XXII

THE ULTIMATE & TWENTY-SECOND LECTURE OF

AUTOMATA & FORMAL LANGUAGES

Thursday, 5 December 2024

$$WP \equiv_m \mathbb{K}_0 = \{(w,v);\ w \in W_v\}$$

$$Emp \equiv_m \mathbb{E}\mathrm{mp} = \{w;\ \emptyset = W_w\}$$

$$Eq \equiv_m \mathbb{E}\mathrm{q} = \{(w,v);\ W_w = W_v\}$$

Note. This identification requires the translation of grammars to RM [C 4.38]

$$G \longmapsto h_0(G)$$

and RM to grammars [not given in lectures]

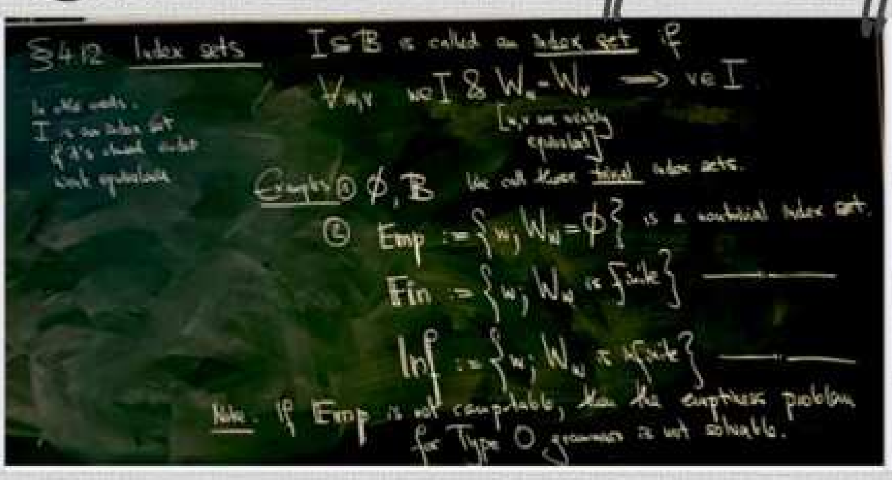
$$M \longmapsto h_1(M)$$

such that

$$w \in \mathcal{L}(G) \iff w \in W_{h_0(G)}$$

$$w \in W_M \iff w \in \mathcal{L}(h_1(M))$$

Still to do: $\mathbb{E}\mathrm{mp}$ & $\mathbb{E}\mathrm{q}$ are not computable.



Index sets

closed under weak equivalence

Trivial $\emptyset$ & $\mathbb{B}$

$\mathbb{E}\mathrm{mp}$ is a non-trivial index set

Theorem 4.48 (RICE)

No non-trivial index set is computable.

Proof. **Preparation** Fix w and consider

$$g_w(u,v) := \begin{cases} f_{w,1}(v) & \text{if } u \in \mathbb{K} \\ \uparrow & \text{if } u \notin \mathbb{K}. \end{cases}$$

Note: This is not an allowed case distinction by the case-distinction lemma.

However, here it works: check $u \in \mathbb{K}$. If answer is Yes, compute $f_{w,1}(v)$. If never receive answer, then $\uparrow$ which is what we want.

Thus g_w is computable, so by s-m-n there is total computable h_w s.t. $f_{h_w(u),1}(v) = g_w(u,v)$

Corollary $\mathbb{E}\mathrm{mp}$ is not computable $\Rightarrow$ the emptiness problem for Type 0 grammars is not solvable.

Case 1 If $v \in \mathbb{K}$, then $f_{h_w(v),1} = f_{w,1}$ $\Longrightarrow W_{h_w(v)} = W_w$ $(\dagger)$

If $v \notin \mathbb{K}$, then $f_{h_w(v),1}$ is nowhere defined. $\Longrightarrow W_{h_w(v)} = \emptyset$ $(\ddagger)$

Get to the proof. Let I be a nontrivial index set. Fix some e s.t. $W_e = \emptyset$.

Two cases: $e \in I$ or $e \notin I$.

Case 1 $e \in I$. Find $w \notin I$. Consider g_w and h_w as in the preparation.

CLAIM h_w is a reduction witnessing $\mathbb{N} \setminus \mathbb{K} \leq_m I$.

[$v \notin \mathbb{K} \xRightarrow{(\ddagger)} W_{h_w(v)} = \emptyset$, so e & $h_w(v)$ are weakly equivalent

$\Longrightarrow h_w(v) \in I$.

$v \in \mathbb{K} \xRightarrow{(\dagger)} W_{h_w(v)} = W_w$, so $h_w(v)$ & w are weakly eq.

$\Longrightarrow h_w(v) \notin I$.]

Case 2: $e \notin I$. Fix $w \in I$ by nontriviality. Consider g_w & h_w as in preparation.

Claim: h_w witnesses $\mathbb{K} \leq_m I$.

$$\Big[\; v \in \mathbb{K} \overset{(\dagger)}{\Longrightarrow} W_{h_w(v)} = W_w \text{, so } h_w(v) \,\&\, w \text{ are weakly eq} \Longrightarrow h_w(v) \in I.$$

$$v \notin \mathbb{K} \overset{(\ddagger)}{\Longrightarrow} W_{h_w(v)} = \emptyset \text{, so } h_w(v) \,\&\, e \text{ are weakly eq} \Longrightarrow h_w(v) \notin I \;\Big]$$

q.e.d.

Remarks

① The proof shows more:

$e \in I \Rightarrow \mathbb{B}\setminus\mathbb{K} \leq_m I$ (i.e., I is not c.e.)

$e \notin I \Rightarrow \mathbb{K} \leq_m I$.

② ① implies that $\mathbb{B}\setminus\mathbb{K} \leq_m$ Emp, Fin.

$\mathbb{K} \leq_m$ Inf

③ In ES#4: Emp $\equiv_m \mathbb{B}\setminus\mathbb{K}$, so Emp is Π_1-complete.

④ Can show (please read the proof in the typed notes) that Fin and Inf are neither Σ_1 nor Π_1 by showing that $\mathbb{K}\oplus\mathbb{B}\setminus\mathbb{K} \leq_m$ Fin, Inf (Prop 4.50).

§ 4.13 Decision Problems

	TYPE	WP	Emp	Eq
regular	3	✓	✓ C 2.27	✓ T 2.30 MINIMISATION
context-free	2	✓	✓ C 3.15 PUMPING LEMMAS	✗ NOT PROVED IN LECTURES
noncontracting	1	✓ Thm 1.21	✗	✗
c.e.	0	✗ C 4.42	✗ C 4.49	✗ Cor 4.51

Goal for Lectures XXI & XXII.

Cor. 4.51 Eq is not computable.

Pf. Suppose e is such that $W_e = \emptyset$ & consider the map $w \longmapsto (w,e)$. This is an operation that can be performed by a RM.

Note
This proof means that solvability of Eq. pr. implies solvability of Emp pr!

$$\chi'(w) := \chi_{\mathbb{E}q}(w,e) = \begin{cases} 1 & \text{if } W_w = W_e = \emptyset \\ 0 & \text{if } W_w \neq \emptyset \end{cases}$$

$$= \chi_{\mathbb{E}mp}(w)$$

Thus $\chi_{\mathbb{E}q}$ computable $\Rightarrow$ $\chi_{\mathbb{E}mp}$ computable

Thus by C4.49, $\chi_{\mathbb{E}q}$ not computable. q.e.d.