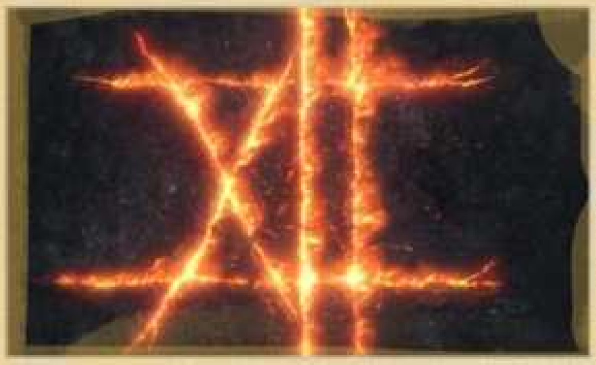


Twelfth Lecture

AUTOMATA & FORMAL LANGUAGES

Saturday, 9 November 2024

RECAP of Chapter 3

Context-free languages $\rightsquigarrow$ PARSE TREES

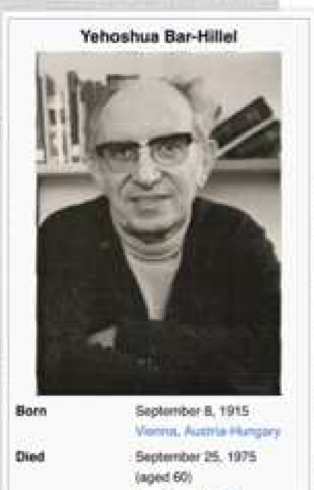
Chomsky Normal Form CNF

Theorem (Chomsky).
For every context-free G there is a G' in CNF s.t. $\mathcal{L}(G) = \mathcal{L}(G')$.

3.3 The pumping lemma for context-free languages

Definition 3.10. Let $L \subseteq \mathbb{W}$ be a language. We say that L *satisfies the (context-free) pumping lemma with pumping number* n if for every word $w \in L$ such that $|w| \geq n$ there are words u, v, x, y, z such that $w = xuyvz$, $|uv| > 0$, $|uyv| \leq n$ and for all $k \in \mathbb{N}$, we have that $xu^k yv^k z \in L$. We say that L *satisfies the (context-free) pumping lemma* if there is some n such that it satisfies the (context-free) pumping lemma with pumping number n.

The first proof of the context-free pumping lemma is usually attributed to Yehoshua Bar-Hillel (1915–1975); the statement is therefore also known as the *Bar-Hillel Lemma.*[12]



Theorem Every context-free language satisfies the CFPL for some n.

[If G is in CNF, $n := 2^m + 1$
where $m := |V|$.]

If $|w| \geq n$, then $w = xuyvz$ with $|uv| > 0$, $|uyv| \leq n$ and $xu^k yv^k z \in L$ $\forall k$.

Application of CFPL

Example 3.13 $L = \{ 0^k 1^k 2^k,\ k > 0 \}$ is not context-free. Easy to get $(01)^k 2^k$ as context-free

Compare ES#1 (5)(iii).

Assume it it. Let N be the PN.

Pick $0^N 1^N 2^N \in L$

By CFPL, $0^N 1^N 2^N = xuyvz$ with $|uv| > 0$, $|uyv| \le N$

$\underbrace{\quad\quad}_{0}\underbrace{\quad\quad}_{1}\underbrace{\quad\quad}_{2}$

So one of two cases applies: either uyv contains no 0 or uyv contains no 2.

So, pumping up or down will change at least one of the numbers, but not the number of the letters not contained in uyv. q.e.d

Comment on storage / memory: We showed that $0^n 1^n$ is not regular.

Automata have finite memory, but as $0^n 1^n$ shows: not unbounded memory.

If there is a model of computation for CFL, it has one unbounded memory space, but it seems to destroy the information when reading it.

Es#2: $0^n 1^m 2^n 3^m$

not context-free; would require two independent memory cells.

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§ 3.4 Closure properties

We already know that CFL are closed under union & concatenation.

If $\mathcal{C}$ is closed under union AND complementation, then closed under intersection.

DE MORGAN: $A \cap B = X \setminus (X \setminus A \cup X \setminus B)$

$A, B \subseteq X$

Note: If CFL not closed under intersection, then not closed under complement.

Prop 3.14 CFL not closed under intersection.

Pf Show that L from E 3.13 is the intersection of two CFL.

$$L_0 = \{ 0^k 1^k 2^n ;\ k,n > 0 \}$$

$$L_1 := \{ 0^n 1^k 2^k ;\ k,n > 0 \}$$

Clearly, $L = L_0 \cap L_1$.

We know that $L_2 := \{ 0^k 1^k ;\ k > 0 \}$ and

$L_3 := \{ 1^k 2^k ;\ k > 0 \}$ are CF.

Then $L_0 = L_2 2^+$ and $L_1 = 0^+ L_3$, so both L_0 & L_1 are CFL, since CFL are closed under concatenation.

qed

§ 3.5 Decision Problems

	regular (type 3)	context-free (type 2)
Closure properties.		
Concatenation	✓	✓
Union	✓	✓
Intersection	✓	×
Complementation	✓	×
Difference	✓	×
Decision problems.		
Word problem	✓	✓
Emptiness problem	✓	✓
Equivalence problem	✓	×

Figure 4: Closure properties and decision problems of regular and context-free grammars in an overview.

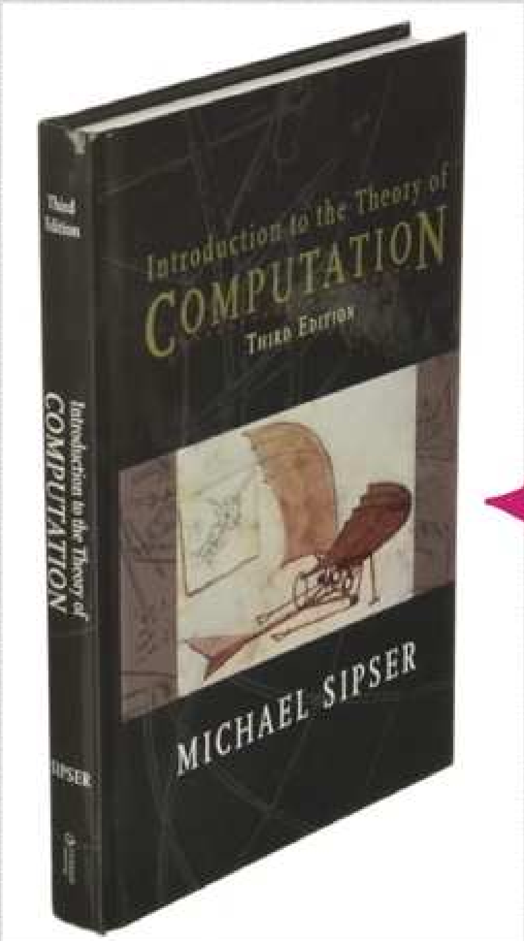
Emptiness

Note that for solving EMPTINESS, we only needed that the shortest word of a non-empty language is shorter than the PN [o/w it can be pumped down].

But that's implied by the CFPL.

So, the same proof works.

The Equivalence problem for context-free grammars



Answer: NO!
Not solvable.

Exercise 5.1
Theorem 5.13

Note Negative answer must require a precise definition of "solvable". This uses §§ 4.8 & 4.11 from our typed notes.

Coda on Chapter 3. Model of computation.

There is a notion called PUSHDOWN AUTOMATON

This is an automaton with one additional memory cell called a STACK with LIFO memory (LAST IN FIRST OUT)

Then we can prove. L is CFL $\iff$ L is accepted by a pushdown automaton.

Note: Failure of intersection closure means that there cannot be a product notion.

Chapter 4.

COMPUTABILITY THEORY

Alan Turing
OBE FRS

Turing c. 1928 at age 16

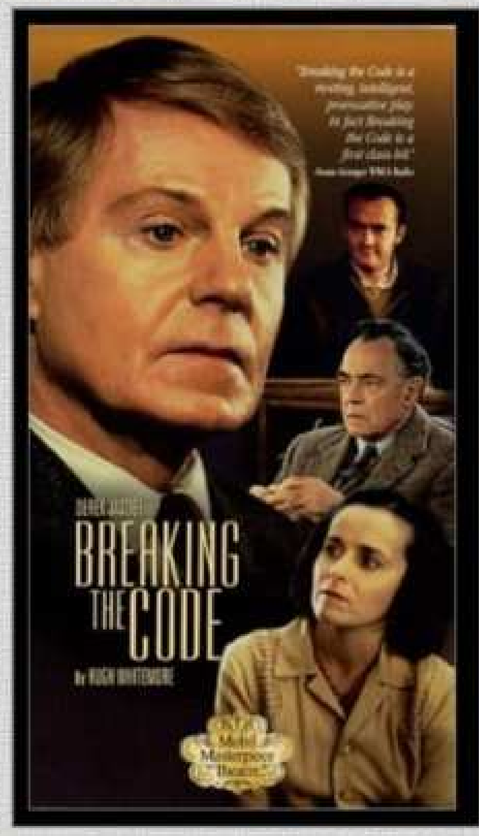
Born	Alan Mathison Turing 23 June 1912 Maida Vale, London, England
Died	7 June 1954 (aged 41) Wilmslow, Cheshire, England

230 A. M. TURING [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

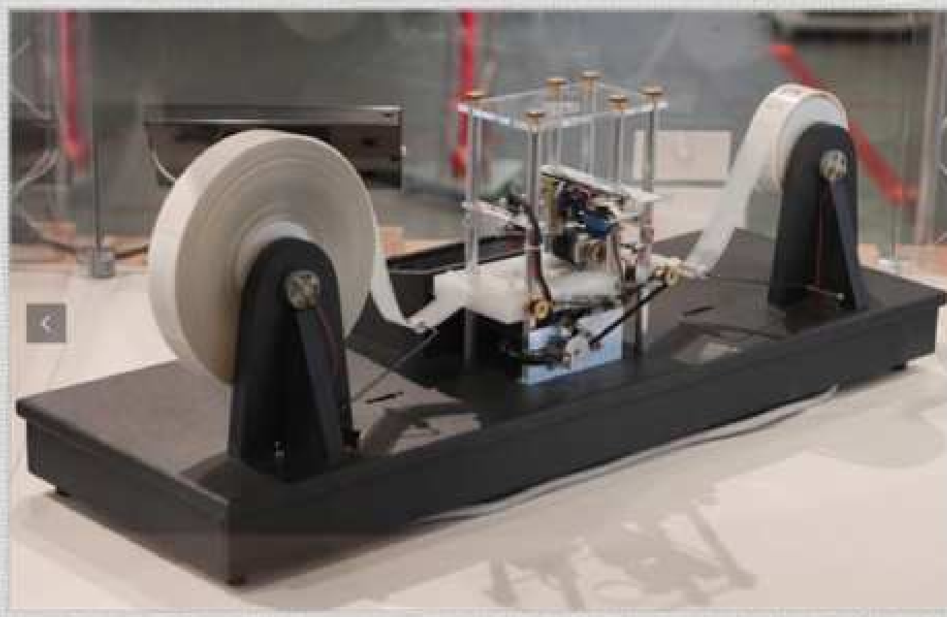
By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]



TURING MACHINES

Single Tape
(infinite):
serves as input,
scratch, &
output



REGISTER MACHINES

Joachim LAMBEK 1922–2014

Zdzislaw MELZAK 1926–?

Marvin MINSKY 1927–2016

John C. SHEPHERDSON
1926–2015

Howard E. STURGIS
1936–1990

von Neumann
architecture

finitely many
memory cells:

REGISTERS

Basic ideas for Register Machines

COMPUTERS CAN ONLY UNDERSTAND BINARY

For the moment $\Sigma := \Sigma_{01} = \{0, 1\}$

$$
\left.
\begin{array}{lll}
+0 & ?0 & - \\
+1 & ?1 & \\
 & ?\varepsilon &
\end{array}
\right\} \text{form INSTRUCTIONS}
$$

Six basic operation types.