Saturday 16 November 2024: **XV**

FIFTEENTH LECTURE OF AUTOMATA & FORMAL LANGUAGES

Performing operations

$F: \mathbb{B}^{n+1} \dashrightarrow \mathbb{B}^{n+1}$

M performs F if M halts on input $\vec{w}$ with output $\vec{v}$ iff $F(\vec{w}) = \vec{v}$

Answering questions

$W = \{A_0, \dots, A_k\}$ disjoint $\mathbb{B}^{n+1} = \bigcup_{i=0}^{k} A_i$

M answers W if $\forall \vec{w}$ M reaches configuration $(\hat{q}_i, \vec{w})$ iff $\vec{w} \in A_i$

Three Lemmas

- L 4.6 SUBROUTINE LEMMA
- L 4.7 CASE DISTINCTION LEMMA
- L 4.8 REPEAT LEMMA

LECTURE XIV

NEVER HALT Operation $F: \mathbb{B}^{n+1} \dashrightarrow \mathbb{B}^{n+1}$ with $\mathrm{dom}(F) = \emptyset$

RM: $q_S \longmapsto +_{\mathbb{0}}(0, q_S)$

Or any RM that never reaches the state q_H.

HALT W/O CHANGING ANYTHING Operation $F: \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$, $F(\vec{w}) = \vec{w}$

RM: $q_S \longmapsto ?_{\varepsilon}(0, q_H, q_H)$

Or any RM that immediately halts without change. OR OTHERS!

DELETE THE FINAL LETTER OF REGISTER i, IF IT EXISTS $q_S \longmapsto -(i, q_H, q_H)$

DELETE THE CONTENT OF REGISTER i $q_S \longmapsto -(i, q_H, q_S)$

ADD $\mathbb{0}$ / $\mathbb{1}$ TO REGISTER i $q_S \longmapsto +_{\mathbb{0}}(i, q_H)$ / $+_{\mathbb{1}}$

LECTURE XIV

IS REGISTER i EMPTY?

$A_0 := \{\vec{w};\ w_i = \varepsilon\}$
$A_1 := \{\vec{w};\ w_i \neq \varepsilon\}$
$q_S \longmapsto ?_{\varepsilon}(i, \hat{q}_0, \hat{q}_1)$

DOES REGISTER i END WITH $\mathbb{0}$ / $\mathbb{1}$?

$q_S \longmapsto ?_{\mathbb{0} / \mathbb{1}}(i, \hat{q}_0, \hat{q}_1)$

$A_0 := \{\vec{w};\ w_i = v\mathbb{0}\}$
$A_1 := \mathbb{B}^{n+1} \setminus A_0$

WHAT IS THE FINAL LETTER OF REGISTER i?

$A_0 := \{\vec{w};\ w_i = v\mathbb{0}\}$
$A_1 := \{\vec{w};\ w_i = v\mathbb{1}\}$
$A_2 := \{\vec{w};\ w_i = \varepsilon\}$

Take machines $M_{\mathbb{0}}$ checking $\mathbb{0}$: $q_S^{\mathbb{0}}\ q_H^{\mathbb{0}}\ \hat{q}_0^{\mathbb{0}}\ \hat{q}_1^{\mathbb{0}}$
$M_{\mathbb{1}}$ checking $\mathbb{1}$: $q_S^{\mathbb{1}}\ q_H^{\mathbb{1}}\ \hat{q}_0^{\mathbb{1}}\ \hat{q}_1^{\mathbb{1}}$

As before, assume $Q_{\mathbb{0}} \cap Q_{\mathbb{1}} = \{q_S^{\mathbb{1}}\}$ with $\hat{q}_1^{\mathbb{0}} = q_S^{\mathbb{1}}$.

Combine $Q := Q_{\mathbb{0}} \cup Q_{\mathbb{1}}$ and P by combining $P_{\mathbb{0}}, P_{\mathbb{1}}$ removing superfluous instructions.

$q_S := q_S^{\mathbb{0}}$ $\hat{q}_0 := \hat{q}_0^{\mathbb{0}}$ $\hat{q}_1 := \hat{q}_0^{\mathbb{1}}$ $\hat{q}_2 := \hat{q}_1^{\mathbb{1}}$

ADD w TO REGISTER i

Apply Subroutine Lemma to "Add $\mathbb{0}$/$\mathbb{1}$ to register i"

MORE EXAMPLES (Example 4.9)

Lecture XIV

REPLACE CONTENT OF REG i WITH w.

Empty register i.
Add w to register i.

COPY FINAL LETTER OF REG i TO REG j, IF IT EXISTS

Determine final letter
If 0, write 0 in j.
If 1, write 1 in j.
If ε, do nothing.

MOVE FINAL LETTER OF REG i TO REG j, IF IT EXISTS

Copy final letter from i to j.
Remove final letter from i.

MOVE CONTENT OF REG i TO REG j IN REVERSE ORDER

Repeat:
move final letter
Until empty.

MOVE CONTENT OF REG i TO REG j.

Take unused reg. k.
Empty it.
Move i to k in reverse order.
Move k to j in reverse order.

COPY CONTENT OF REG i TO REG j IN REVERSE ORDER

Take unused register k
Empty it.
Repeat: Copy final letter from i to k.
Move final letter from i to j.
Until i is empty.
Move k in reverse order into i.

COPY CONTENT OF REG i TO REG j.

Take unused reg. k
Empty it.
Copy i into k in reverse order.
Move k into j in reverse order.

REMARKS

(1) The three lemmas are constructive : explicit constructions of a RM.

(2) Therefore: descriptions of how to build RM are precise enough to identify the RM.

(3) Unused registers are SCRATCH SPACE : building RM with operations that use "unused registers" increases the size of the machines and needs to be done in a principled way to have a "concrete description" as in (2).

§4.3 Computable functions & sets

If M RM, $k > 0$ define $f_{M,k} : \mathbb{B}^k \dashrightarrow \mathbb{B}$

$f_{M,k}(\vec{w})\downarrow \;:\Longleftrightarrow$ the computation of M w/ input $\vec{w}$ halts

$f_{M,k}(\vec{w}) = v \;:\Longleftrightarrow$ the output is $\vec{v}$ and $v = v_0$.

[All other registers are considered "scratch".]

A partial function $f : \mathbb{B}^k \dashrightarrow \mathbb{B}$ is called COMPUTABLE if there is M RM s.t. $f = f_{M,k}$.

Remarks
1. If M, M' are str. eq., then $f_{M,k} = f_{M',k}$.
2. Converse doesn't hold (ES#3).
3. By L 4.3, only ctbly many computable fns.
4. By L 4.4, each computable f has ∞ many M s.t. $f = f_{M,k}$.

Examples
1. $\mathrm{id} : \mathbb{B} \longrightarrow \mathbb{B}$ is computable [Halt]
2. Constant functions are computable. [Empty reg. 0; write v in reg. 0, halt.]
3. Projections $\vec{w} \longmapsto w_i$ [if $i=0$, this is just "Halt". if $i>0$, empty 0; move i to 0; halt.]

Def. If $A \subseteq \mathbb{B}^k$, write

$$\chi_A(\vec{w}) := \begin{cases} \mathbb{1} & \text{if } \vec{w} \in A \\ \mathbb{0} & \text{o/w} \end{cases}$$

CHARACTERISTIC FUNCTION OF A

$$\psi_A(\vec{w}) := \begin{cases} \mathbb{1} & \text{if } \vec{w} \in A \\ \uparrow & \text{o/w} \end{cases}$$

PSEUDOCHARACTERISTIC FUNCTION

Def. A is <u>computable</u> if χ_A is computable.
A is <u>computably enumerable</u> if ψ_A is computable
c.e.

<u>Q</u>: Is every c.e. set computable ?!?
We will prove later: NO!

Remark
① Historically: these were called "recursive" and "recursively enumerable" (r.e.)
② We explain the name of "c.e." later.

<u>Link back to Chapter 2</u>

<u>T 4.13</u> Every regular language $L \subseteq \mathbb{B}$ is computable.

<u>Proof.</u> Fix $D = (\Sigma_{01}, Q, \delta, q_0, F)$ s.t. $L = \mathcal{L}(D)$.
Define RM $(\hat{Q}, \hat{P})$ mimicking the behavior of D.
for each $q \in Q$ have $Q_q \subseteq \hat{Q}$ pairwise disjoint "mimicking state q"

Start by reversing content of ⓪ into reg. 1.

Go into some state in Q_{q_0} and repeat until reg 1 is empty:

[If we're in some state in Q_q, read top letter in reg. 1, say b, remove it and go to some state in $Q_{q'}$ where $q' = \delta(q, b)$.

After this, we're in some state in some Q_q for $q \in Q$

Empty reg. 0, write ⓪ if $q \notin F$ and halt.
write $\mathbb{1}$ if $q \in F$

q.e.d.

Remark More generally, all Type 1 languages are computable (ES#3).

§4.3 Computable Functions & Sets

Proposition 4.12. Let $X \subseteq \mathbb{B}^k$.

(a) If X is computable, then so is $\mathbb{B}^k \backslash X$, i.e., being computable is closed under complementation.

(b) A set X is computably enumerable if and only if it is the domain of a computable partial function. (If $k = 1$, this is equivalent to "there is an M such that $X = \mathrm{W}_M$".)

(c) Every computable set is computably enumerable.

Proof (a) $g(\vec{w}) := \begin{cases} \mathbb{1} & \text{if } \vec{w} = \mathbb{0} \\ \mathbb{0} & \text{o/w} \end{cases}$

is computable: Check whether reg. 0 is $\mathbb{0}$.
If so, empty it, write $\mathbb{1}$, halt.
If not, empty it, write $\mathbb{0}$, halt.

Now:
$\chi_{\mathbb{B}^k \backslash X} = g \circ \chi_X$. SUBROUTINE LEMMA!

(b) "$\Longrightarrow$" definition

"$\Longleftarrow$" Let c be the constant fn with value $\mathbb{1}$.
Let $X = \mathrm{dom}(f)$, then
$\psi_X = c \circ f$ SUBROUTINE LEMMA

(c) $h(\vec{w}) := \begin{cases} \uparrow & \text{if } w = \mathbb{0} \\ \mathbb{1} & \text{o/w} \end{cases}$ is computable as in (a).

Then $\psi_X = h \circ \chi_X$.

q.e.d.

§ 4.4 Coding numbers

$\mathbb{B}^n \rightsquigarrow$ read them as binary number $b(w) < 2^n$

$$\#w := 2^{|w|} + b(w) - 1$$

bijection

w	$\|w\|$	$b(w)$	#w
ε	0	0	$2^0 + 0 - 1 = 0$
0	1	0	$2^1 + 0 - 1 = 1$
1	1	1	$2^1 + 1 - 1 = 2$
00	2	0	$2^2 + 0 - 1 = 3$
01	2	1	$2^2 + 1 - 1 = 4$
10	2	2	$2^2 + 2 - 1 = 5$
11	2	3	$2^2 + 3 - 1 = 6$
000	3	0	$2^3 + 0 - 1 = 7$
⋮	⋮	⋮	⋮

$\# : (\mathbb{B}, <_{\text{shortlex}}) \longrightarrow (\mathbb{N}, <)$ isomorphism

This corresponds to ordering all binary words first by length and words of the same length lexicographically. This order is usually called the *shortlex order*:

$$w <_{\text{shortlex}} v \quad :\iff \quad \begin{array}{l} |w| < |v| \text{ or} \\ |w| = |v| \text{ and } w \neq v \text{ and} \\ \quad \text{if } i \text{ is minimal such that } w(i) \neq v(i), \text{ then } w(i) = \mathbf{0} \neq \mathbf{1} = v(i). \end{array}$$

decodes a string and $\#^{-1}$ encodes a number

$\# : \mathbb{B} \longrightarrow \mathbb{N}$ $\qquad \#^{-1} : \mathbb{N} \longrightarrow \mathbb{B}$

Lecture $\overline{\text{XVI}}$: encoding numerical functions

$f : \mathbb{N}^k \longrightarrow \mathbb{N}$