# XXI

Twenty-first and (what we all hope to be) the PENULTIMATE Lecture of AUTOMATA & FORMAL LANGUAGES

Tuesday, 3 December 2024

**Finally**

$X \subseteq \mathbb{N}^k$ is a "problem"

$X$ is SOLVABLE iff $X$ is computable

Thus: $\mathbb{K} \subseteq \mathbb{B}$ the HALTING PROBLEM is not solvable

$\mathrm{WP} := \{(w,v) ;\ w \in \mathcal{L}(G_v)\}$ is not solvable (Corollary 4.42)

| TYPE | | WP | Emp | Eq |
|---|---|---|---|---|
| regular | 3 | ✓ | ✓ C 2.27 | ✓ T 2.30 MINIMISATION |
| context-free | 2 | ✓ | ✓ C 3.15 PUMPING LEMMAS | ✗ NOT PROVED IN LECTURES |
| noncontracting | 1 | ✓ Thm 1.21 | ✗ | ✗ |
| c.e. | 0 | ✗ C 4.42 | ? | ? |

Goal for Lectures XXI & XXII.

# § 4.11 Reductions

$(X, \leq)$ partial order if $\leq$ is reflexive, transitive, & anti-symmetric.

$(X, \leq)$ partial preorder if $\leq$ is reflexive and transitive

If $(X, \leq)$ is a partial preorder, define $x \equiv y :\Leftrightarrow x \leq y \;\&\; y \leq x$

$\equiv$ is an equivalence relation respecting $\leq$

Then $(X/_{\equiv}, \leq)$ is a partial order. ES#4.

Eq. classes are called **≡-degrees**.

**Def.** A total computable function $f : \mathbb{B} \longrightarrow \mathbb{B}$ is called a reduction from $L'$ to $L$ (of $L$ to $L'$) if $\forall w \quad w \in L \Longleftrightarrow f(w) \in L'$.

We say $L$ is many-one reducible to $L'$. Symbols: $L \leq_m L'$.

Clearly, $\leq_m$ is a partial preorder. It's $\underline{\equiv_m\text{-degrees}}$ are called DEGREES OF UNSOLVABILITY.

<u>Proposition</u> 4.43.

(a) If $L \leq_m L'$ & $L'$ is computable $\Rightarrow$ $L$ is computable.

(b) ―――――― c.e. $\Rightarrow$ ―― c.e.

<u>Proof.</u> If $f$ is the red. witnessing $L \leq_m L'$, then

$$\chi_L = \chi_{L'} \circ f$$

$$\psi_L = \psi_{L'} \circ f \qquad \text{q.e.d.}$$

<u>Remarks</u>

① If $L \leq_m L'$, then $\mathbb{B}\setminus L \leq_m \mathbb{B}\setminus L'$.

② Main application.

$\mathbb{K} \leq_m L \Rightarrow L$ is not computable

Similarly,

$\mathbb{B}\setminus\mathbb{K} \leq_m L \rightarrow L$ is not c.e.

$W_w := \mathrm{dom}(f_{w,1})$

③ If the coding fns are computable, then

$Emp = \{w;\ \mathcal{L}(G_w) = \phi\} \equiv_m \{w;\ W_w = \phi\}$

$Eq = \{(w,v);\ \mathcal{L}(G_w) = \mathcal{L}(G_v)\} \equiv_m \{(w,v);\ W_w = W_v\}$

If $\mathcal{C}$ is a class of languages, we say that $X$ is **$\mathcal{C}$-hard** if

$$\forall L \in \mathcal{C} \quad L \leq_m X.$$

$X$ is at least as complicated as every element in $\mathcal{C}$.

and $X$ is **$\mathcal{C}$-complete** if it's $\mathcal{C}$-hard and $X \in \mathcal{C}$.

$X$ is the most complicated elt in $\mathcal{C}$.

**Prop 4.45** If $L$ is computable, $L \neq \emptyset, \mathbb{B}$, then $L$ is $\Delta_1$-complete.

**Proof** Note $\Delta_1 =$ computable.
By assumption, take $v \in L$, $u \notin L$, let $X$ be computable.

$$g(w) := \begin{cases} v & \text{if } w \in X \\ u & \text{if } w \notin X \end{cases}$$

Clearly, $g$ is total and computable and witnesses $X \leq_m L$.

q.e.d.

Theorem 4.26 allows us to simplify our notation in a natural way: instead of using the register machine $M$ as parameter of our computable functions, we can define for arbitrary words $v$

$$f_{v,k}(\vec{w}) := f_{U,2}(v, \mathrm{code}(q_S, \vec{w})).$$

If $v = \mathrm{code}(M)$, this partial function coincides with $f_{M,k}$; if $v$ is not the code of a register machine, it'll give the nowhere defined partial function. We extend this notation to the computably enumerable sets $\mathrm{W}_M$ by writing $\mathrm{W}_w := \mathrm{dom}(f_{w,1})$ and to truncations by writing $T_v := T_M$ for $v = \mathrm{code}(M)$.

**Theorem 4.27** (The *s-m-n* Theorem). Let $g\colon \mathbb{B}^{k+1} \dashrightarrow \mathbb{B}$ be any partial computable function. Then there is a total computable function $h\colon \mathbb{B} \to \mathbb{B}$ such that for all $v \in \mathbb{B}$ and all $\vec{w} \in \mathbb{B}^k$, we have $f_{h(v),k}(\vec{w}) = g(\vec{w}, v)$.

The curious name of this theorem derives from the notation $S^m_n$ used for the

**Haskell Brooks Curry**

| | |
|---|---|
| **Born** | September 12, 1900<br>Millis, Massachusetts, US |
| **Died** | September 1, 1982 (aged 81)<br>State College, Pennsylvania, US |
| **Nationality** | American |

Currying.

**Thm 4.46** $\mathbb{K}$ is $\Sigma_1$-complete.

Proof. Note $\Sigma_1 = $ c.e.
Pick $f$ s.t. $X = \mathrm{dom}(f)$.

Define

$$g : \mathbb{B}^2 \dashrightarrow \mathbb{B}$$
$$(w,v) \longmapsto f(w)$$

Clearly computable.

Apply s-m-n to get total computable $h$ s.t.

$$f_{h(w),1}(v) = g(w,v) = f(w)$$

Claim $h$ reduces $X$ to $\mathbb{K}$.

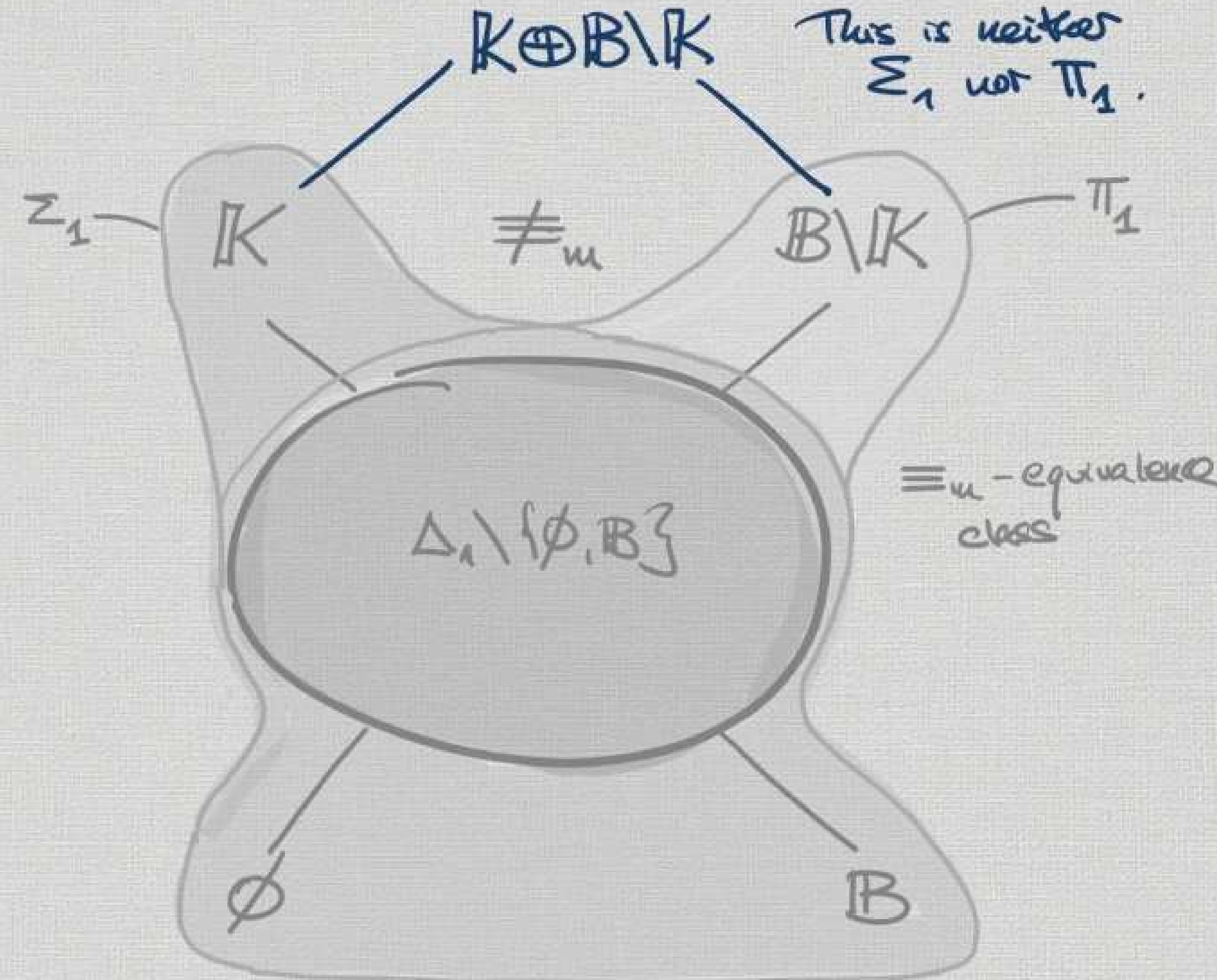
(1) $w \in X \iff w \in \mathrm{dom}(f) \iff f_{h(w),1}$ is everywhere defined

$\implies f_{h(w),1}(h(w))\downarrow \iff h(w) \in \mathbb{K}$.

(2) $w \notin X \iff w \notin \mathrm{dom}(f) \iff f_{h(w),1}$ is nowhere defined

$\implies f_{h(w),1}(h(w))\uparrow \iff h(w) \notin \mathbb{K}$.

q.e.d.




$\emptyset, \mathbb{B}$ are discussed on ES#4.

<u>Q</u> Is that it?

<u>A</u> No.

If $X, Y \subseteq \mathbb{B}$, define

$$X \oplus Y := 0X \cup 1Y.$$

TURING JOIN OF X AND Y

$X \leq_m X \oplus Y$ [via $w \longmapsto 0w$]

$Y \leq_m X \oplus Y$ [via $w \longmapsto 1w$]

$$\mathbb{K}, \mathbb{B} \setminus \mathbb{K} <_m \mathbb{K} \oplus \mathbb{B} \setminus \mathbb{K}$$

## §4.12 Index sets

$I \subseteq \mathbb{B}$ is called an index set if

$$\forall w,v \quad w \in I \;\&\; W_w = W_v \implies v \in I.$$

[$w,v$ are weakly equivalent]

In other words, $I$ is an index set if it's closed under weak equivalence

Examples ① $\emptyset$, $\mathbb{B}$ We call those trivial index sets.

② $\mathrm{Emp} := \{w;\ W_w = \emptyset\}$ is a nontrivial index set.

$\mathrm{Fin} := \{w;\ W_w \text{ is finite}\}$ —— " ——

$\mathrm{Inf} := \{w;\ W_w \text{ is infinite}\}$ —— " ——

Note: If Emp is not computable, then the emptiness problem for Type 0 grammars is not solvable.