AUTOMATA & FORMAL LANGUAGES

Fifth Lecture : 24 OCTOBER 2024

RECAP

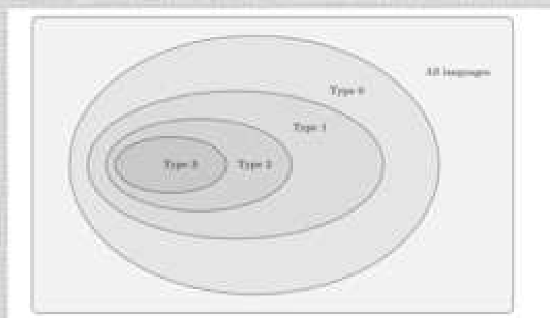


Figure 1: The Chomsky hierarchy

Type 0: all grammars
Type 1: noncontracting
Type 2: context-free
Type 3: regular

Theorem The classes of Type 0, 1, 2 languages are closed under concatenation and union.

Q: What about Type 3 = REGULAR?

RECAP

Chapter 2 : REGULAR LANGUAGES

Lecture IV

<u>Regular grammar</u>

	$A, B \in V$ & $a \in \Sigma$	# letters	# variable	length
TERMINAL RULES	$A \longrightarrow a$	+1	−1	0
NONTERMINAL RULES	$A \longrightarrow aB$	+1	0	+1

$$S \xrightarrow{G} w$$

1. There is precisely terminal rule: at the very end.
2. There are precisely $|w|-1$ nonterminal rules.

$$S \xrightarrow[1]{G} a_0 A_0 \xrightarrow[1]{G} a_0 a_1 A_1 \xrightarrow[1]{G} \cdots \xrightarrow[1]{G} a_0 \cdots a_{n-1} A_{n-1} \xrightarrow[1]{G} a_0 \cdots a_n.$$

THINK OF a regular grammar

as a machine that can replace variables by letters and either stops or continues (and provides a new variable to continue).

Back to closure under concatenation and union:

Now that we understand what regular derivations look like, we can re-visit the question of closure under union and concatenation. The union and concatenation grammars we used in § 1.7 were not regular, so we need to give alternative constructions. Let $G = (\Sigma, V, P, S)$ and $G' = (\Sigma, V', P', S')$ be regular grammars.

(a) The *regular concatenation grammar of G and G'* is $(\Sigma, V \cup V', P^*, S)$ where $P^* := P' \cup (P \backslash \{A \to a\,;\, A \to a \in P\}) \cup \{A \to aS'\,;\, A \to a \in P\}$.

(b) The *regular union grammar of G and G'* is $(\Sigma, V \cup V' \cup \{T\}, P^*, T)$ with a new variable T and $P^* := P \cup P' \cup \{T \to \alpha\,;\, S \to \alpha \in P\} \cup \{T \to \alpha\,;\, S' \to \alpha \in P'\}$.

Clearly, if G and G' are regular, then so are the regular concatenation and regular union grammars.

Regular concatenation grammar RCG $G=(\Sigma,V,S,P)$

$$G'=(\Sigma,V',S',P')$$

$$P^+ := P' \cup P \setminus \{A\to a;\ A\to a\in P\} \cup \{A\to aS';\ A\to a\in P\}$$

Regular union grammar RUG

$$P^+ := P\cup P' \cup \{T\to\alpha;\ S\to\alpha\in P\}\cup\{T\to\alpha,\ S'\to\alpha\in P'\}.$$

Note: If G, G' are regular, then so are the RCG & RUG.

<u>Prop 2.2</u> If G, G' are regular and $V\cap V'=\emptyset$, then

(i) if H RCG, then $\boxed{\mathcal{L}(H)=\mathcal{L}(G)\,\mathcal{L}(G')}$

(ii) if H RUG, then $\mathcal{L}(H)=\mathcal{L}(G)\cup\mathcal{L}(G')$.

<u>Proof</u> Just do calculation.

"$\supseteq$": $S \xrightarrow{G} w \quad S' \xrightarrow{G'} v$

$$\Rightarrow S \xrightarrow{G} \underbrace{w'A \xrightarrow{G}_1 w'a = w}_{\text{terminal rule}}$$

$$A \to a \in P$$

$$\Rightarrow A \to aS' \in P^+$$

$$S \xrightarrow{H} wS' \xrightarrow{H} wv$$

"$\subseteq$": $S \xrightarrow{H} v$

$$S \xrightarrow{H} \underbrace{v'A \xrightarrow{H}_1 v'a = v}_{\text{terminal rule}}$$

$$A \to a \in P'$$

$$\Rightarrow A \in V'$$

therefore

$$\underbrace{S \xrightarrow{H} w'B}_{\text{rules in } P} \underbrace{\xrightarrow{H}_1 wS'}_{\text{new } A \to aS' \text{ rule}} \underbrace{\xrightarrow{H} wv = v}_{\text{rules in } P'}$$

Thus $S \xrightarrow{G} w$ & $S' \xrightarrow{G'} v$.

qed

§ 2.2 Deterministic Automata

Fix Σ. $D = (\Sigma, Q, \delta, q_0, F)$ is called **deterministic automaton** if:

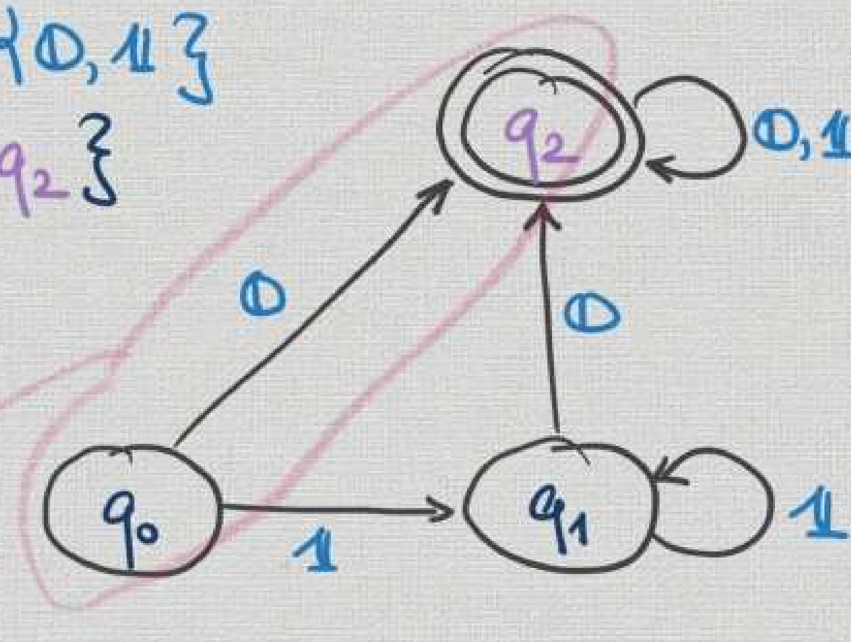
1. Q is a finite set of states
2. $q_0 \in Q$ start state,
3. $F \subseteq Q \setminus \{q_0\}$ accept states
4. $\delta : Q \times \Sigma \longrightarrow Q$
   transition function

GRAPHICAL REPRESENTATION:

$\Sigma = \Sigma_{01} = \{0, 1\}$

$Q = \{q_0, q_1, q_2\}$

$F = \{q_2\}$



$\delta(q_0, 0) = q_2$

INTERPRETATION

① Computations receives word $w \in W$ as input.

② It starts in state q_0.

③ Read the word letter by letter:
if you're in state q and read a,
move to $\delta(q, a)$.

④ After reading all of w, you're in some state q.
If $q \in F$: D ACCEPTS w.
$q \notin F$: D REJECTS w.

Properly, define
$\hat{\delta} : Q \times W \to Q$
by recursion:
$\hat{\delta}(q, \varepsilon) = q$
$\hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a)$.

$\mathcal{L}(D) := \{ w \,;\, \hat{\delta}(q_0, w) \in F \}$

THE LANGUAGE ACCEPTED BY D.

If $D = (\Sigma, Q, \delta, q_0, F)$ and $D' = (\Sigma, Q', \delta', q_0', F')$ are deterministic automata over the same alphabet Σ, we say that a map $f : Q \to Q'$ is a *homomorphism from D to D'* if

(i) for all $q \in Q$ and $a \in \Sigma$, we have that $\delta'(f(q), a) = f(\delta(q, a))$,

(ii) we have $f(q_0) = q_0'$, and

(iii) for all $q \in Q$, $q \in F$ if and only if $f(q) \in F'$.

As usual, bijective homomorphisms are called *isomorphisms* and automata that have an isomorphism between them are called *isomorphic*. Note that if f is a bijection, then f^{-1} satisfies (i) to (iii) and thus is a homomorphism.

If f is a homomorphism, property (i) extends by induction to $\widehat{\delta}'(f(q), w) = f(\widehat{\delta}(q, w))$ for $w \in \mathbb{W}$.

(î)

Remarks

① If $f : Q \longrightarrow Q'$ is a homom. and $q' \in Q'$ can be reached from q_0' (i.e., there is $w \in \mathbb{W}$ s.t. $\hat{\delta}'(q_0', w) = q'$), then $q' \in \mathrm{ran}(f)$.

ACCESSIBLE STATES

② If $f(p) = f(q)$, p and q must agree on many things:

INDISTINGUISHABLE STATES

e.g., for all w $\hat{\delta}(p, w) \in F \iff \hat{\delta}(q, w) \in F$

[This will be used next week!]

__Remarks__ ① $D =$  , then $\mathcal{L}(D) = \{ w ;\ w \text{ contains a } \}$.

② $\varepsilon \notin \mathcal{L}(D)$ by definitions of F & $\hat{\delta}$.

__Prop 2.5__ If f is a homomorphism from D to D', then $\mathcal{L}(D) = \mathcal{L}(D')$.

__Proof.__

$$
\begin{aligned}
w \in \mathcal{L}(D) &\Longleftrightarrow \hat{\delta}(q_0, w) \in F \\
&\overset{(iii)}{\Longleftrightarrow} f(\hat{\delta}(q_0, w)) \in F' \\
&\overset{(i)}{\Longleftrightarrow} \hat{\bar{\delta}}(f(q_0), w) \in F' \\
&\overset{(ii)}{\Longleftrightarrow} \hat{\bar{\delta}}(q_0', w) \in F' \\
&\Longleftrightarrow w \in \mathcal{L}(D').
\end{aligned}
$$

q.e.d.

Application

Without loss of generality, $q_0 \notin \mathrm{ran}(\delta)$

[I.e., $\forall D \; \exists D'$ s.t. $\mathcal{L}(D) = \mathcal{L}(D')$ and D' has $q_0 \notin \mathrm{ran}(\delta)$]

Define $Q' := Q \cup \{q^*\}$ where $q^* \notin Q$

$D' = (\Sigma, Q', \delta', q_0, F)$ with

$$\delta'(q,a) := \begin{cases} \delta(q,a) & \text{if } q \in Q \text{ and } \delta(q,a) \neq q_0 \\ \delta(q_0,a) & \text{if } q = q^* \text{ and } \delta(q_0,a) \neq q_0 \\ q^* & \text{otherwise.} \end{cases}$$

Could prove by hand that $\mathcal{L}(D) = \mathcal{L}(D')$.

However, easier to observe that $f : Q' \longrightarrow Q$ with $f(q^*) = f(q_0) = q_0$ and identity o/w is a homomorphism.