AUTOMATA AND FORMAL LANGUAGES (C) **24 lectures, Michaelmas Term**

Part IA Numbers and Sets is essential.

Recursively enumerable languages

Register machines. Recursive functions. Recursively enumerable sets. Church's thesis. Undecidability of the halting problem. Universal register machines. The recursion theorem. The *s-m-n* theorem. Reductions. Rice's theorem. Degrees of unsolvability. Hardness and completeness. [10]

Regular languages

Deterministic and non-deterministic finite-state automata. Regular languages. Regular expressions. Limitations of finite-state automata: closure properties; the pumping lemma; examples of non-regular languages. Minimisation. [9]

Context-free languages

Context-free grammars. Context-free languages. Chomsky normal form. Regular languages are context-free. Limitations of context-free grammars: the pumping lemma for context-free languages; examples of non-context-free languages. [5]

LECTURE I

15 OCTOBER 2024

Topic:

Computation / Computability

1. What is computation?
2. What can be computed?
3. What is not computable?

EXAMPLES.

DECISION PROBLEMS

Fix a domain X and a property Φ of elements of X.

GIVEN $x \in X$, DOES x HAVE PROPERTY Φ?

(1) Given a group with n elements, is it Abelian?

n elements $\longrightarrow$ n^2 pairs of elements

So $2n^2$ pairs of pairs $((x,y),(y,x))$

Check for all whether they're equal.

BRUTE FORCE ALGORITHM

(2) Given ax^2+bx+c with $a,b,c \in \mathbb{Z}$, is there an integer solution?

Brute Force Algorithm gives positive answers in finite time.

Does not give negative solutions.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Calculate x and check whether $x \in \mathbb{Z}$.

We did provide UNIFORM solutions:

algorithmic solvability.

HILBERT'S TENTH PROBLEM

David Hilbert

Hilbert in 1912

Born	23 January 1862 Königsberg or Wehlau, Kingdom of Prussia
Died	14 February 1943 (aged 81) Göttingen, Nazi Germany
Education	University of Königsberg (PhD)

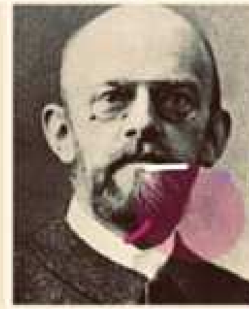
Mathematische Probleme.

Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900.

Von

D. Hilbert.

Wer von uns würde nicht gern den Schleier lüften, unter dem die Zukunft verborgen liegt, um einen Blick zu werfen auf die bevorstehenden Fortschritte unserer Wissenschaft und in die Geheimnisse ihrer Entwickelung während der künftigen Jahrhunderte! Welche besonderen Ziele werden es sein, denen die führenden mathematischen Geister der kommenden Geschlechter nachstreben? welche neuen Methoden und neuen Thatsachen werden die neuen Jahrhunderte entdecken — auf dem weiten und reichen Felde mathematischen Denkens?



10. DETERMINATION OF THE SOLVABILITY OF A DIOPHANTINE EQUATION.

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

Example (3)

H10: Given $p \in \mathbb{Z}[X_1, \ldots, X_n]$, is there an integer solution.

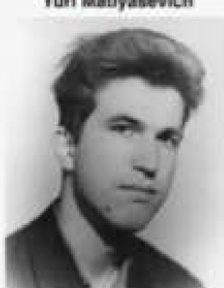
Davis-Matiyasevich-Putnam-Robinson: There is no decision procedure for H10.

Martin Davis

Davis in 1996

Born	Martin David Davis March 8, 1928 New York City, U.S.
Died	January 1, 2023 (aged 94) Berkeley, California, U.S.

Yuri Matiyasevich



Born	2 March 1947 (age 77) Leningrad, Soviet Union
Nationality	Soviet Russian
Alma mater	Leningrad State University

Hilary Putnam

Putnam in 2006

Born	Hilary Whitehall Putnam July 31, 1926 Chicago, Illinois, U.S.
Died	March 13, 2016 (aged 89) Arlington, Massachusetts, U.S.

Julia Hall Bowman Robinson

Julia Robinson in 1975

Born	Julia Hall Bowman December 8, 1919 St. Louis, Missouri, United States
Died	July 30, 1985 (aged 65) Oakland, California, United States

!! What does that even mean ?!?

ASYMMETRY OF DECISION PROBLEMS

YES: Provide an algorithm.

NO: This requires a formal definition of "decision procedure" and a proof that it doesn't exist.

The lecture course is mainly about this formal definition of COMPUTATION that will allow us to prove negative results!

One model of computation (a very simple one): AUTOMATA.

Automata & Formal Languages

Contents

Lemma 1.1. If X and Y are countable, then so is $X \times Y$.

Proof. First of all, if either X or Y is empty, then $X \times Y$ is empty, so we can assume that they are both non-empty and pick surjections $\pi_X \colon \mathbb{N} \to X$ and $\pi_Y \colon \mathbb{N} \to Y$. Remember from *Numbers & Sets* that there is a bijection $z \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, e.g., Cantor's *zigzag bijection*

$$(i, j) \mapsto \frac{(i+j)(i+j+1)}{2} + j$$

which will feature prominently in §4.5. Given any $n \in \mathbb{N}$, find i and j such that $n = z(i, j)$ and define $f(n) := (\pi_X(i), \pi_Y(j))$. It is easy to check that this is a surjection onto $X \times Y$.

Q.E.D.

Proposition 1.2. If $X \neq \varnothing$ is countable, then X^* is infinite and countable.

Proposition 1.3 (Cantor's Theorem)**.** If X is infinite, then the power set of X, i.e., the set of all subsets of X, denoted by $\wp(X)$, is uncountable.

Proposition 1.4. If X is countable, then the set of finite subsets of X is countable.

§ 1.1 Notation (N&S)

(1) $\mathbb{N} \ni 0$. Set-theoretic convention: $n = \{0, \dots, n-1\}$; $0 = \emptyset$

(2) X finite set of symbols

We call X^n the set of X-strings of length n

Set-theoretic convention:

$$\alpha \in X^n \Longrightarrow \alpha : \underset{\{0,\dots,n-1\}}{n} \longrightarrow X \text{; write } |\alpha| = n$$

(3) Empty string $\varepsilon : \emptyset \longrightarrow X$

("the empty function")

Thus: $X^0 = \{\varepsilon\}$

(4) $X^* := \bigcup_{n \in \mathbb{N}} X^n$ the set of X-strings

(5) If $|\alpha| = n$ and $k \leq n$, then $\alpha \restriction k$ is the unique initial segment of α with length k.

(6) $X^+ := X^* \setminus \{\varepsilon\}$. The set of nonempty X-strings.

(7) $\alpha\beta$ denotes the concatenation of α and β

Write αx for $\alpha\beta$ with $|\beta| = 1$ and $\mathrm{ran}(\beta) = \{x\}$

(8) x^n is the unique seq of length n with $\mathrm{ran}(x^n) = \{x\}$

(9) Recursive definition:

$$\alpha^0 := \varepsilon$$

$$\alpha^{n+1} := \alpha^n \alpha$$

(10) If $f : X \longrightarrow Y$, there is a natural $\hat{f} : X^* \longrightarrow Y^*$, the lift of f.

(1.1) A set X is called <u>infinite</u> if there is an injection from $\mathbb{N}$ to X.

(1.2) A set X is called <u>countable</u> if either $X = \emptyset$ or there is a surj from $\mathbb{N}$ …

<u>Proof of P1.2</u>

X^* is infinite: $n \longmapsto x^n$ where $x \in X$ is an injection.

To show countability, show that each X^n is ctble.

[That's enough by N&S as ctble unions of ctble sets are ctble]

By induction. $X^0 = \{\varepsilon\}$ is ctble.

If X^n is ctble, there is a bij between $\underbrace{X^n \times X}_{\text{is ctble by P 1.1}}$ and X^{n+1}

<u>Sketch of P1.3.</u>

Show that power set of $\mathbb{N}$ is uncountable.

Let π be any function from $\mathbb{N}$ to its power set.

$D := \{ n \,;\, n \notin \pi(n) \}$

If $\pi(d) = D$, then

$d \in D = \pi(d) \Longleftrightarrow d \notin \pi(d)$

Contradiction!

<u>Proof of P1.4</u>

Find surjection from X^* to the set of finite subse…

$F = \{x_0, \ldots, x_n\}$

Define $\alpha_F := (x_0, \ldots, x_n)$

Let $\pi : X^* \longrightarrow \mathrm{Fin}(X)$

$\alpha_F \longmapsto F$