Automata & Formal Languages

Eighth Lecture — 31 October 2024

RECAP

GETTING TO THE END OF CHAPTER 2

Goals

1. Prove strict inclusion of Chomsky classes. — Done Ex 2.12, Ex 3.1
2. Closure properties
   - Concatenation ✓ P2.2
   - Union ✓ P2.2
   - Complementation ✓ P2.17
   - Intersection ✓ P2.18
3. Decision problems:
   - (a) Word problem — Done T1.21
   - (b) Emptiness problem — Done C2.27
   - (c) Equivalence problem — ? TODAY

Lecture V: HOMOMORPHISMS

If $D = (\Sigma, Q, \delta, q_0, F)$ and $D' = (\Sigma, Q', \delta', q_0', F')$ are deterministic automata over the same alphabet Σ, we say that a map $f : Q \to Q'$ is a *homomorphism from D to D'* if

(i) for all $q \in Q$ and $a \in \Sigma$, we have that $\delta'(f(q), a) = f(\delta(q, a))$,

(ii) we have $f(q_0) = q_0'$ and

(iii) for all $q \in Q$, $q \in F$ if and only if $f(q) \in F'$.

As usual, bijective homomorphisms are called isomorphisms and automata that have an isomorphism between them are called isomorphic. Note that if f is a bijection, then f^{-1} satisfies (i) to (iii) and thus is a homomorphism.

If f is a homomorphism, property (i) extends by induction to $\hat{\delta}'(f(q), w) = f(\hat{\delta}(q, w))$ for $w \in W$.

Remarks

1. If $f : Q \to Q'$ is a homom. and $q' \in Q'$ can be reached from q_0' (i.e., there is $w \in W$ s.t. $\hat{\delta}'(q_0', w) = q'$), then $q' \in \mathrm{ran}(f)$.

   ACCESSIBLE STATES

2. If $f(p) = f(q)$, p and q must agree on many things: e.g., for all w $\hat{\delta}(p, w) \in F \leftrightarrow \hat{\delta}(q, w) \in F$

   INDISTINGUISHABLE STATES

[This will be used next week!]

- If $f : D \to D'$ homomorphism, then $\mathcal{L}(D) = \mathcal{L}(D')$
- If $q' \in Q'$ is **accessible** then $q' \in \mathrm{ran}(f)$.
- If $f(q) = f(q')$, then q and q' are **indistinguishable**.

$\mathcal{D} = (\Sigma, Q, \delta, q_0, F)$

$q \in Q$ <u>accessible</u> if $\exists w \quad \hat{\delta}(q_0, w) = q$

$q, q' \in Q$ <u>distinguishable</u> if $\exists w \quad \hat{\delta}(q, w) \in F$ and $\hat{\delta}(q', w) \notin F$ | w <u>distinguishes</u> q & q'

or vice versa

Define $q \sim q' :\Leftrightarrow q, q'$ indistinguishable.

This is an equivalence relation; as usual: $[q] := \{ q' ;\ q' \sim q \}$.

$\boxed{\text{QUOTIENT AUTOMATON}}$ $\quad \mathcal{D}/\sim \ := (\Sigma, Q/\sim, [\delta], [q_0], [F])$ with

① $Q/\sim \ := \{ [q] ;\ q \in Q \}$

② $[\delta]([q], a) := [\delta(q, a)]$

③ $[F] := \{ [q] ;\ q \in F \}$

<u>NOTE</u> ① This is well-defined. [Check!]

② $\widehat{[\delta]}([q], w) = [\hat{\delta}(q, w)]$ [Induction!]

③ No two states in $\mathcal{D}/\sim$ are indistinguishable.

Prop 2.20

Prop 2.21 $\mathcal{L}(D) = \mathcal{L}(D/\sim)$.

Proof. Check that $q \longmapsto [q]$ is a homomorphism from D to $D/\sim$ (!) and apply P 2.5. q.e.d.

Let's call D irreducible if there are no inaccessible and no indistinguishable states.

Prop 2.22 If $f: D \longrightarrow D'$ homomorphism, then

(i) if D is irr., f is injective

(ii) if D' is irr., f is surjective

(iii) if D, D' are irr., f is iso.

Note: if D, D' irr. and there is a homom., then $D \cong D'$.

Theorem 2.24 If I, I' are irreducible and $\mathcal{L}(I) = \mathcal{L}(I')$, then $I \cong I'$.

Proof By last remark, only need homomorphism. As usual, $Q \cap Q' = \emptyset$.
Define a relation $\sim$ on $Q \times Q'$ called INDISTINGUISHABILITY:

$$q \sim q' :\Longleftrightarrow \{w;\ \hat\delta(q,w) \in F\} = \{w;\ \hat\delta'(q',w) \in F'\}$$

Note ① If $q_1 \sim q' \sim q_2 \Longrightarrow q_1 = q_2$ since I is irred.

② $q_0 \sim q_0'$ since $\{w;\ \hat\delta(q_0,w) \in F\} = \mathcal{L}(I)$ and $\{w;\ \hat\delta'(q_0',w) \in F'\} = \mathcal{L}(I')$.

③ If $q_1' \sim q \sim q_2' \Longrightarrow q_1' = q_2'$ since I' is irred.

CLAIM $\forall q\, \exists q'\ \ q \sim q'$.

[Proof by induction on shortest length of w s.t. $\hat\delta(q_0,w) = q$. Note that this exists since I is irr.
If $|w| = 0 \Rightarrow w = \varepsilon$, so the claim is just Note ②.
Suppose $|w| = n+1$, say $w = va$ where $|v| = n$.
$\hat\delta(q_0, va) = q$
Define $p := \hat\delta(q_0, v)$. By IH, there is $p' \in Q'$, $p \sim p'$.
Let $q' := \hat\delta'(p', a)$ and then check that

$q \sim q'$.]

!

Now ③ & claim imply:

$$f(q) := \text{the unique } q' \text{ s.t. } q \sim q'$$

is properly defined. Check ∇ that f is a homomorphism

q.e.d.

<u>Thm 2.23</u> For each $\mathcal{D}$, there is an irr. $\mathcal{I}$ s.t.

$\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{I})$ which is unique up to iso

and has at most as many states as $\mathcal{D}$.

THE MINIMAL AUTOMATON FOR $\mathcal{L}(\mathcal{D})$

<u>Proof</u> Start with $\mathcal{D}$; remove inaccessible states to get $\mathcal{D}'$

Note $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}')$.

Let $\mathcal{I} := \mathcal{D}'/\sim$. Then $\mathcal{I}$ is irreducible

By P 2.21 $\mathcal{L}(\mathcal{I}) = \mathcal{L}(\mathcal{D}') = \mathcal{L}(\mathcal{D})$.

By T 2.24 unique up to iso.

q.e.d.

Back to the EQUIVALENCE PROBLEM

What does all of this have to do with it?

Proposition 2.28. There is an algorithm that determines which states of an automaton are inaccessible.

Proof. Let $D = (\Sigma, Q, \delta, q_0, F)$ and $n := |Q|$. By Corollary 2.14, a state q is inaccessible if and only if there is no word w of length $\leq n$ such that $\widehat{\delta}(q_0, w) = q$. Since there are finitely many such words, we can just check $\widehat{\delta}(q_0, w)$ for all such words to determine which states are accessible; the remaining states must be inaccessible. Q.E.D.

Proposition 2.29. There is an algorithm that determines whether two states of an automaton are equivalent.

Proof. We determine whether the states are indistinguishable; from this, we can easily determine equivalence. This algorithm is known as the *table filling algorithm*. We write $Q \times Q$ as a table; note that due to the fact that indistinguishability is an equivalence relation, we only need to fill half of the table, so we can ignore the lower left triangle.

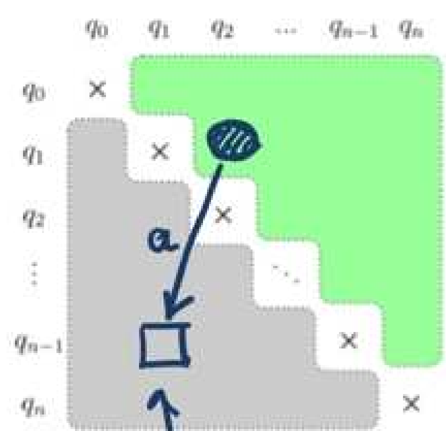
EXAMPLE

$q_2 \xrightarrow{a} q_{n-1}$

$q_1 \xrightarrow{a} q_2$

IDEA:

We mark (q, q') with w if w distinguishes q and q'.



If (q_{n-1}, q_2) is distinguished by w, then (q_2, q_1) is distinguished by aw!

<u>Remark on P 2.28</u> This is just the (proof of the) pumping lemma.
The shortest word s.t. $\hat{\delta}(q_0, w) = q$ must have length $< |Q|$. So just check all words of these lengths.

<u>Proof of Prop 2.29</u> Algorithm: TABLE FILLING ALGORITHM

<u>STEP 1</u> Check all (q, q'). If $q \in F \,\&\, q' \notin F$, then write ε in (q, q')
$q \notin F \,\&\, q' \in F$ ——— " ———

<u>STEP $n > 1$</u> Have partially filled table. Check all unmarked (q, q') and for each $a \in \Sigma$ consider $(\delta(q, a), \delta(q', a))$.
If unmarked, do nothing.
If marked by w, write aw in (q, q').
After that, if nothing was marked in STEP n, TERMINATE.
if sth was marked, go to STEP $n+1$.

This terminates, since there are only $|Q|^2$ pairs, so it'll stop before step $|Q|^2 + 1$.

We obtain a partially filled table:

if (q, q') is marked, q, q' are distinguished

Still need to show:

if (q, q') is unmarked, q, q' are indistinguishable

$\longrightarrow$ SATURDAY.