection $c \colon \Sigma \cup V \to \omega$. Give the recursive definition that specifies which finite ω-sequences are (codes for) well-formed formulas and argue that this set is absolute for transitive models of some finite $T \subseteq \mathsf{ZFC}$. Discuss what T is.

(13) Fix a set X and $E \subseteq X \times X$. Using your encoding from (12), provide a recursive definition of "$(X, E) \models \varphi$" that is absolute for transitive models of some finite $T \subseteq \mathsf{ZFC}$ containing X and E. Discuss what T is. Highlight why it is important that X is an element of the model. Use this to argue that $D(\varphi, p, X)$ and $\mathcal{D}(X)$, as defined in Lecture VI, are absolute operations for transitive models of some finite T^*. Again, discuss what T^* is.

(14) Show that $\alpha \mapsto \mathbf{L}_\alpha$ is a hierarchy in the sense of Lecture IV.