TWENTY-FOURTH & ULTIMATE LECTURE

AUTOMATA & FORMAL LANGUAGES

29 November 2023

Remaining goals: Show that the emptiness & equivalence problems for type 0 grammars are unsolvable.

$\{v \,;\, \mathcal{L}(G_v) = \emptyset\}$

$\{(w,v) \,;\, \mathcal{L}(G_w) = \mathcal{L}(G_v)\}$

Last time: Reduction functions: $A \leq_m B$

at least as unsolvable as

$A \equiv_m B$

precisely as unsolvable as

Method for proving that

(i) A is not computable : $\mathbb{K} \leq_m A$

(ii) A is not c.e. : $\mathbb{N} \setminus \mathbb{K} \leq_m A$

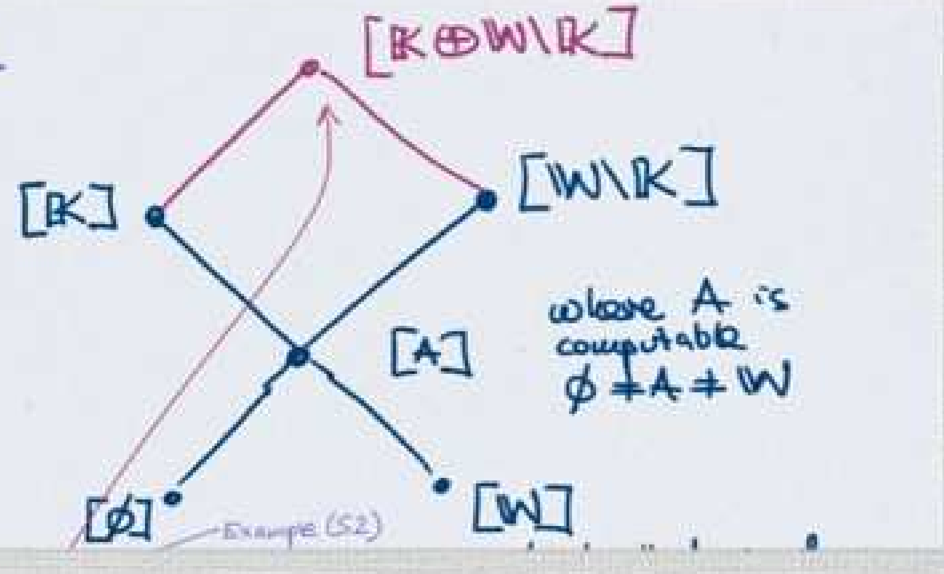
Apply this to $\le_m$ & $\equiv_m$:

$$\left(\mathcal{P}(\mathbb{W})/_{\equiv_m}, \le_m\right)$$

called DEGREES OF UNSOLVABILITY

Lecture $\overline{\text{XXIII}}$
pages 8 & 9

Picture



Index sets

I is an index set if it's closed under weak equivalence
[w, v weakly eq. if $W_w = W_v$]

Examples

Lecture $\overline{\text{XXIII}}$
page 10

Examples

$\mathrm{Emp} := \{ v ; W_v = \emptyset \}$

[!!! That's the emptiness problem.]

$\mathrm{Fin} := \{ v ; W_v \text{ is finite} \}$

$\mathrm{Inf} := \{ v ; W_v \text{ is infinite} \}$

$\mathrm{Tot} := \{ v ; W_v = \mathbb{W} \}$

are all index sets.

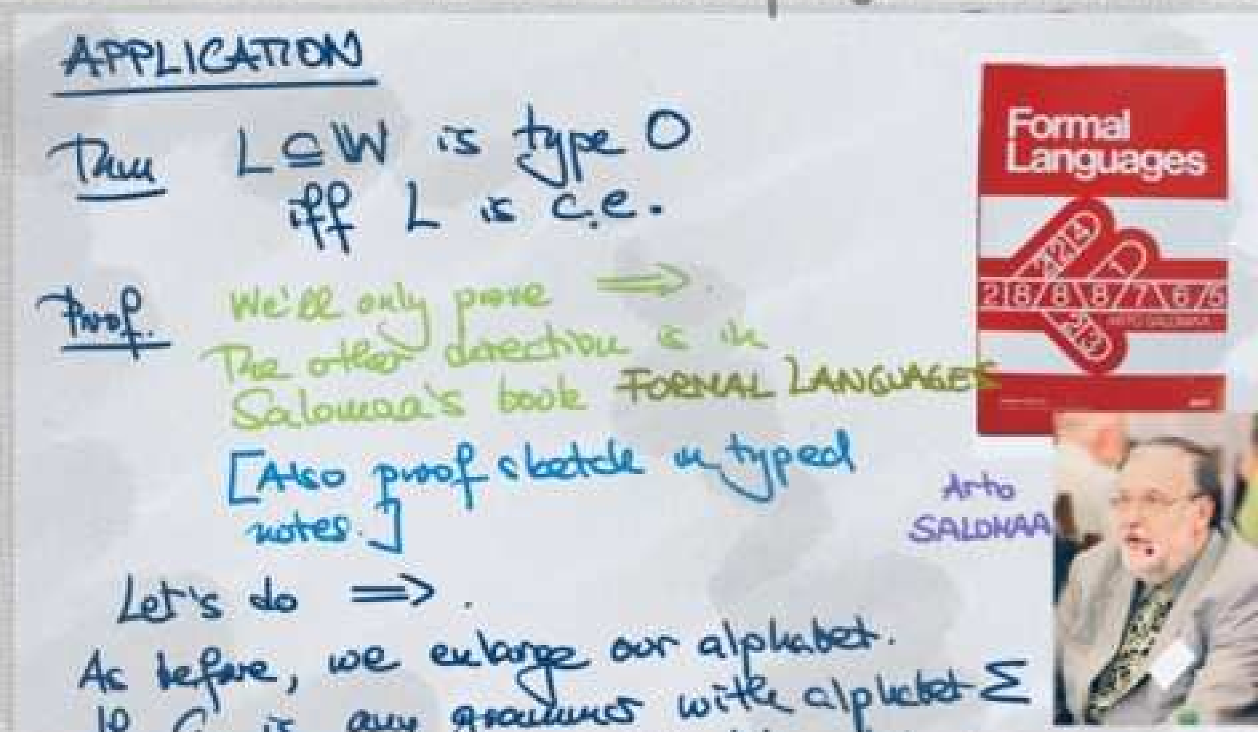
Observe:
1. If $\emptyset \neq I$, then I must be infinite [padding lemma]
2. Non-example: $\{ v ; f_{v,1} \text{ is constant} \}$ is not an index set.
3. $\mathbb{K}$ is not an index set → ES#4 See also next page

Use of the Recursion Theorem →

$$\mathrm{Emp} = \{ v ; W_v = \emptyset \}$$
$$E = \{ v ; \mathcal{L}(G_v) = \emptyset \}$$

THE EMPTINESS PROBLEM FOR TYPE 0 GRAMMARS

Lecture $\overline{\text{XXI}}$, page 7


APPLICATION

Thm $L \subseteq \mathbb{W}$ is type 0 iff L is c.e.

Proof. We'll only prove $\Longrightarrow$. The other direction is in Salomaa's book FORMAL LANGUAGES

[Also proof sketch in typed notes.]

Let's do $\Longrightarrow$.

As before, we enlarge our alphabet. As before, we enlarge our alphabet with alphabet Σ

If G is any grammar ...

In the theorem type 0 $\Longleftrightarrow$ c.e. we proved the $\Longrightarrow$-direction by providing a computable function $h_1 : \mathbb{W} \longrightarrow \mathbb{W}$ s.t.

$$\mathcal{L}(G_v) = W_{h_1(v)}$$

This takes a code for a grammar v and produces the code $h_1(v)$ of a RM that accepts the same words as G_v.

The proof of the $\Longleftarrow$-direction (not seen in lectures) does precisely the same the other way round: i.e., $h_2 : \mathbb{W} \longrightarrow \mathbb{W}$ s.t.

$$W_v = \mathcal{L}(G_{h_2(v)})$$

Then h_1 witnesses that $E \leq_m \mathrm{Emp}$
h_2 witnesses that $\mathrm{Emp} \leq_m E$

$\Big\}\ E \equiv_m \mathrm{Emp}.$

Therefore, it is perfect reasonable to say

$\mathbb{E}mp$ $\underline{\underline{\text{is}}}$ the emptiness problem for type 0 grammars

While $\mathbb{E}mp \neq E$, their solvability is equivalent.

<u>Remark</u> Nothing special about $\mathbb{E}mp$.

Similarly

$$\mathbb{F}in \equiv_m \{v ; \mathcal{L}(G_v) \text{ is finite}\}$$

$$\mathbb{I}nf \equiv_m \{v ; \mathcal{L}(G_v) \text{ is infinite}\}$$

In particular:

$$\mathbb{E}quiv := \{(w,v) ; W_w = W_v\}$$
$$\equiv_m$$
$$\{(w,v) ; \mathcal{L}(G_w) = \mathcal{L}(G_v)\}$$

Slight caution here: official definition only of $\mathbb{N}$, not $\mathbb{N}^2$.

We call an index set I trivial if $I = \emptyset$ or $I = \mathbb{N}$.

Others: non-trivial.

Henry G. Rice 1920-2003

Theorem (Rice's Theorem)

No nontrivial index set is computable.

Corollary Emp, Inf, Fin, Tot are all not computable.

Corollary Emptiness problem for type 0 grammars is unsolvable.

Proof. Uses the technique we've seen in the proof of

$\mathbb{K}$ is Σ_1-complete

Structure of the proof –

① Find the right $g : \mathbb{N}^2 \dashrightarrow \mathbb{N}$ computable

② Apply s-m-n to get h total computable with $g(w,v) = f_{h(w),1}(v)$

③ Claim and prove that h is the desired reduction.

Lecture XXIII, page 7

Theorem $\mathbb{K}$ is Σ_1-complete.

Proof. NTS if X is c.e., $X \leq_m \mathbb{K}$.

Let $X = \mathrm{dom}(f)$ for some comp f

$$g(w,v) := f(w)$$

Apply s-m-n theorem to g:
There is a total computable h s.t.

$$f_{h(w),1}(v) = g(w,v) = f(w)$$

Note that $f_{h(w),1}$ is either constant or nowhere defined.

Claim h reduces X to $\mathbb{K}$.

[Proof: Suppose $w \in X = \mathrm{dom}(f)$, so $f(w)\downarrow$, and thus $f_{h(w),1}$ is a constant function.

Thus $f_{h(w),1}(h(w))\downarrow \implies h(w) \in \mathbb{K}$.

Suppose now $w \notin X = \mathrm{dom}(f)$, so $f(w)\uparrow$ and thus $f_{h(w),1}$ is nowhere defined.

Thus $f_{h(w),1}(h(w))\uparrow \implies h(w) \notin \mathbb{K}$.]

q.e.d.

Almost all proofs of $A \leq_m B$ work along those lines.

Proof of Rice Let I be a nontrivial index set:

Fix $w_0 \notin I$ and $w_1 \in I$ (nontriviality).

Fix $e \in \mathbb{N}$ s.t. $W_e = \emptyset$.

We have two cases: $e \in I$ or $e \notin I$.

Case 1. Case 2

We're going to show:

if $e \in I \implies \mathbb{N} \setminus \mathbb{K} \leq_m I$ → I is not c.e., so not computable

if $e \notin I \implies \mathbb{K} \leq_m I$ → I is not computable

Case 1 : $e \in I$.

$$w_0 \notin I$$

$$g(w,v) := \begin{cases} f_{w_0,1}(v) & \text{if } w \in \mathbb{K} \\ \uparrow & \text{o/w} \end{cases}$$

<u>Important</u> : This is not an application of the case distinction lemma; instead we have to argue that if we run the check $f_w(w)\downarrow$, and fail, the result is the desired value of g.

g is computable: check whether $w \in \mathbb{K}$, if not $\uparrow$ is correct; if $w \in \mathbb{K}$, then compute $f_{w_0,1}(v)$.

<u>s-m-n</u> : ex. h s.t. $f_{h(w),1}(v) = g(w,v)$.

<u>Claim</u> h reduces $\mathbb{W}\setminus\mathbb{K}$ to I.

$$w \in \mathbb{K} \longrightarrow g(w,v) = f_{w_0,1}(v)$$

$$g(w,v) = f_{h(w),1}(v)$$

$$\longrightarrow f_{h(w),1} = f_{w_0,1} \longrightarrow W_{h(w)} = W_{w_0}$$

$[w_0 \notin I]$

$\longrightarrow h(w) \notin I$

[since $h(w)$, w_0 are weakly eq.]

$w \notin \mathbb{K} \longrightarrow g(w,v)\uparrow$ for all v

$\longrightarrow f_{h(w),1}(v)\uparrow$ for all v

$\longrightarrow W_{h(w)} = \emptyset = W_e$

[$e, h(w)$ are weakly equivalent]

$\longrightarrow h(w) \in \mathcal{I}$.

Together: $w \in \mathbb{K} \Longleftrightarrow h(w) \notin \mathcal{I}$

$\Longrightarrow \mathbb{W}\setminus\mathbb{K} \leq_m \mathcal{I}$

<u>Case 2</u> : $e \notin \mathcal{I}$. $\quad w_1 \in \mathcal{I}$

$$g(w,v) := \begin{cases} f_{w_1,1}(v) & \text{if } w \in \mathbb{K} \\ \uparrow & \text{o/w} \end{cases}$$

By same argument, g is computable, so by s-m-n find h s.t.

$f_{h(w),1}(v) = g(w,v)$.

<u>Claim</u> h reduces $\mathbb{K}$ to $\mathcal{I}$.

$w \in \mathbb{K} \longrightarrow f_{h(w),1} = f_{w_1,1} \longrightarrow W_{h(w)} = W_{w_1}$

[$w_1 \in \mathcal{I}$]

$\longrightarrow h(w) \in \mathcal{I}$.

$w \notin \mathbb{K} \longrightarrow f_{h(w),1}$ is nowhere defined

$\longrightarrow W_{h(w)} = \emptyset = W_e$

[since in Case 2, $e \notin \mathbb{I}$]

$\longrightarrow h(w) \notin \mathbb{I}$.

q.e.d.

Remark We prove more than claimed:

$e \in \mathbb{I} \Longrightarrow \mathbb{N} \setminus \mathbb{K} \leq_m \mathbb{I}$

$e \notin \mathbb{I} \Longrightarrow \mathbb{K} \leq_m \mathbb{I}$

$\mathbb{N} \setminus \mathbb{K} \leq_m \mathbb{E}\mathrm{mp}, \mathbb{F}\mathrm{in}$

$\mathbb{K} \leq_m \mathbb{I}\mathrm{nf}, \mathbb{T}\mathrm{ot}$.

On ES #4, you'll show that $\mathbb{E}\mathrm{mp} \equiv_m \mathbb{N} \setminus \mathbb{K}$ (54).

The other three are even more unsolvable.

Theorem $\mathbb{F}\mathrm{in}$ is neither Σ_1 nor Π_1.

Proof Rice's Thm (more precisely, proof of Rice's Thm) implies

$$\mathbb{F}\mathrm{in} \text{ is not } \Sigma_1.$$

It's enough to show $\mathbb{K} \leq_m \mathbb{F}\mathrm{in}$.

We use the same strategy again:

$$g(w,v) := \begin{cases} \uparrow & \text{if } (w,v) \in T_W \\ \varepsilon & \text{o/w} \end{cases}$$

the truncated computation

This function is computable since T_W is, so by s-m-n find h s.t. $f_{h(w),1}(v) = g(w,v)$.

Claim: h reduces $\mathbb{K}$ to $\mathbb{F}\mathrm{in}$.

$$w \in \mathbb{K} \longrightarrow f_{w,1}(w)\downarrow \longrightarrow \exists v\ (w,v) \in T_W$$
$$\longrightarrow \exists v\ \forall u \geq v\ (w,u) \in T_W$$
$$\longrightarrow W_{h(w)} \text{ is finite}$$
$$\longrightarrow h(w) \in \mathbb{F}\mathrm{in}.$$

$$w \notin \mathbb{K} \longrightarrow \forall v\ (w,v) \notin T_W$$
$$\longrightarrow W_{h(w)} = \mathbb{W} \longrightarrow h(w) \notin \mathbb{F}\mathrm{in}.$$

q.e.d.

CONCLUSION

Theorem $\mathbb{E}\mathrm{quiv}$ is not solvable.

Proof. Fix e s.t. $W_e = \emptyset$.

The function $g : w \longmapsto (w, e)$ can be performed by a RM.

$$\chi_{\mathbb{E}\mathrm{mp}} = \chi_{\mathbb{E}\mathrm{quiv}} \circ g$$

And therefore, if $\mathbb{E}\mathrm{quiv}$ is computable, then so is $\mathbb{E}\mathrm{mp}$. But $\mathbb{E}\mathrm{mp}$ wasn't.

q.e.d.

	Word problem	Emptiness problem	Equivalence problem
regular (type 3)	✓	✓	✓
context-free (type 2)	✓	✓	×
context-sensitive (type 1)	✓	×	×
computably enumerable (type 0)	×	×	×

Figure 8: The decision problems of all classes of languages we discussed in an overview

Type 1 results were not proved in the lectures.