XVI

Sixteenth Lecture
of Automata & Formal Languages

10 November 2023

Lecture XV pp. 5 & 7

THE SUBROUTINE LEMMA (L4.6)

If M performs F and M' performs F', then there is a machine $\hat{M}$ performing $F' \circ F$.

... assume $Q \cap Q' = \emptyset$.

Lemma 4.7 (Case Distinction Lemma). Let $Q = \{A_i\,;\, i \leq k\}$ be a question with $k+1$ answers and $f_i : \mathbb{W}^{n+1} \dashrightarrow \mathbb{W}^{n+1}$ be operations for $i \leq k$. If Q is answered by a register machine $M = (\Sigma, Q, P)$ and f_i is performed by $M_i := (\Sigma, Q_i, P_i)$ (for $i \leq k$), then we can construct a register machine that performs the operation defined by $g(\vec{w}) := f_i(\vec{w})$ if and only of $\vec{w} \in A_i$.

Lecture XV

Lots of examples

Example 1 Operation:

$F : \mathbb{W}^{n+1} \dashrightarrow \mathbb{W}^{n+1}$

$\mathrm{dom}(F) = \emptyset$

Example 2 ALWAYS HALT, DON'T CHANGE INPUT

$F = \mathrm{id} : \mathbb{W}^{n+1} \longrightarrow \mathbb{W}^{n+1}$

Example 3 Is register i empty?

Example 4 Does register i end in a?

Example 5

$$F(\vec{w}) := \begin{cases} \vec{w} & w_i \neq \varepsilon \\ \uparrow & w_i = \varepsilon \end{cases}$$

"undefined"

6. REMOVE FINAL LETTER IN i (IF EXISTS)
7. EMPTY REG. i
8. ADD a to REG i
9. ADD $w \in \mathbb{W}$ to REG i
10. What is the final letter of i? (if any)

11. Copy final letter of reg. i (if exists) to reg. j.

Ask: WHAT IS THE FINAL LETTER 10.

$l<k$ $\hat{a}_l$ → ADD a_l to reg. j. 8.

$\hat{a}_k$ (EMPTY) → DO NOTHING 2.

12. Move final letter of i to j.

First: Copy final letter from i to j. 11.

Then: Remove final letter from i 6.

13. Move content from i to j in reverse order.

Apply 12. repeatedly until empty.

14. Move content from i to j in correct order.

Take an <u>UNUSED</u> <u>REGISTER</u> k.

First: Empty k. 7.

Then: Reverse i into k. 13.

Reverse k into j. 13.

15. Copy content from i to j in reverse order.

Take UNUSED REGISTER k.

SUBROUTINE [Copy final letter from i to j
Move final letter from i to k.

Do the subroutine until i is empty.

<u>Then</u>: Move content from k to i in reverse order.

16. Copy content from i to j in correct order.

Take UNUSED REGISTER k.

Copy content from i to k in reverse order (15)

Move content from k to j in reverse order. (13.)

17. Do i & j have the same # of symbols?

NEED TWO UNUSED REGISTERS k & l.

Copy i to k (16.)

Copy j to l (16.)

Check whether k empty $\xrightarrow{\text{YES}}$ Check whether l empty

Check whether l empty $\xrightarrow{\text{YES}}$ YES

Check whether l empty $\xrightarrow{\text{NO}}$ NO

Check whether k empty $\xrightarrow{\text{NO}}$ Remove final letter of k. Check whether l is empty.

NO → Remove final letter of l → Check whether k empty

YES → NO

18. Is the reg. content of i the word w?

$w = a_0 \dots a_n$

NEED UNUSED REGISTER k.

Afterwards clean up by moving k into i.

Copy i into k.

Check whether final letter is a_n $\xrightarrow{\text{NO}}$ NO

Delete final letter

↓ YES

Check whether final letter is a_{n-1} $\xrightarrow{\text{NO}}$ NO

Delete final letter

↓ YES

⋮

↓ YES

Check whether final letter is a_0 $\xrightarrow{\text{NO}}$ NO

Delete final

$\xleftarrow{\text{YES}}$ Check whether empty.

YES ↑ YES

NO ↗ NO

§ 4.3 Computability

If M is a register machine, $k \in \mathbb{N}$,
define

$$f_{M,k} : \mathbb{W}^k \dashrightarrow \mathbb{W}$$

as follows

$$\mathrm{dom}(f_{M,k}) := \{\vec{w} \,;\, M \text{ halts with input } \vec{w}\}$$

$$f_{M,k}(\vec{w}) = v_0$$

where $(v_0, \ldots, v_{n-1})$ is the reg. content at time of halting.

Notational convention: If k is smaller than the upper register index of M, we consider $\vec{w}$ to mean a vector of length $n+1$ (where n is u.r.i.) and $w_j = \varepsilon$ if $j \geq k$.

Definition A partial function $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ is called computable if there is a RM M s.t. $f = f_{M,k}$.

Remark By padding lemma, the machine M in the def. is far from unique.

Examples

1. $id : \mathbb{W} \longrightarrow \mathbb{W}$ is computable by Example 2.

2. $c_{k,v} : \mathbb{W}^k \longrightarrow \mathbb{W}$, $\vec{w} \longmapsto v$ — constant function — is computable by Example 9.

3. $\pi_{k,i} : \mathbb{W}^k \longrightarrow \mathbb{W}$, $\vec{w} \longmapsto w_i$ — projection functions — is computable:
   - if $i = 0$ Example 2.
   - if $i \neq 0$ Example 16.

Remark

By Subroutine Lemma.

1. If $f : \mathbb{W} \dashrightarrow \mathbb{W}$, $g : \mathbb{W}^k \dashrightarrow \mathbb{W}$ are computable then $f \circ g$ is comp.

2. If $F : \mathbb{W}^k \dashrightarrow \mathbb{W}^k$ is operation performed by RM, $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ computable then $f \circ F$ is computable

Definition Let $A \subseteq W$

$f : W \longrightarrow W$ is called a characteristic fn of A if

$$f(w) = \varepsilon \iff w \notin A .$$

$f : W \dashrightarrow W$ is called a pseudocharacteristic (p.) function of A if $\mathrm{dom}(f) = A$

Fix some $a \in \Sigma$. Then we call

$$\chi_A(w) := \begin{cases} a & \text{if } w \in A \\ \varepsilon & \text{if } w \notin A \end{cases}$$

THE characteristic fn of A

and

$$\psi_A(w) := \begin{cases} a & \text{if } w \in A \\ \uparrow & \text{if } w \notin A \end{cases}$$

THE pseudocharacteristic fn of A.

Definition $A \subseteq W$ is called computable if χ_A is computable

$A \subseteq W$ is called computably enumerable (c.e.) if ψ_A is computable.

Remarks

Example 5

$$F(\vec{w}) := \begin{cases} \vec{w} & w_i \neq \varepsilon \\ \uparrow & w_i = \varepsilon \end{cases}$$

"undefined"

can be performed by RM as follows:

This looks like natural language, but the pink refinements make it precise

Check WHETHER i IS EMPTY;
if so, HALT W/O CHANGES;
otherwise, NEVER HALT.

① A is computable iff <u>any</u> char. fn of A is computable.

Consider $f(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ a & \text{o/w.} \end{cases}$

This is computable:

Check whether reg. 0 is empty

- Yes → Do nothing
- No → Empty reg. 0. Write a into reg. 0.

If g is any char. fn of A, then $f \circ g = \chi_A$.

Similarly:
A is c.e. iff <u>any</u> pseudocharacteristic fn is computable.

② A is computable $\Longrightarrow$ A is c.e.

If g is any char. fn of A, then $F \circ g$ is a pseudocharact. fn of A.

from Ex. 5