

XVI

Sixteenth Lecture of Automata & Formal Languages

10 November 2023

Lecture XV →
pp. 587

THE SUBROUTINE LEMMA

(L 4.6)

If M performs F and M' performs F' ,
then there is a machine M performing
 $F' \circ F$.
... assume $Q \cap Q' = \emptyset$.

Lemma 4.7 (Case Distinction Lemma). Let $Q = \{A_i; i \leq k\}$ be a question with $k + 1$ answers and $f_i : W^{n+1} \rightarrow W^{n+1}$ be operations for $i \leq k$. If Q is answered by a register machine $M = (\Sigma, Q, P)$ and f_i is performed by $M_i := (\Sigma, Q_i, P_i)$ (for $i \leq k$), then we can construct a register machine that performs the operation defined by $g(\vec{w}) := f_i(\vec{w})$ if and only of $\vec{w} \in A_i$.

Lecture XV
Lots of examples

Example 1

operation:

$$F: W^{n+1} \rightarrow W^{n+1}$$

$$\text{dom}(F) = \emptyset$$

Example 2

ALWAYS HALT, DON'T
CHANGE INPUT

$$F = \text{id}: W^{n+1} \rightarrow W^{n+1}$$

Example 3

Is register i empty?

Example 4

Does register i end in a a ?

Example 5

$$F(\vec{w}) = \begin{cases} \vec{w} & w_i \neq \epsilon \\ \uparrow & w_i = \epsilon \\ \text{"undefined"} & \end{cases}$$

6. REMOVE FINAL LETTER IN i (IF EXISTS)

7. EMPTY REG. i

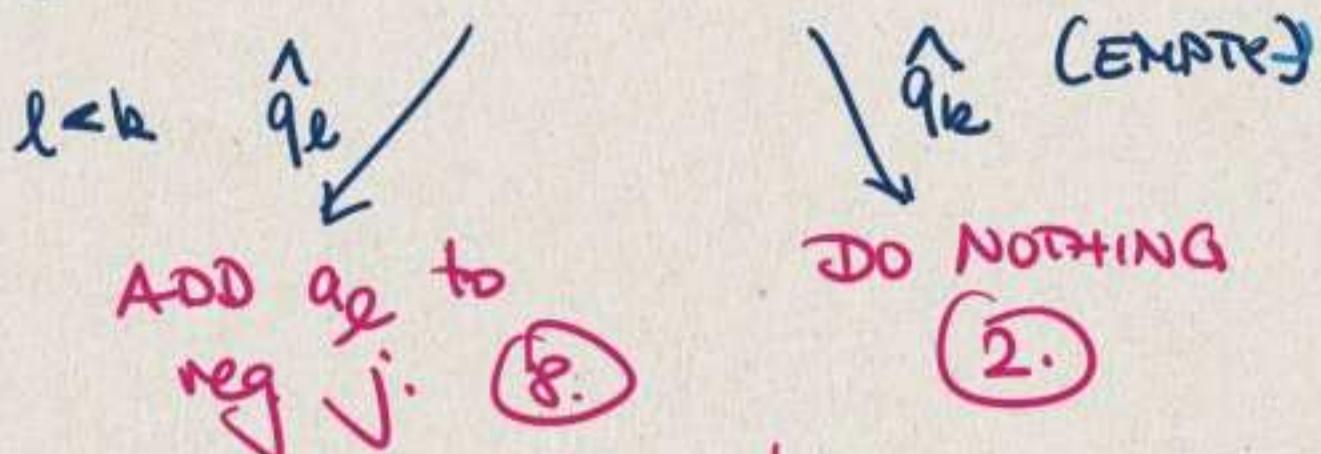
8. ADD a to REG i

9. ADD $w \circ w$ to REG i

10. What is the final letter of i ? (If any)

11. Copy final letter of reg. i (if exists)
to reg. j.

Ask: WHAT IS THE FINAL LETTER 10.



12. Move final letter of : to j.

First: Copy final letter from i to j. (11.)

Then: Remove final letter from i

(6.)

13. Move content from i to j in reverse order.

Apply (12.) repeatedly until empty.

14. Move content from i to j in correct order.

Take an UNUSED REGISTER k.

First: Empty k. (7.)

Then: Reverse i into k.

(13.)

Reverse k into j.

(13.)

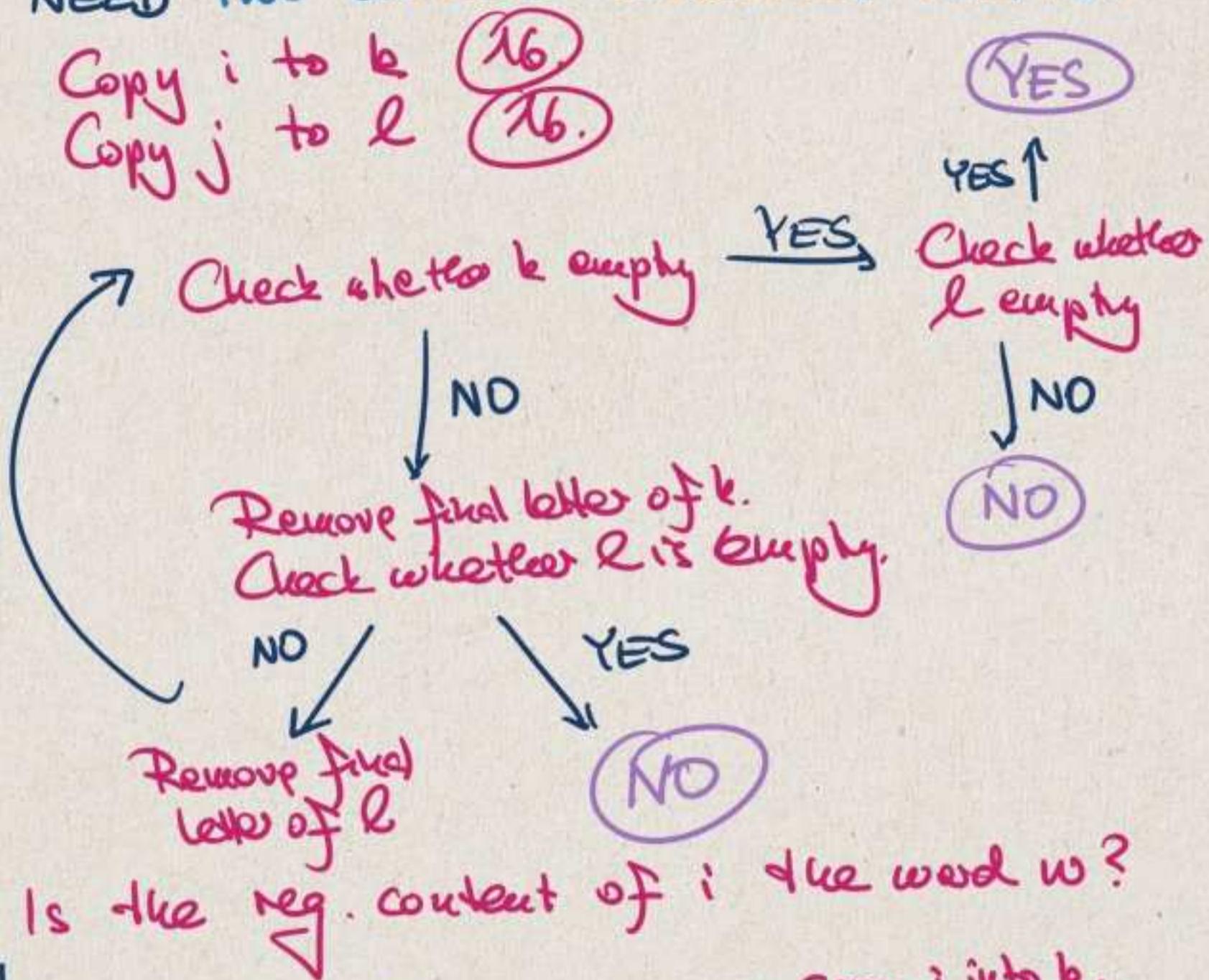
15. Copy content from i to j in reverse order.
Take UNUSED REGISTER k .

SUBROUTINE [Copy first letter from i to j
Move first letter from i to k
Do the subroutine until i is empty.
Then: Move content k to i in
from reverse
order.

16. Copy content from i to j in correct order.

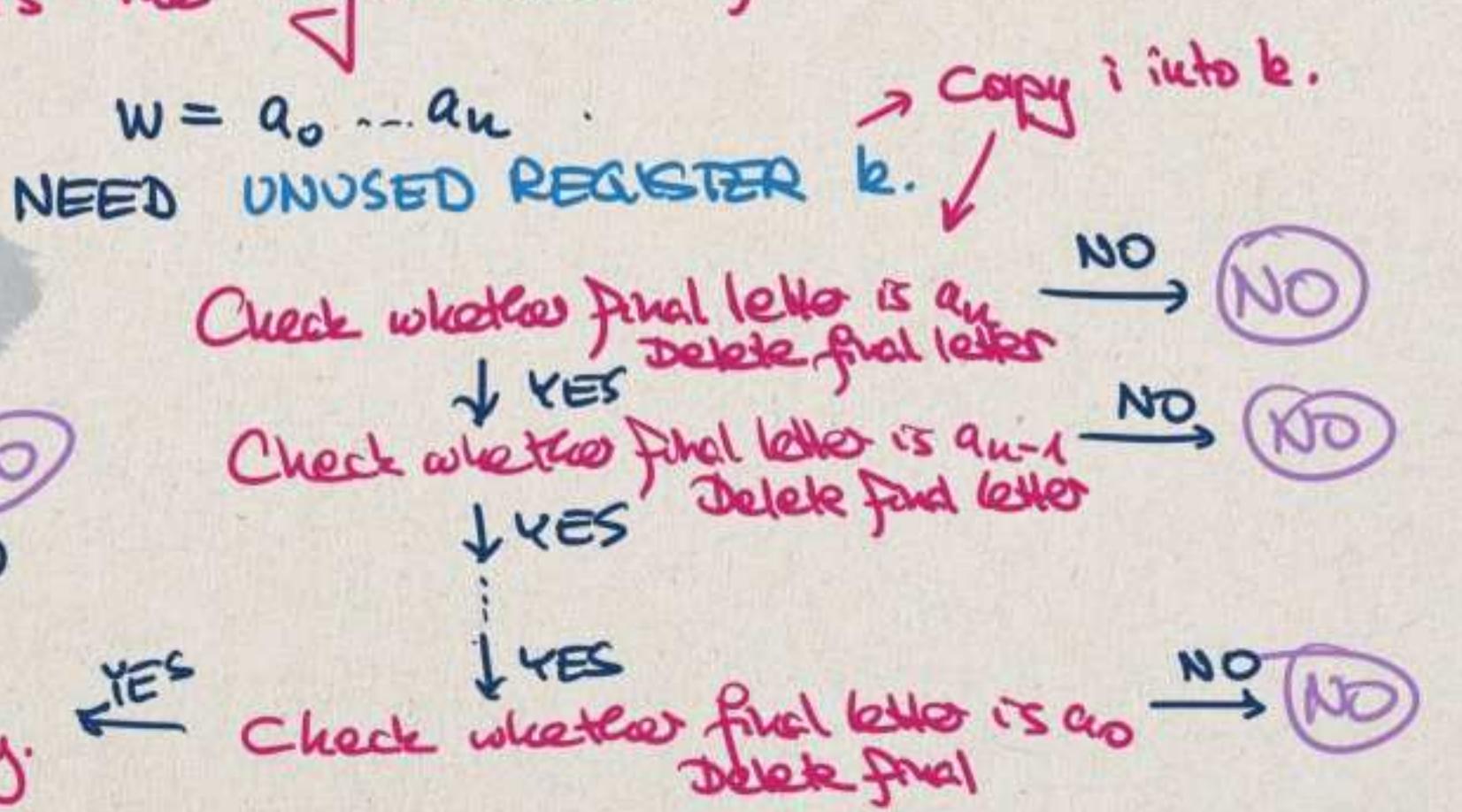
Take UNUSED REGISTER k .
Copy content from i to k in reverse order 15
Move content from k to j in reverse order 13.

17. Do $i \geq j$ have the same # of symbols?
 NEED TWO UNUSED REGISTERS $b \neq l$.



18. Is the neg. content of i the word w ?

Afterwards
clean up by
moving
 b into l .



§ 4.3 Computability -

If M is a register machine, $k \in \mathbb{N}$,
define

$$\rho_{M,k} : \mathbb{W}^k \dashrightarrow \mathbb{W}$$

as follows

$$\text{done}(\rho_{M,k}) := \{ \vec{w} ; M \text{ halts with input } \vec{w} \}$$

$$\rho_{M,k}(\vec{w}) = v_0$$

where (v_0, \dots, v_{k-1}) is the reg. content at time of halting.

Notational convention If k is smaller than the upper register index of M , we consider \vec{w} to mean a vector of length $n+1$ (where n is o.r.i.) and $w_j = \epsilon$ if $j \geq k$.

Definition A function $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ is called computable if there is a RM M s.t. $f = \rho_{M,k}$.

Remark

By padding lemma, the machine M in the def. is far from unique.

Examples

1. $\text{id}: W \rightarrow W$ is computable
by Example 2.

2. $c_{k,v}: \overbrace{W^k}^w \rightarrow W$ constant function
is computable by Example 9.

3. $\pi_{k,i}: \overbrace{W^k}^w \rightarrow W$ projection functions
 $w \mapsto w_i$

is computable:

if $i = 0$ Example 2.

if $i \neq 0$ Example 16.

Remark

By
Structive
Lemma.

1. If $f: W \rightarrow W$ and $g: W^k \rightarrow W$ are comp.

are computable

2. If $F: W^k \rightarrow W^k$ is operation performed

by RNY

$f: W^k \rightarrow W$ computable

then

$f \circ F$ is computable

Definition Let $A \subseteq \mathbb{W}$

$f: \mathbb{W} \rightarrow \mathbb{W}$ is called a characteristic fn of A if

$$f(w) = \varepsilon \iff w \notin A$$

$f: \mathbb{W} \rightarrow \mathbb{W}$ is called a pseudocharacteristic (p.) function of A if $\text{dom}(f) = A$

Fix some $a \in \Sigma$. Then we call

$$\chi_A(w) := \begin{cases} a & \text{if } w \in A \\ \varepsilon & \text{if } w \notin A \end{cases}$$

THE characteristic fn of A

$$\text{and } \psi_A(w) := \begin{cases} a & \text{if } w \in A \\ \uparrow & \text{if } w \notin A \end{cases}$$

THE pseudocharacteristic fn of A .

A $\subseteq \mathbb{W}$ is called computable

if χ_A is computable

A $\subseteq \mathbb{W}$ is called computably-enumerable (c.e.)

if ψ_A is computable.

Remarks

(1) A is computable iff any char. fn of A is computable.

$$\text{Consider } f(w) := \begin{cases} \epsilon & : w = \epsilon \\ a & \text{o/w.} \end{cases}$$

This is computable:

Check whether reg. 0 is empty

Yes

Do nothing

Example 5

$$F(\vec{w}) = \begin{cases} \vec{w} & : w_i \neq \epsilon \\ \vec{w} & : w_i = \epsilon \end{cases}$$

can be performed by RM as follows:

Check WHETHER i IS EMPTY;
if so, HALT w/o CHANGES;
otherwise, NEVER HALT.

The while loop
will never
terminate,
but the pink
represents like
it precise.

If $\downarrow g$ is any char. fn of A ,
then $f \circ g = \chi_A$.

Similarly:

A is c.e. iff any pseudocharacteristic fn is computable

(2) A is computable $\implies A$ is c.e.

If g is any char. fn of A , then

$f \circ g$ is a pseudocharact. fn of A .

from G.S.